

TOWERS OF SEMISIMPLE ALGEBRAS, THEIR GRAPHS AND JONES INDEX

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Dedicated to the memory of Professor Marc Krasner

ABSTRACT. The paper presents an algebraic approach to the theory of pairs of semisimple algebras, the Jones fundamental construction of a tower and its index. The methods of the representation theory of hereditary algebras allows to classify the pairs in terms of a bimodule and a vector space and to describe the values of the Jones index.

INTRODUCTION

The goal of this presentation is to underline the very nature of Mathematics when abstract theories often have a direct impact on concrete problems that seemed to had been unrelated. Such an occasion, related to the objectives of this Conference, is an application of the algebraic methods developed in the theory of representation of hereditary algebras in the 70s (see [6]) to the questions concerning the „mysteries” of the discrete nature of the set of all possible values of the Jones index raised in the 80s of the last century in functional analysis.

A FEW REMARKS ON REPRESENTATIONS OF HEREDITARY ALGEBRAS

Let us start with a short summary of the basic concepts used in what follows. Throughout, K is a fixed field and \mathcal{A} a finite dimensional K -algebra. Denote by $\text{Rad}\mathcal{A}$ the radical of \mathcal{A} and note that the factor algebra $\bar{\mathcal{A}} = \mathcal{A}/\text{Rad}\mathcal{A}$ is a semisimple finite dimensional K -algebra and thus, by Wedderburn-Artin theorem, a finite direct product of the full matrix rings

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over division K -algebras. In the case that $\bar{\mathcal{A}}$ is a product of division K -algebras we call \mathcal{A} basic.

Given a K -algebra \mathcal{A} , let $\{S(1), S(2), \dots, S(n)\}$ be the set of all simple (right) \mathcal{A} -modules and $\{P(1), P(2), \dots, P(n)\}$ the set of the respective indecomposable projective \mathcal{A} -modules, the projective covers of $S(t)$, $1 \leq t \leq n$.

Define the dimension type $\dim X$ of a given \mathcal{A} -module $X_{\mathcal{A}}$ by

$$\dim X = (x_1, x_2, \dots, x_n),$$

where x_t is the multiplicity of the simple module $S(t)$ in a composition series of X . Thus, x_t is the length of the E_t -module $\text{Hom}(P(t), X)$, where $E_t = \text{End}P(t)$.

Consider the (right) regular representation $\mathcal{A}_{\mathcal{A}}$ of \mathcal{A} and recall that with appropriately defined multiplication of the endomorphisms,

$$\mathcal{A} \simeq \text{End}\mathcal{A}_{\mathcal{A}}.$$

Write

$$\mathcal{A}_{\mathcal{A}} = \bigoplus_{t=1}^n \bigoplus_{i=1}^{r_t} P_i(t) \quad \text{with } r_t \geq 1 \text{ for all } 1 \leq t \leq n.$$

The algebra is basic if $r_t = 1$ for all $1 \leq t \leq n$. A K -algebra \mathcal{B} is said to be Morita equivalent to \mathcal{A} if

$$\mathcal{B} \simeq \text{End} \left(\bigoplus_{t=1}^n \bigoplus_{i=1}^{k_t} P_i(t) \right) \quad \text{for some integers } k_t \geq 1, 1 \leq t \leq n.$$

The fundamental statement concerning this concept is the following theorem.

Theorem 1. *Two finite dimensional K -algebras \mathcal{A} and \mathcal{B} are Morita equivalent if and only if their representation categories (i.e. their categories of right finite dimensional modules) $\text{mod-}\mathcal{A}$ and $\text{mod-}\mathcal{B}$ are equivalent.*

Let us point out that each Morita equivalent class of K -algebras contain a (unique) basic K -algebra $\text{End}(\bigoplus_{t=1}^n P(t))$. Theorem 1 allows us to restrict our considerations to basic algebras.

Given a basic K -algebra \mathcal{A} , the semisimple K -algebra $\Lambda = \mathcal{A}/\text{Rad}\mathcal{A}$ decomposes (uniquely) into the product

$$\Lambda = F_1 \times F_2 \times \dots \times F_n \quad \text{with division } K\text{-algebras } F_t, 1 \leq t \leq n.$$

