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ON CODIMENSION GROWTH OF GRADED PI-ALGEBRAS

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Dedicated to the memory of Professor Marc Krasner

ABSTRACT. Two finite dimensional simple Lie algebras over an algebraically closed field are isomorphic if and only if they satisfy the same polynomial identities (A. Koshkulei, Y. Razmyslov, 1983). As an alternative approach to the characterization of finite dimensional simple Lie algebras, some numerical invariants of the algebra identities can be used. We associate with a finite dimensional Lie algebra L a sequence of integers $c_n(L)$, called the n-th codimensions of L. It appears that these quantities grow asymptotically like k^n, for some nonnegative integer $k \leq \dim L$ (A. Giambruno, M. Zaicev, 1999). Moreover, $k = \dim L$ iff algebra L is simple.

1. INTRODUCTION

Color Lie superalgebras as a natural generalisation of Lie algebras and Lie superalgebras play an important role in mathematics and some parts of physics. To the first significant results in their investigation belong the J. Feldvoss's description of the representations of color Lie superalgebras [5], as well as R. Farnsteiner's studies of derivations and central extensions $H^2(L, L)$ of graded Lie algebras over finite fields $\mathrm{GF}(p^k)$ [4] in 90s. Some years later M. Wilson [19] and K. Price [13] investigated the universal enveloping algebras of color Lie superalgebras, while M. Scheunert and R. Zhang introduced and studied their cohomology groups [17, 18]. In this survey we list some latest results related to various graded algebras and

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their polynomial identities. At the beginning we shortly recall the main definitions.

Let F denote an arbitrary field, $F^* = F \setminus \{0\}$ the group of its units and $(G, +)$ some abelian group. An algebra A over F is called *G-graded* if we have the grading of the related vector space $A = \bigoplus_{g \in G} A_g$ and the condition $A_g A_h \subset A_{g+h}$ holds for all $g, h \in G$. Traditionally, in case of a Lie algebra L the product of two elements $x, y \in L$ will be written with a Lie bracket: $[x, y]$.

An element x of a $G-$graded algebra A is said to be *homogeneous of the degree g* when there exists such $g \in G$ that $x \in A_g$. Evidently, each element of a $G-$graded algebra A has a unique presentation as the sum of homogeneous elements $x = \sum_{i=1}^{k} x_{g_i}$, where each x_{g_i} belongs to some component A_{g_i}.

We call a subspace $U \subset A$ *homogeneous (graded)* if for each element $x = \sum_{i=1}^{k} x_{g_i}$ from U this subspace contains all its homogeneous components x_{g_i}. In a similar way a *graded subalgebra $K \subset A$* and *graded ideal $I \triangleleft A$* of a $G-$graded algebra A are defined.

Finally, to introduce color Lie superalgebras we need some special scalar forms, acting on the abelian group G.

A map $\beta : G \times G \to F^*$ is called *a skew-symmetric bicharacter* if it satisfies the following properties for arbitrary elements g, h, k from the group G:

(1) $\beta(g + h, k) = \beta(g, k)\beta(h, k)$;
(2) $\beta(g, h + k) = \beta(g, h)\beta(g, k)$ and
(3) $\beta(h, g) = \beta(g, h)^{-1}$.

Evidently, the following properties hold for all skew-symmetric bicharacters.

Corollary 1. *For every bicharacter β, acting on abelian group G and all elements g of the group G we have:*

(a) $\beta(0, g) = \beta(g, 0) = 1$ *and*
(b) $\beta(g, g) = \pm 1$.

A $G-$graded algebra L is called *a color Lie superalgebra* or, more precisely, a (G, β)-color Lie superalgebra for some abelian group G and a bicharacter β on G if for all homogeneous elements $x \in L_g, y \in L_h, z \in L_k$ we have:

(1) $[x, y] = -\beta(x, y)[y, x]$ and
(2) $\beta(z, x)[x, [y, z]] + \beta(x, y)[y, [z, x]] + \beta(y, z)[z, [x, y]] = 0$.

