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A NOTE ON RADICALS OF PARAGRADED RINGS

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Dedicated to the memory of Professor Marc Krasner

ABSTRACT. In this paper we prove that there exist paragraded rings which are not graded and we discuss prime and Jacobson radicals of paragraded rings. In particular, we prove that paragraded counterparts of prime and Jacobson radicals are the largest paragraded ideals contained in them.

1. INTRODUCTION

The homogeneous part of the direct product of graded rings needs not to be the direct product of the homogeneous parts of those graded rings, which was the motivation for introducing the notion of a *paragraded ring* [14, 15, 16, 17]. A ring R is called *paragraded* if there exists a mapping $\pi : \Delta \to \mathrm{Sg}(R,+)$, $\pi(\delta) = R_\delta$ ($\delta \in \Delta$), of a partially ordered set $(\Delta, <)$, which is from below complete semi-lattice and from above inductively ordered, to the set $\mathrm{Sg}(R,+)$ of subgroups of $(R,+)$, called *paragrading*, and the following axioms are satisfied:

$r1)$ $\pi(0) = R_0 = \{0\}$, where $0 = \inf \Delta$; $\delta < \delta' \Rightarrow R_\delta \subseteq R_{\delta'}$;

> **Remark 1.1.** $A = \bigcup_{\delta\in\Delta} R_\delta$ is called *the homogeneous part* of R with respect to π, and elements of A are called *homogeneous elements*.
>
> **Remark 1.2.** If $x \in A$, we say that $\delta(x) = \inf\{\delta \in \Delta \mid x \in R_\delta\}$ is the *degree of* x. We have $\delta(x) = 0$ if and only if $x = 0$. Elements $\delta(x)$, $x \in A$, are called *principal degrees* and they form a set which is denoted by Δ_p.

$r2)$ $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} R_\delta = R_{\inf \theta}$;

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- $r3)$ Homogeneous part A is a generating set of $(R, +)$ with the set of A-inner relations: $x + y = z$;
- $r4)$ Let $B \subseteq A$ be a subset such that for all $x, y \in B$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x)$, $x \in B$;
- $r5)$ For all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$.

This definition gives a binary operation on Δ, namely $\xi\eta = \sup\{\delta(x) \mid x \in R_\xi R_\eta\}$, called the *minimal multiplication* [16, 17], and so, $R_\xi R_\eta \subseteq R_{\xi\eta}$. An ideal I of a paragraded ring R is called *homogeneous* [17] if I is generated by $I \cap A$ by A-inner relations, where A is the homogeneous part of R.

One of the aims of this note is to show that there are paragraded rings which are not graded. We also discuss prime and Jacobson radicals of paragraded rings and as main results we obtain that paragraded counterparts of prime and Jacobson radicals are the largest paragraded ideals contained in them. Note that the notion of the Jacobson radical introduced here is more general than that from [7] where we considered the category of paragraded rings with the same paragrading set. In the process, we will use the notions of a *quasianneid* [16], *paragraded module* and of a *quasimoduloid* [16], which we recall in the sequel. Origins of this "homogeneous" approach can be found in [12].

If R is a paragraded ring with homogeneous part A, then we may observe restrictions of operations from R to A. Induced addition is partial and we write $x\#y$ if and only if $x + y \in A$. The obtained structure is called a *paraanneid* [16]. If $x \in A$, let $A(x) = \{y \in A \mid x\#y\}$. Paraanneid certainly satisfies the following axioms:

- $a1)$ There exists an element $0 \in A$ such that $A = A(0)$ and such that for all $x \in A$ we have $0 + x = x$;
- $a2)$ If $a \in A$, $x + y$ is always defined on $g(a) = \{x \in A \mid A(x) \supseteq A(a)\}$ and $(g(a), +)$ is an Abelian group;
- $a3)$ If $B \subseteq A$ is such that for all $x, y \in B$ we have $x\#y$, then there exists $G \subseteq A$ such that $x + y \in G$ for all $x, y \in G$, $x \in G$ implies $g(x) \subseteq G$ and $B \subseteq G$;
- $a4)$ $A^2 \subseteq A$;
- $a5)$ $x\#x'$ and $y\#y'$ imply $xy\#x'y'$.

Structure $(A, +, \cdot)$ which satisfies axioms $a1) - a5)$ is called a *quasianneid* [16]. Quasianneid does not have to be a paraanneid; it is under a few more assumptions (see [16]), in which case A can be *linearized* [16] to a paragraded ring, denoted by $\overline{A}$, whose homogeneous part it is.