Moreover, the Λ - Λ -bimodule ${}_{\Lambda}M_{\Lambda} = \text{Rad}\mathcal{A}/(\text{Rad}\mathcal{A})^2$ has a unique decomposition

$$M = \bigoplus_{1 \leq i, j \leq n} {}_iM_j$$

into the F_i - F_j - spaces ${}_iM_j = F_iMF_j$. The data $\mathcal{S}(\mathcal{A}) = (F_i, {}_iM_j; 1 \leq i, j \leq n)$ is said to be the K -species of \mathcal{A} . Recall the homological meaning if the ingredients of $\mathcal{S}(\mathcal{A})$:

$$F_i = \text{End}(S(i)) \text{ and } {}_jM_i^* = \text{Hom}_{F_j}({}_iM_j, F_j) = \text{Ext}_{\mathcal{A}}^1(S(j), S(i)).$$

Furthermore, assign every K -algebra \mathcal{A} an oriented valued graph $\Gamma(\mathcal{A})$. By definition, an (oriented, symmetrizable) valued graph is a pair of non-negative integral $n \times n$ matrices

$$(U = (u_{i,j}), V = (v_{i,j}))$$

such that there is a positive invertible diagonal $n \times n$ matrix $D = (d_{i,j})$ that satisfies the equality $UD = DV$. The index set $\{1, 2, \dots, n\}$ is the set of the vertices of the graph and if $u_{i,j} \neq 0$ (and thus $v_{i,j} \neq 0$), there is an arrow from i to j with valuation $(u_{i,j}, v_{i,j})$. The graph $\Gamma(\mathcal{A}) = \Gamma(\mathcal{S}(\mathcal{A}))$ of the K -algebra \mathcal{A} is defined by

$$u_{i,j} = \dim({}_iM_j)_{F_j}, v_{i,j} = \dim_{F_i}({}_iM_j) \text{ and } d_{i,i} = \dim_K F_i.$$

Clearly, \mathcal{A} is a semisimple K -algebra if and only if its graph $\Gamma(\mathcal{A})$ consists of a number of discrete vertices, with no arrows.

Now, starting with an (abstract) K -species $\mathcal{S} = (F_i, {}_iM_j; 1 \leq i, j \leq n)$ (i.e. with a K -modulation of a given oriented valued graph consisting of division K -algebras F_i and F_i - F_j -bimodules ${}_iM_j$ with K operating centrally) we may define the tensor algebra $\mathcal{T}(\mathcal{S})$ of \mathcal{S} of the Λ - Λ -bimodule

$${}_{\Lambda}M_{\Lambda} = \bigoplus_{1 \leq i, j \leq n} {}_iM_j$$

with the canonical operation by $\Lambda = F_1 \times F_2 \times \dots \times F_n$. Hence

$$\mathcal{T} = \bigoplus_{t \geq 0} N^{(t)},$$

where $N^{(0)} = \Lambda$, $N^{(1)} = M$ and $N^{(t+1)} = N^{(t)} \otimes_{\Lambda} M$ for $t \geq 1$ with componentwise addition and multiplication induced by taking tensor products. In general, $\mathcal{T}(\mathcal{S})$ is an infinite dimensional K -algebra. In fact, $\mathcal{T}(\mathcal{S})$ is finite dimensional if and only if there is no oriented cycle in the graph $\Gamma(\mathcal{S})$. In this case, $\mathcal{T}(\mathcal{S})$ is hereditary in the sense that $\text{Rad} \mathcal{T}(\mathcal{S})$ is a projective $\mathcal{T}(\mathcal{S})$ -module, and thus every submodule of a projective module is projective.

The basic results on the representation type of hereditary algebras can be summarized as follows (see [6] and [7]).

Theorem 2. (i) *A finite dimensional hereditary connected K -algebra (i.e. a K -algebra whose graph is connected) is of finite representation type if and*

only if it is Morita equivalent to a tensor K -algebra over a Dynkin graph (see Figure 1).

(ii) A finite dimensional hereditary connected K -algebra \mathcal{A} is of tame representation type (i.e. \mathcal{A} is not of finite representation type and there is no exact full embedding of the category of finite dimensional modules over the polynomial algebra in two noncommuting variables into $\text{mod-}\mathcal{A}$) if and only if \mathcal{A} is of type $\tilde{\mathcal{A}}(n, \delta)$ or is Morita equivalent to a tensor K -algebra over a Euclidean graph (see Figure 2).

The hereditary K -algebras \mathcal{A} that appear in our application satisfy the condition $(\text{Rad}\mathcal{A})^2 = 0$, and thus the algebras of type $\tilde{\mathcal{A}}(n, \delta)$ will not appear.

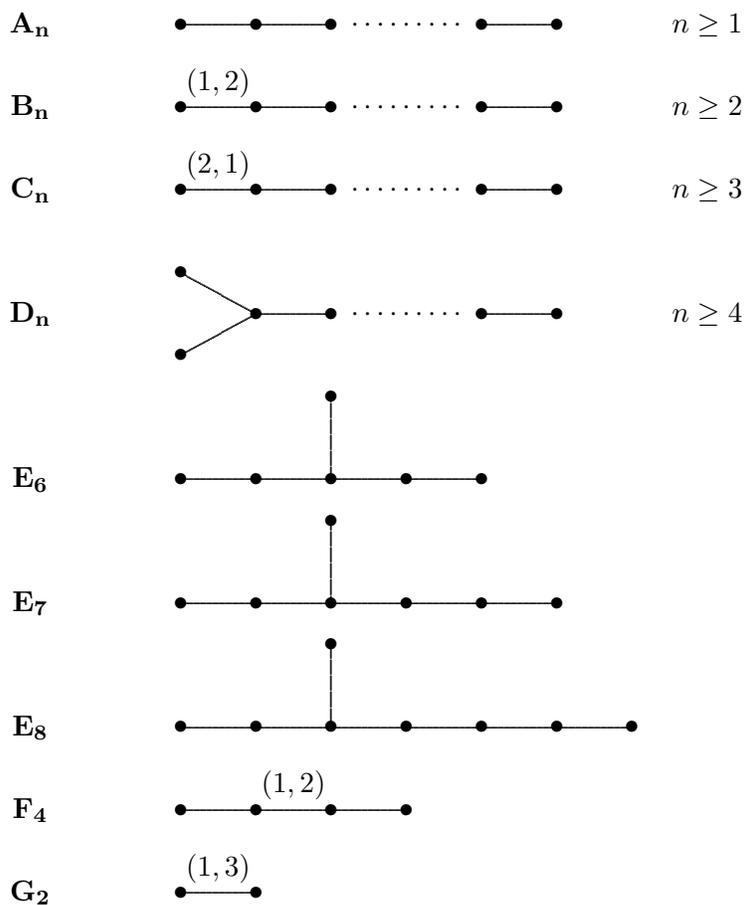


Figure 1. Dynkin graphs

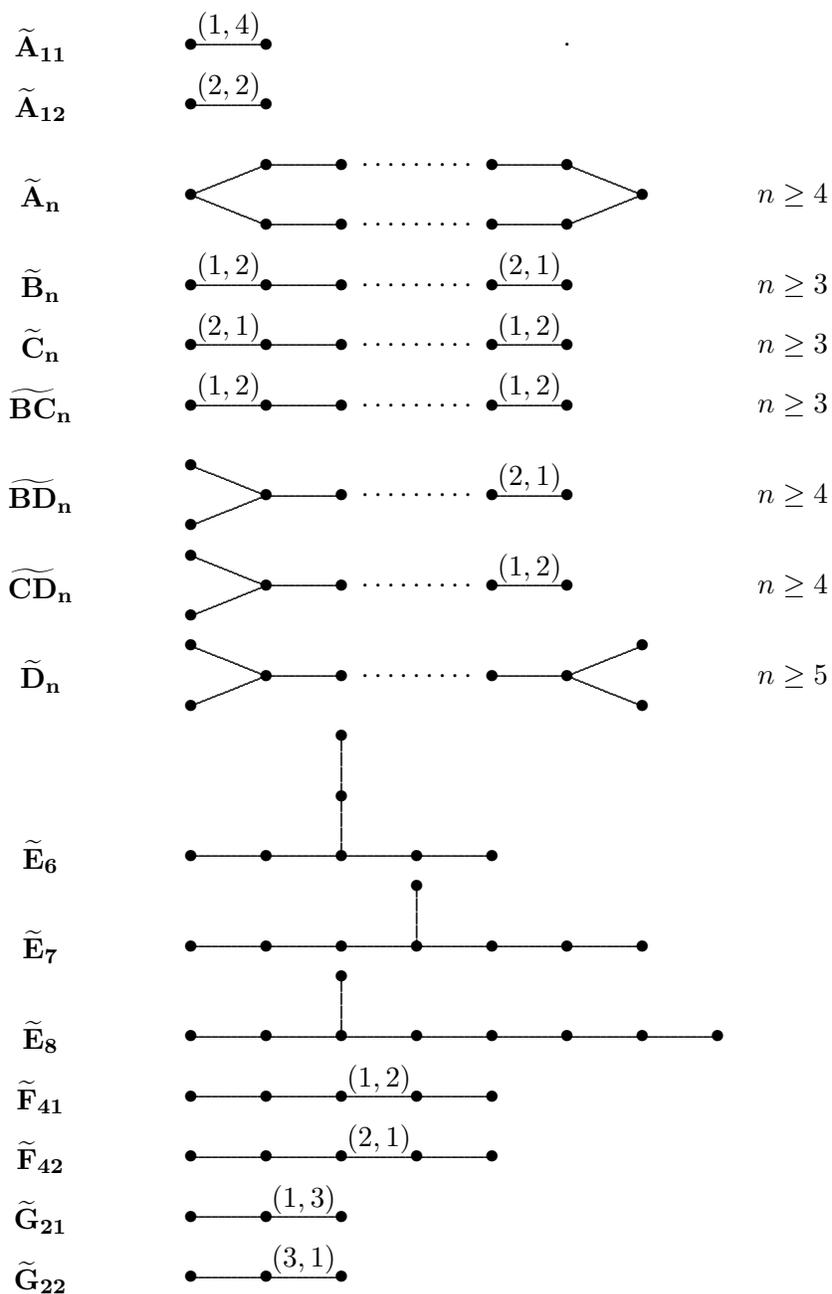


Figure 2. Euclidean graphs

Let $\mathcal{S} = (F_{i,i} M_j; 1 \leq i, j \leq n)$ be a K -species whose graph $\Gamma(\mathcal{S})$ has no oriented cycles. Then the tensor algebra $\mathcal{A} = \mathcal{T}(\mathcal{S})$ is a finite dimensional hereditary K -algebra. There is a pair of endofunctors C^-, C^+ of the category $\text{mod-}\mathcal{A}$ called Coxeter functors having the following properties:

If $X \in \text{mod-}\mathcal{A}$ is an indecomposable representation, then

- (i) $C^-X = 0$ if and only if X is injective; otherwise C^-X is indecomposable and $C^+C^-X \simeq X$;
- (ii) $C^+X = 0$ if and only if X is projective; otherwise C^+X is indecomposable and $C^-C^+X \simeq X$.

The elements of the set

$$\mathcal{P} = \{C^{-k}P(t) \mid k \geq 0, P(t) \text{ indecomposable projective}, 1 \leq t \leq n\}$$

are the (indecomposable) preprojective \mathcal{A} -modules. The set \mathcal{P} is finite if and only if the graph $\Gamma(\mathcal{S})$ is Dynkin; in this case, \mathcal{P} is the set of all indecomposable \mathcal{A} -modules.