If R is a paragraded ring with paragrading $\pi : \Delta \ni \delta \mapsto R_\delta$, and M a right R-module, then M is called a *paragraded R-module* [16, 17] if $(M, +)$ satisfies axioms $r1) - r4)$ for π' instead of π, D instead of Δ, M instead of R and

$N = \bigcup_{d\in D} M_d$ instead of A, and if moreover, for all $d \in D$ and $\delta \in \Delta$, there exists $t \in D$ such that $\pi'(d)R_\delta \subseteq \pi'(t)$. If we observe restrictions of addition from M to N and of external multiplication $M \times R \to M$ to $N \times A \to N$, we obtain a structure called a *paramoduloid* [16]. A paramoduloid N over a paraanneid A certainly satisfies the following axioms:

$m1)$ $x(ab) = (xa)b$ $(a, b \in A, x \in N)$;
$m2)$ If $a, a' \in A$ and $x, x' \in N$ are such that $a\#a'$ and $x\#x'$, then $xa\#x'a'$;
$m3)$ If $a\#a'$ $(a, a' \in A)$ and $x \in N$, then $x(a + a') = xa + xa'$;
$m4)$ If $x\#x'$ $(x, x' \in N)$ and $a \in A$, then $(x + x')a = xa + x'a$.

A *quasimoduloid* N over a quasianneid A is a structure which satisfies axioms $m1) - m4)$. It does not have to be a paramoduloid. It will be a paramoduloid under a few more assumptions (see [16, 17]), in which case N can be linearized to a paragraded module, which we denote by $\overline{N}$, whose homogeneous part it is.

If N is an A-quasimoduloid, then $K \subseteq N$ is called a *subquasimoduloid* [17] if: *a)* $x \in K \Rightarrow -x \in K$; *b)* $x, y \in K \wedge x\#y \Rightarrow x + y \in K$; *c)* $a \in A \wedge x \in K \Rightarrow xa \in K$. A subquasimoduloid of a quasianneid A, observed as an A-quasimoduloid, is called a *right ideal* of a quasianneid A [17]. Factor structures are defined as usual (for more details, one may consult [17]; see also [20]).

2. Examples of paragraded rings

It is known from [17] that there is a large class of paragraded rings which are graded, but here we provide a class of examples of paragraded rings which are not graded. By a *graded ring* [1, 5, 13, 17, 9, 10, 11] R we mean $R = \bigoplus_{\delta\in\Delta} R_\delta$ if for all $\xi, \eta \in \Delta$ there exists $\zeta \in \Delta$ such that $R_\xi R_\eta \subseteq R_\zeta$, where R_δ are additive subgroups of R, and Δ is a nonempty set.

Theorem 2.1. *A paragraded ring which is not graded exists.*

Proof. Let R be a ring and $M_2(R)$ the set of all 2×2 matrices over R. As we know, $M_2(R)$ is a ring under the usual matrix addition and multiplication. Let binary sequences $a_1^1 a_1^2 a_2^1 a_2^2$ of length four correspond to the set of matrices over R which have as (i, j)-entry an arbitrary element from R if $a_i^j = 1$, and a zero (i, j)-entry if $a_i^j = 0$. Also, for convenience, let us denote that set by $R_{a_1^1 a_1^2 a_2^1 a_2^2}$. For instance, 1101 corresponds to $R_{1101} = \left(\begin{smallmatrix} R & R \\ 0 & R \end{smallmatrix}\right)$. Obviously, $R_{1111} = M_2(R)$ and $R_{0000} = O$, where O denotes the zero-matrix. Denote by Δ the set of sequences $a_1^1 a_1^2 a_2^1 a_2^2$, $a_i^j \in \{0, 1\}$, $i, j = \{1, 2\}$, and let us define $a_1^1 a_1^2 a_2^1 a_2^2 \le b_1^1 b_1^2 b_2^1 b_2^2$ if and only if $R_{a_1^1 a_1^2 a_2^1 a_2^2} \subseteq R_{b_1^1 b_1^2 b_2^1 b_2^2}$, $a_1^1 a_1^2 a_2^1 a_2^2, b_1^1 b_1^2 b_2^1 b_2^2 \in \Delta$. For every $a_1^1 a_1^2 a_2^1 a_2^2, b_1^1 b_1^2 b_2^1 b_2^2 \in \Delta$, we have