The representations $X \in \mathcal{P}$ are uniquely defined by their dimension type $\dim X \in \mathbb{Q}^n$. The Coxeter transformation of \mathbb{Q}^n , i.e. a linear transformation $c = c_\Gamma : \mathbb{Q}^n \rightarrow \mathbb{Q}^n$ defined in terms of the graph $\Gamma(\mathcal{S})$ satisfies

$$c^{-1}(\dim X) = \dim(C^-X) \text{ for every } X \in \mathcal{P}.$$

The transformation c can be expressed explicitly as a composition of n reflections defining the respective Weyl group of $\Gamma(\mathcal{S})$ (see [8]).

The matrix of the Coxeter transformation c with respect to the standard basis has the form

$$-H(H^{-1})^{tr}, \text{ with } H = PD^{-1},$$

where P is the matrix whose columns are the vectors $\dim P(1), \dim P(2), \dots, \dim P(n)$. Hence the characteristic polynomial χ_c of c is reciprocal: If r is a root of χ_c , then so is $1/r$ (with the same multiplicity). Here is the description of the eigenvalues of c_Γ (see [2]).

Theorem 3. *If Γ is a Dynkin graph, then all eigenvalues λ of c_Γ satisfy $|\lambda| = 1$ and $\lambda \neq 1$. If Γ is a Euclidean graph, then all eigenvalues λ of c_Γ satisfy $|\lambda| = 1$ and 1 is an eigenvalue. Otherwise, the spectral radius of c_Γ is greater than 1 and is itself an eigenvalue.*

THE VALUES OF THE JONES INDEX

Consider a tower of finite dimensional semisimple K -algebras

$$A_1 \subseteq A_2 \subseteq \dots \subseteq A_t \subseteq A_{t+1} \subseteq A_{t+2} \subseteq \dots, \quad (1)$$

where $A_1 = A$, $A_2 = B$ and

$$A_{t+2} = \text{End}(A_{t+1})_{A_t} \text{ for } t \geq 1.$$

Such tower is fully determined by a K -pair, i.e. a pair $A \subseteq B$ of finite dimensional semisimple K -algebras (with unital embedding) and the Jones fundamental construction (see [12]) applied recursively. Such towers and their inductive limits have been objects of study in the theory of C^* -algebras (see e.g. [4], [5], [11], [13]), and the respective methods and results are well described in the case of multi-matrix algebras in the monograph [11]. In particular, the Jones index $[B : A]$ of A in B defines the growth rate

$$[B : A] = \limsup_{t \rightarrow \infty} [\text{rank}(A_t | A_1)]^{\frac{1}{t}},$$

that is shown to be the square of the norm of a certain matrix of natural numbers and thus, Kronecker theorem restricts its values. While in the case of multi-matrix algebras the pairs are fully characterized by graphs known as the weighted Bratteli diagrams, in general situation, one has to consider certain defining bimodules in order to characterize the K -pairs fully. To illustrate this point, consider the following simple example of a central and non-central embedding of the real algebra of the complex numbers \mathbb{C} into the real algebra of the 2×2 complex matrices. The two \mathbb{R} -pairs

$$\mathbb{C} \subseteq \text{Mat}(2, \mathbb{C}),$$

defined

$$\text{either by } c \mapsto \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \text{ or by } c = a + bi \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

have in both cases the same weighted Bratteli diagram

$$(1) \xrightarrow{(2,2)} (2);$$

the bimodules that provide a characterization of these embeddings are

$$\mathbb{C}^{\mathbb{C}}\mathbb{C} \oplus \mathbb{C}^{\mathbb{C}}\mathbb{C} \text{ and } \mathbb{C}^{\mathbb{C}}\mathbb{C} \oplus \mathbb{C}^{\mathbb{C}}\overline{\mathbb{C}} \text{ (with the operations: } c_1 \cdot c \cdot c_2 = c_1 c \overline{c_2}),$$

respectively.

Our train of thought may be encapsulated in the following display:

$$\begin{aligned} A \subseteq B \iff \mathcal{A}(A, B) &= \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} \xleftrightarrow{\text{Morita}} \mathcal{A} \\ &= \begin{pmatrix} F & FM_G \\ 0 & G \end{pmatrix} \Leftrightarrow (FM_G, X_F), \text{End} X_F = A \end{aligned}$$

Two K -pairs $A \subseteq B$ and $A' \subseteq B'$ are considered to be equivalent if there is a K -algebra isomorphism $\varphi : B \rightarrow B'$ such that $\varphi(A) = A'$. If we assign

to every K -pair $A \subseteq B$ the hereditary K -algebra of the upper triangular 2×2 matrices

$$\mathcal{A}(A, B) = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix} = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a \in A; b, c \in B \right\},$$

then

(a) the K -pairs $A \subseteq B$ and $A' \subseteq B'$ are equivalent if and only if $\mathcal{A}(A, B)$ and $\mathcal{A}(A', B')$ are isomorphic K -algebras.

Furthermore,

(b) $\mathcal{A}(A, B)$ is Morita equivalent to a K -algebra of 2×2 matrices

$$\begin{pmatrix} F & FM_G \\ 0 & G \end{pmatrix} = \left\{ \begin{pmatrix} f & m \\ 0 & g \end{pmatrix} \mid f \in F, g \in G, m \in M \right\},$$

where F, G and M are described below. If

$$A = \prod_{i=1}^m \text{Mat}(x_i, F_i), \quad B = \prod_{j=1}^n \text{Mat}(y_j, G_j),$$

then there are finite dimensional F_i -spaces X_i and F_i - G_j -bimodules ${}_iM_j$ such that

$$A \simeq \text{End } X_F \quad \text{and} \quad B \simeq \text{End}(X_F \otimes_F M_G),$$

where $X_F = \bigoplus_{i=1}^m X_i$ and $FM_G = \bigoplus_{i=1}^m \bigoplus_{j=1}^n {}_iM_j$ with the canonical operations by the basic K -algebras

$$F = F_1 \times F_2 \times \cdots \times F_m \quad \text{and} \quad G = G_1 \times G_2 \times \cdots \times G_n.$$

In addition,

(c) referring to the Jones fundamental construction $C = \text{End } B_A$, let us add that we obtain a new K -pair $B \subseteq C$ and

$$C \simeq \text{End}(X_F \otimes_F M_G \otimes_G M_F^*),$$

where ${}_G M_F^* = \text{Hom}({}_F M_G, {}_G G)$.