$R_{a_1^1a_1^2a_2^1a_2^2} \cap R_{b_1^1b_1^2b_2^1b_2^2} = R_{c_1^1c_1^2c_2^1c_2^2}$, for some $c_1^1c_1^2c_2^1c_2^2 \in \Delta$, and R_{0000} is contained in $R_{a_1^1a_1^2a_2^1a_2^2}$, for every $a_1^1a_1^2a_2^1a_2^2 \in \Delta$. Also, if $\{R_{a_1^1a_1^2a_2^1a_2^2}\}$ is a chain of subgroups with respect to the set inclusion, their union is their upper bound. Hence, Δ is from below a complete semi-lattice and from above inductively ordered with respect to $\leq$, with 0000 as the least element. Also, the mapping $\pi : \Delta \to \mathrm{Sg}(M_2(R), +)$, defined by $\pi(a_1^1a_1^2a_2^1a_2^2) = R_{a_1^1a_1^2a_2^1a_2^2}$, where $\mathrm{Sg}(M_2(R), +)$ denotes the set of subgroups of $(M_2(R), +)$, is paragrading, since, as it can be easily verified, it satisfies all axioms $r1$)-$r5$). Hence, $M_2(R)$ is a paragraded ring. Notice also that $M_2(R)$ is not graded with respect to π, since, for instance, $R_{1000} \cap R_{1001} = R_{1000} \neq O$. $\square$

Reasoning from the proof of the previous theorem can be generalized into the following theorem.

Theorem 2.2. *If R is a ring, then every matrix ring $M_n(R)$, where n is a natural number, can be regarded as a paragraded ring.*

Example 2.3. Let A be a ring and observe the ring of upper triangular matrices $R = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$. Also, let $R_{\delta_1} = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $R_{\delta_2} = \left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$, $R_{\delta_3} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right)$, and $R_{\delta_4} = \left(\begin{smallmatrix} A & A \\ 0 & 0 \end{smallmatrix}\right)$. If $R_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, denote by Δ the set $\{0\} \cup \{\delta_i \mid i = 1, 2, 3, 4\}$. For convenience, let $\delta_0 = 0$. Set Δ is a partially ordered set, which is from below complete semi-lattice and from above inductively ordered, with respect to $\delta_i < \delta_j \Leftrightarrow R_{\delta_i} \subseteq R_{\delta_j}$, $i, j = 0, 1, 2, 3, 4$. It is easy to see that R is a paragraded ring with respect to $\pi : \delta_i \mapsto R_{\delta_i}$, $\delta_i \in \Delta$. Note that R is not graded with respect to π since for instance, $R_{\delta_1} \cap R_{\delta_4} = R_{\delta_1} \neq R_0$. It is interesting to notice that every element $\delta \in \Delta$ is either an idempotent or nilpotent of degree of nilpotency 2 with respect to the minimal multiplication "$\cdot$". The following table shows this property.

$\cdot$	0	δ_1	δ_2	δ_3	δ_4
0	0	0	0	0	0
δ_1	0	δ_1	δ_2	0	δ_4
δ_2	0	0	0	δ_2	0
δ_3	0	0	0	δ_3	0
δ_4	0	δ_1	δ_2	δ_2	δ_4

Example 2.4. Let us observe a subring $\left(\begin{smallmatrix} A & A \\ 0 & 0 \end{smallmatrix}\right)$ of the paragraded ring $\left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$ from the previous example. It is also paragraded with respect to paragrading $R_{\delta_1} = \left(\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right)$, $R_{\delta_2} = \left(\begin{smallmatrix} A & A \\ 0 & 0 \end{smallmatrix}\right)$. It is also worth to notice that this paragrading induces the minimal multiplication which makes the paragrading set a cancellative partial groupoid.

3. On prime and Jacobson radicals

The *prime spectrum of a paragraded ring* is introduced in [6]. Here we reformulate it for quasianneids.

Let A be a quasianneid. A proper ideal P of A is said to be *prime* if for all ideals I, J of A, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. As in the case of rings, one may prove that an ideal P of A is prime if and only if $xAy \subseteq P$ implies $x \in P$ or $y \in P$, where $x, y \in A$. The set of all prime ideals of A is denoted by $\operatorname{Spec}(A)$ and it is called the *prime spectrum of a quasianneid A*. The *prime or the Baer radical of a quasianneid A* is defined to be the set $\bigcap(P \mid P \in \operatorname{Spec}(A))$ and is denoted by $\beta(A)$.