Hence we may conclude that

(d) writing $\dim({}_iM_j)_{G_j} = u_{ij}$ and $\dim_{F_i}({}_iM_j) = v_{ij}$, we have

$$\dim(X_i)_{F_i} = x_i \quad \text{and} \quad \sum_{i=1}^n x_i u_{ij} = y_j.$$

Moreover,

$$C = \prod_{k=1}^m \text{Mat}(z_k, F_k), \quad \text{where} \quad z_k = \sum_{i=1}^m x_i \sum_{j=1}^n u_{ij} v_{kj} = \sum_{k=1}^n y_j v_{kj}.$$

Let us summarize these results in the following theorems.

Theorem 4. *The K -algebras that are Morita equivalent to the K - algebra*

$$\mathcal{A} = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$$

are of the form

$$\begin{pmatrix} \text{End} X_F & \text{End}(X_F \otimes {}_F M_G) \\ 0 & \text{End}(X_F \otimes {}_F M_G) \end{pmatrix},$$

where X_F is a direct sum of x_i -dimensional F_i -spaces for arbitrary $x_i \geq 1$, $1 \leq i \leq m$, with the canonical operation of the K -algebra $F = \prod_{i=1}^m F_i$. The K -species of these algebras is

$$(F_1, F_2, \dots, F_m, G_1, G_2, \dots, G_n, {}_i M_j = F_i M G_j, 1 \leq i \leq m, 1 \leq j \leq n)$$

and their (bipartite) valued graph is

$$\Gamma(\mathcal{A}) = \left(\begin{pmatrix} 0_{m \times m} & U \\ 0_{n \times m} & 0_{n \times n} \end{pmatrix}, \begin{pmatrix} 0_{m \times m} & V \\ 0_{n \times m} & 0_{n \times n} \end{pmatrix} \right). \quad (2)$$

Theorem 5. *There is a bijective correspondence between the pairs $A \subseteq B$ of finite dimensional semisimple K -algebras and the pairs $({}_F M_G, X_F)$ of finite dimensiona F - G -bimodules and finite dimensional F -vector spaces over basic semisimple K -algebras F and G . The pair corresponding to $({}_F M_G, X_F)$ is*

$$\text{End}(X_F) \subseteq \text{End}(X_F \otimes {}_F M_G).$$

Theorem 6. *Given a K -pair $A \subseteq B$, consider $B \subseteq C$ with $C = \text{End } B_A$. If $({}_F M_G, X_F)$ defines $A \subseteq B$, then*

$$({}_G M_F^*, X_F \otimes {}_F M_G)$$

defines $B \subseteq C$, where ${}_G M_F^ = \text{Hom}_G({}_F M_G, {}_G G)$. If*

$$(F_1, F_2, \dots, F_m, G_1, G_2, \dots, G_n, {}_i M_j = F_i M G_j, 1 \leq i \leq m, 1 \leq j \leq n),$$

is the K -species of $A \subseteq B$, then

$$(F_1, F_2, \dots, F_m, G_1, G_2, \dots, G_n, {}_j M_i^* = G_j M^* F_i, 1 \leq i \leq m, 1 \leq j \leq n),$$

is the K -species of $B \subseteq C$ and its valued graph is

$$\left(\begin{pmatrix} 0_{m \times m} & 0_{m \times n} \\ V^{tr} & 0_{n \times n} \end{pmatrix}, \begin{pmatrix} 0_{m \times m} & 0_{m \times n} \\ U^{tr} & 0_{n \times n} \end{pmatrix} \right). \quad (3)$$

Observe that both graphs (2) and (3) have $m + n$ vertices. The valued graph of the K -pair $A_t \subseteq A_{t+1}$ of the tower (1) is given by (2) if t is even and by (3) if t is odd. The vectors $\mathbf{x} = (x_1, x_2, \dots, x_m)$ and $\mathbf{y} = \mathbf{x}U = (y_1, y_2, \dots, y_m)$ define the weighting (\mathbf{x}, \mathbf{y}) of the graphs. The weighting of the valued graph of the entire tower (1) is obtained by successive applications

of the K -linear transformation of the $(m+n)$ -dimensional space \mathbb{R}^{m+n} given by

$$(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, \mathbf{y}) \begin{pmatrix} UV^{tr} & 0_{m \times n} \\ 0_{n \times m} & V^{tr}U \end{pmatrix}.$$

The Jones index $[B : A]$ of the K -pair $A \subseteq B$ can be in our general setup expressed in terms of the tower transformation $\tau_{AB} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ associated with the K -pair $A \subseteq B$ by setting

$$\tau_{AB}(\mathbf{x}) = \mathbf{x}UV^{tr}.$$

and

$$[B : A] = \limsup_{p \rightarrow \infty} \sqrt[p]{|\tau_{AB}^p(\mathbf{x})|},$$

where the norm $|\cdots|$ is given by the sum of all (positive) components of the vector $\tau_{AB}^p(\mathbf{x})$. Applying the Perron-Frobenius theory, we conclude that $[B : A]$ is the largest real eigenvalue of τ_{AB} and equals to $\lim_{p \rightarrow \infty} |\tau_{AB}^{p+1}(\mathbf{x})| \cdot |\tau_{AB}^p(\mathbf{x})|^{-1}$.

The Coxeter transformation $c = c_{AB} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$ defined earlier, can be expressed by

$$\begin{aligned} c_{AB}(\mathbf{x}, \mathbf{y}) &= (\mathbf{x}, \mathbf{y}) \begin{pmatrix} -I_{m \times m} & 0_{m \times n} \\ V^{tr} & I_{n \times n} \end{pmatrix} \cdot \begin{pmatrix} I_{m \times m} & U \\ 0_{n \times m} & -I_{n \times n} \end{pmatrix} \\ &= (\mathbf{x}, \mathbf{y}) \begin{pmatrix} -I_{m \times m} & -U \\ V^{tr} & V^{tr}U - I_{n \times n} \end{pmatrix}. \end{aligned}$$

Hence, if $\lambda \neq -1$ is an eigenvalue of the transformation c_{AB} , then $\lambda + 2 + \lambda^{-1}$ is an eigenvalue of the transformation τ_{AB} . In particular, the largest real part of the eigenvalue of the transformation c_{AB} corresponds to the largest (real) eigenvalue of the transformation τ_{AB} .

Now, the following statements follow easily.

Theorem 7. *Let $A \subseteq B$ be a connected pair of finite dimensional semisimple K -algebras and c the respective Coxeter transformation (defined by its valued graph $\Gamma = \Gamma(A, B)$).*

If Γ is a Dynkin graph, then the Jones index of the K -pair $A \subseteq B$ equals

$$[B : A] = 2(r + 1),$$

where r is the largest value of the real parts of the eigenvalues of the transformation c . Thus, we get the well-known values $\sigma_n = 4 \cos^2 \frac{\pi}{n}$ for $n \geq 3$ in the case of a Dynkin graph $\neq A_1$, when $r = \cos^2 \frac{2\pi}{n}$.

Otherwise,

$$[B : A] = \lambda + 2 + \lambda^{-1},$$

where λ is the largest real eigenvalue of the transformation c . Hence, $[B : A] = 4$ if and only if Γ is a Euclidean graph.

If the graph is neither Dynkin nor Euclidean,

$$[B : A] \geq 4.026417949\dots = \rho^* = \lambda^* + 2 + \frac{1}{\lambda^*}$$

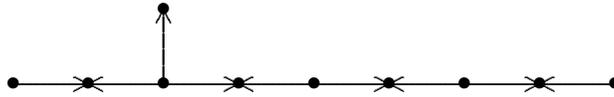
where $\lambda^* = 1.176280818\dots$ is the largest real root of the polynomial

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$$

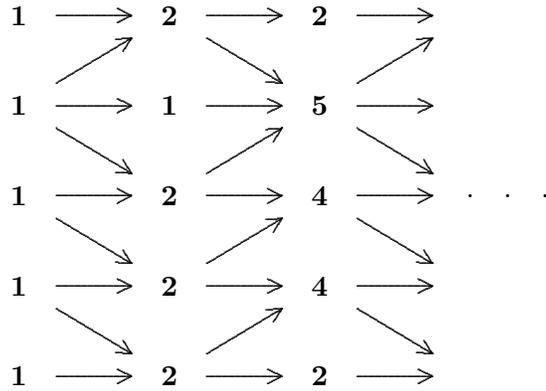
Of course, ρ^* is the largest real root of the polynomial

$$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1.$$

The value $[B : A] = \rho^*$ is attained for the pair defined by the oriented graph



described by the Bratteli graph



These conclusions can be found in the papers [13] and [14]. Similar considerations lead to determine the special values of the Jones index

$$\rho_n = \beta_n + 2 + \beta_n^{-1}, \text{ where } \beta_n \text{ is the largest root of the equation}$$

$$x^{n+1} = x^{n-1} + x^{n-2} + \dots + x + 1 \text{ and}$$

$$\bar{\rho} = \tau + 2 + \tau^{-1}, \text{ where } \tau = \frac{1}{2}(1 + \sqrt{5}); \text{ thus } \bar{\rho} = 2 + \sqrt{5}.$$

The set of the relevant values of the Jones index can be described by the following Figure 3.