Proposition 3.1. *Let A be a paraaneid, that is, let A be linearizable to a paragraded ring $\overline{A}$.*

a) *If $\overline{P} \in \operatorname{Spec}(\overline{A})$, then $\overline{P} \cap A \in \operatorname{Spec}(A)$, where $\operatorname{Spec}(\overline{A})$ denotes the spectrum of $\overline{A}$ regarded as a ring.*

b) *If $Q \in \operatorname{Spec}(A)$, then there exists $\overline{P} \in \operatorname{Spec}(\overline{A})$ such that $Q = \overline{P} \cap A$.*

c) *Prime radical of A coincides with $\beta(\overline{A}) \cap A$, where $\beta(\overline{A})$ denotes the prime radical of $\overline{A}$ regarded as a ring.*

Proof. Statements *a*) and *b*) can be proved by the same means as in the case of ordinary graded rings (see [2] or [18]), while *c*) follows directly from *a*) and *b*). $\square$

The Jacobson radical of an anneid, the homogeneous part of a graded ring with induced operations, is thoroughly examined in [5], inspired by [8] (see also [3]). In this section we will see that the similar theory may be established for quasianneids under some appropriate modifications in formulations and in proofs of relevant results. In this section, quasimoduloids are, for convenience, denoted by M.

Let A be a quasianneid, M a right A-quasimoduloid, N a subquasimoduloid of M, and S a subset of M. If $(N : S)$ denotes the set of elements $a \in A$ such that $Sa \subseteq N$, then it can be easily proved that $(N : S)$ is a right ideal of A. Particularly, if $x \in M$, then $(0 : x)$ is a right ideal of A.

The *heart* [17] of an A-quasimoduloid M, denoted by C, is defined to be the set $\{x \in M \mid (\forall y \in M)\ x\#y\}$.

An A-quasimoduloid M is called *regular* [17] if for every $a, b \in A$ and $x \in M$, $xa, xb \notin C$, where C is the heart of M, and $xa\#xb$, imply $a\#b$. A quasianneid A is called *right regular* [17] if it is regular as a right A-quasimoduloid. The notion of a *left regular* and of a *regular* quasianneid is clear enough.

If M and M' are two A-quasimoduloids, then the mapping $f : M \to M'$ is called a *quasihomomorphism* [17] if for all $x, y \in M$ and $a \in A$: *a*)

$x\#y$ implies $f(x)\#f(y)$ and $f(x+y) = f(x) + f(y)$; *b)* $f(xa) = f(x)a$. If, moreover, $f(x)\#f(y)$ and $f(x), f(y) \notin C'$ imply $x\#y$, where C' denotes the heart of M', then f is called a *homomorphism.* The paragraded versions of these mappings are called *quasihomogeneous homomorphism* and *homogeneous homomorphism*, respectively (see [17]).

Lemma 3.2. *If M is a regular A-quasimoduloid and $0 \neq x \in M$, then $A/(0 : x) \cong xA$.*

Proof. Let $f : A \to xA$ be the mapping defined by $f(a) = xa$ $(a \in A)$. If $a, b \in A$ and $a\#b$, then $xa\#xb$ and $x(a + b) = xa + xb$, so $f(a + b) = f(a) + f(b)$. Also, $f(ab) = x(ab) = (xa)b = f(a)b$. Hence, f is a quasihomomorphism of A-quasimoduloids. It is a homomorphism, since, if $f(a)\#f(b)$ and $f(a), f(b) \notin C$, then $xa\#xb$, $xa, xb \notin C$, and since M is regular, we have $a\#b$. The kernel of f is a right ideal of A, namely, $(0 : x)$, while its image is obviously xA. According to the first isomorphism theorem [17], $A/(0 : x) \cong xA$. □

Definition 3.3. An A-quasimoduloid M is called *irreducible* if $MA \neq 0$ and if 0 and M are its only subquasimoduloids.

Definition 3.4. The *Jacobson radical* of a quasianneid (paraanneid) A is defined to be the intersection of annihilators of all irreducible regular A-quasimoduloids (A-paramoduloids). The intersection of annihilators of all irreducible A-quasimoduloids (A-paramoduloids) is called the *large Jacobson radical* of a quasianneid (paraanneid) A.

Remark 3.5. It is clear that the large Jacobson radical of a quasianneid is contained in the Jacobson radical of a quasianneid. Also, the prime radical of a quasianneid is contained in the large Jacobson radical of an anneid.

Definition 3.6. A right ideal I of a quasianneid A is called *modular* if there exists an element $u \in A$ such that $a \sim ua \mod I$, for all $a \in A$. We say that I is a modular ideal with respect to u.

Remark 3.7. Let us notice that, if I is a proper modular ideal with respect to u, then $\delta(u)$ and $\delta(u^2)$ have a common upper bound. Indeed, since I is proper, $u \notin I$, and so, $u\#u^2$ and $u - u^2 \in I$. Hence, $u^2 \notin I$, and so, $u^2 \neq 0$, and the claim follows.

Definition 3.8. A quasimoduloid M over a quasianneid A is called *strictly cyclic* if there exists $x \in M$ such that $M = xA$. Such an element x is called a *strict generator of M.*

An A-quasimoduloid M (see [17]) is said to be *without heart* if

$$C = \{x \in M \mid (\forall y \in M)\ x\#y\} = 0.$$

Lemma 3.9. *A regular A-quasimoduloid M without heart which is strictly cyclic is isomorphic to A/I, where I is a modular right ideal of A. Every modular right ideal I of A is of the form $(0:x)$, where x is a strict generator of an A-quasimoduloid M.*

Proof. Let M be a strictly cyclic regular A-quasimoduloid without heart, and let x be its strict generator. Then, according to Lemma 3.2, $M = xA \cong A/(0:x)$. We hence need to prove that $(0:x)$ is modular. Since $x \in xA$ and since M is regular, there exists $u \in A$ such that $x = xu$. For an arbitrary $a \in A$ we have $xa = xua$. If $xa = 0$, then both a and ua belong to $(0:x)$. If $xa = xua \neq 0$, then, since M is without heart, and since M is regular, we have $a\#ua$ and $0 = xa - xua = x(a - ua)$, and so, $a - ua \in (0:x)$. Therefore, u is a left identity modulo $(0:x)$, and so, $(0:x)$ is modular. The second assertion is clear. □

Similarly to the case of rings, one may now prove the following theorem.

Theorem 3.10. *If M is an irreducible regular A-quasimoduloid without heart, then $M \cong A/I$, where I is a maximal modular right ideal of A. Conversely, if I is a maximal modular right ideal of a regular quasianneid A without heart, then A/I is an irreducible regular A-quasimoduloid without heart.*

Theorem 3.11. *The Jacobson radical of a regular quasianneid A without heart coincides with the intersection of all maximal modular right ideals of A.*

Proof. If A is a regular quasianneid A without heart, and I is a maximal modular right ideal of A, then A/I is a regular A-quasimoduloid without heart, and so, the claim follows from the previous theorem. □

The proof of the following theorem is analogous to the proof of the corresponding theorem for anneids given in [5], but we give it here for the sake of completeness.

Theorem 3.12. *Let A be a paraanneid, $\overline{A}$ its linearization. If $J_l(A)$ is the large Jacobson radical of A and $J(\overline{A})$ the ordinary Jacobson radical of the ring $\overline{A}$, then $J_l(A) = J(\overline{A}) \cap A$.*

Proof. Since every irreducible $\overline{A}$-module M may be regarded as an irreducible A-paramoduloid M, and since $(0:M)_A \subseteq (0:M)_{\overline{A}}$, we have $J_l(A) \subseteq J(\overline{A}) \cap A$. Now, let $a \in J(\overline{A}) \cap A$ and let M be an irreducible A-paramoduloid. It is enough to prove that $a \in (0:M)$. Suppose $a \notin (0:M)$. Then there exists $x \neq 0$ such that $xa \neq 0$. Since M is irreducible, xa is a strict generator of M, and hence, $\overline{M} = xa\overline{A}$, where $\overline{M}$ is the linearization of M. Let $\bar{b} \in \overline{A}$ such that $x = xa\bar{b}$. Then, for all $\bar{y} \in \overline{A}$, $x(\bar{y} - a\bar{b}\bar{y}) = 0$, i.e., $\bar{y} - a\bar{b}\bar{y} \in (0 : x)$, for all $\bar{y} \in \overline{A}$. Since $a \in J(\overline{A})$, $a\bar{b}$ also belongs to $J(\overline{A})$. If $\bar{z}$ is a quasi-inverse of $a\bar{b}$ in $\overline{A}$, then, since $\bar{z} - a\bar{b}\bar{z} \in (0 : x)$, we have $a\bar{b} \in (0 : x)$, which implies $xa\bar{b} = 0$, and therefore $x = 0$, a contradiction. Hence, $J(\overline{A}) \cap A \subseteq J_l(A)$. □