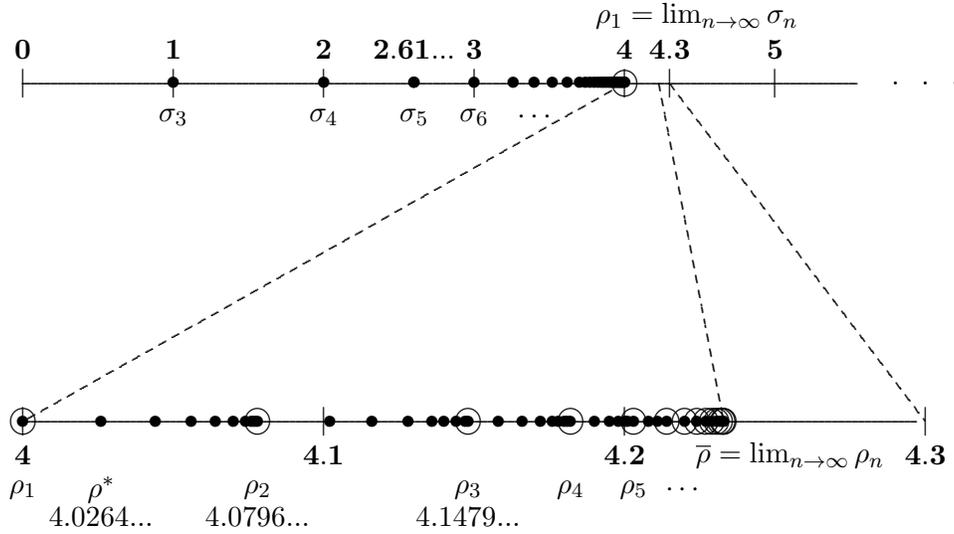


Figure 3. The values of the Jones index

In conclusion, illustrate a calculation of the Jones index on a simple example.

Example. Consider the following unital algebra embedding

$$A = F_1 \times F_2 \subseteq B = \text{Mat}(3, G_1) \times G_2 \times \text{Mat}(2, G_3), \text{ where}$$

$$F_1 = G_2 = G_3 = H \text{ and}$$

$$G_1 \subset H, [H : G_1] = 3, F_2 \subset H, [H : F_2] = 2,$$

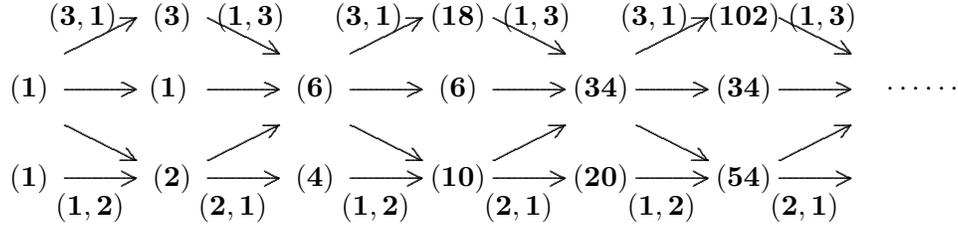
that is defined by the bimodule ${}_F M_G = {}_{F_1} H_{G_1} \oplus {}_{F_1} H_{G_2} \oplus {}_{F_1} H_{G_3} \oplus {}_{F_2} H_{G_3}$ and the space $X_F = F_1 \oplus F_2$. Hence, the algebra $\mathcal{A}(A, B)$ is Morita equivalent to the algebra

$$\mathcal{A} = \begin{pmatrix} F_1 & 0 & H & H & H \\ 0 & F_2 & 0 & 0 & H \\ 0 & 0 & G_1 & 0 & 0 \\ 0 & 0 & 0 & G_2 & 0 \\ 0 & 0 & 0 & 0 & G_3 \end{pmatrix}$$

and the graph of this pair is therefore

$$\Gamma(A \subseteq B) = \left(\left(\begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \right) \right).$$

The (generalized) Bratteli graph of the corresponding tower of algebras is



The matrix of the respective Coxeter transformation is

$$\begin{pmatrix} -1 & 0 & -3 & -1 & -1 \\ 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 3 & 0 & 1 \\ 1 & 2 & 3 & 1 & 2 \end{pmatrix}$$

and its eigenvalues are

$$-1, \frac{3 + \sqrt{17} \pm \sqrt{6\sqrt{17} + 10}}{4}, \frac{3 - \sqrt{17} \pm i\sqrt{6\sqrt{17} - 10}}{4};$$

hence, $[B : A] = \frac{7+\sqrt{17}}{2} = 5.561552815\dots$

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