Lemma 3.13. *Let A be a regular paraanneid without heart such that for each element δ of the corresponding paragrading set Δ^* we have either $\delta^2 = 0$ or $\delta^2 = \delta$ with respect to the minimal multiplication. Then degrees of all unities modulo a proper modular right ideal of A have a common idempotent upper bound.*

Proof. Let I be a proper modular right ideal of A and let u and u' be two unities modulo I. Observe the canonical mapping $f : A \to A/I$. Clearly, $0 \neq f(x) = f(ux) = f(u'x)$ for all $x \in A \setminus I$. Since f is a homomorphism and A is without heart, it follows that $ux\#u'x$. A is regular, without heart and both ux and $u'x$ are nonzero together imply $u\#u'$. This means that $\delta(u)$ and $\delta(u')$ have a common upper bound, denote it by ξ. Then $0 < \delta(u) < \xi$ and $0 < \delta(u') < \xi$. Also, $u\#u^2 \neq 0$. Therefore $0 < \delta(u) = \delta(u)\delta(u) < \xi^2$. This and our assumption imply $\xi^2 = \xi$. □

Remark 3.14. A paraanneid satisfying the conditions from the previous lemma exists as Example 2.3 shows.

Definition 3.15. Let I be a proper modular right ideal of a regular paraanneid without heart such that for each element δ of the corresponding paragrading set Δ^* we have either $\delta^2 = 0$ or $\delta^2 = \delta$ with respect to the minimal multiplication. The least common upper bound of all unities modulo I is called the *degree of I*.

Theorem 3.16. *Let A be a regular paraanneid without heart such that for each element δ of the corresponding paragrading set Δ^* we have either $\delta^2 = 0$ or $\delta^2 = \delta$ with respect to the minimal multiplication and let ξ be an idempotent element of Δ^*. Also, assume that Δ^* is a cancellative partial groupoid with respect to the minimal multiplication. Then there exists a one-to-one correspondence between the maximal modular right ideals of A of degree ξ and the maximal modular right ideals of the ring $A(\xi) = \overline{A}_\xi$.*

Proof. With these assumptions, we may proceed as in [5] for regular anneids. Namely, correspondence is given in the following way. If I is a maximal modular right ideal of $A(\xi)$, let $\hat{I} = \{x \in A \mid xA \cap A(\xi) \subseteq I\}$. Now, as in [5], it may be proved that $\hat{I}$ is a maximal modular right ideal of A of degree ξ. Conversely, if I is a maximal modular right ideal of A of degree ξ, then it can be easily verified that $I \cap A(\xi)$ is a maximal modular right ideal of $A(\xi)$. □

Remark 3.17. A paraanneid satisfying the conditions from the previous theorem exists according to Example 2.4.

Definition 3.18. An element a of a paraanneid A is called *right* (*left*) *quasi-regular* if a is not a left (right) unity modulo any proper right ideal of A. A right ideal of A is called *quasi-regular* if every of its elements is right quasi-regular.

The following characterization of a right quasi-regular element is a corollary of the previous theorem (for an analogous result for regular anneids, see [4, 5]).

Theorem 3.19. *Let A be a regular paraanneid without heart such that for each element δ of the corresponding paragrading set Δ^* we have either $\delta^2 = 0$ or $\delta^2 = \delta$ with respect to the minimal multiplication and assume that Δ^* is a cancellative partial groupoid with respect to the minimal multiplication. An element $a \in A$ is right quasi-regular if and only if one of the following two conditions is satisfied:*

i) *The degree of a is not an idempotent element of Δ^*;*
ii) *If the degree of a is an idempotent element ξ of Δ^*, then a is a right quasi-regular element of the ring $A(\xi)$.*

As in the classical case [8] and in the case of regular anneids [5], one may now prove that, under the assumptions of Theorem 3.16, the Jacobson radical of a regular paraanneid A is a quasi-regular ideal which contains all right quasi-regular ideals of A. This and Theorem 3.16 together imply the following theorem (for the case of regular anneids, see [4, 5]).

Theorem 3.20. *Under the assumptions of Theorem 3.16, we have*

$$J(A(\xi)) = J(A) \cap A(\xi),$$

where $J(A(\xi))$ denotes the Jacobson radical of the ring $A(\xi)$, and $J(A)$ denotes the Jacobson radical of A.

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