Here and later on we are always shortening the expressions of the form $\beta(\deg x, \deg y) = \beta(g, h)$ to a more convenient form $\beta(x, y)$.

The following examples illustrate that all Lie algebras and Lie superalgebras are special cases of the just introduced structure.

1. Every Lie algebra is a color Lie superalgebra, if we choose a trivial group $G = \{0\}$ and trivial bicharacter $\beta(0,0) = 1$.

2. For an arbitrary abelian group G every G-graded Lie algebra L is a color Lie superalgebra if we assume that $\beta(g,h) = 1$ for all $g, h \in G$.

3. In case of $G = \mathbb{Z}_2$ and $\beta(i,j) = (-1)^{i\,j}$ for all $i, j \in G$ our definition of color Lie superalgebra turns into the usual definition of a Lie superalgebra.

4. Every Lie color algebra is by the definition a (G, β)-color Lie superalgebra with the bicharacter β satisfying the condition $\beta(g,g) = 1$ for all $g \in G$.

5. Finally, if A is an arbitrary G-graded associative algebra and β is a bicharacter on G, the bilinear map defined as $(x,y) \mapsto xy - \beta(x,y)\,yx$ leads to a color Lie superalgebra structure on A, called the associated color Lie superalgebra $\mathcal{L}(A)$.

The following constructions play an essential role in the studies of color Lie superalgebras.

For every color Lie superalgebra by *a color L-module V* we mean a G-graded vector space $V = \bigoplus_{g\in G} V_g$ such that

(1) $\forall g, h \in G\ \ L_g V_h \subset V_{g+h}$ and
(2) $\forall x, y \in L,\ v \in V\ [x,y]v = x(yv) - \beta(x,y)\,y(xv)$.

If both V, W are color L-modules of a color Lie superalgebra L, a mapping $f : V \to W$, consistent with their gradings, is called *a color homomorphism of degree $g \in G$* of the L-modules V and W when we have $\forall x \in L_h,\ v \in V\ \ f(xv) = \beta(g,h)x\,f(v)$.

For any color Lie superalgebra L we call a linear mapping $D : L \to L$ its *homogeneous color derivation of degree $g \in G$* if $\forall x \in L_h, y \in L\ \ D([x,y]) = [D\,x, y] + \beta(g,h)\,[x, D\,y]$.

For any color L-module V of a color Lie superalgebra L we call a linear mapping $D : L \to V$ *a homogeneous color derivation of degree $g \in G$* if and only if $\forall x \in L_h, y \in L_k\ \ D([x,y]) = \beta(g,h)\,x\,D(y) - \beta(g+h,k)\,y\,D(x)$.

Corollary 2. *With each element v of some color L-module V a so called* inner derivation *is associated, defined as $ad\,v : x \mapsto xv$ for all $x \in L$. The set of all inner derivations $Inn(L,V)$ is a subspace of the graded space of all color derivations $Der(L,V) = \bigoplus_{g\in G} Der(L,V)_g$ and, by definition, the factor space $Der(L,V)/Inn(L,V) = H^1(L,V)$ is the first cohomology group of algebra L with coefficients in the module V.*

For an arbitrary color Lie superalgebra L and two color L-modules V, W we call a bilinear form $b : V \times W \to F$ *invariant* (more precisely, L-invariant) if $\forall x \in L_g,\ v \in V_h,\ w \in W$ we have: $b(xv, w) = -\beta(g,h)\ b(v, xw)$. In the case of $V = W = L$ the bilinear form b is called *associative* if $b([x, y], z) = b(x, [y, z])$ for all elements x, y, z of algebra L.

Let $f : V \to W$ be a linear function (consistent with the gradings of modules V, W) of degree $g \in G$. We call f *a skew map* in relation to a given bilinear form b if for all $v_1 \in V_h, v_2 \in V_k$ we have $b(v_1, f(v_2)) = -\beta(g, k)\, \beta(h, g+k)\, b(v_2, f(v_1))$. The subspace of all skew color derivations of a color Lie superalgebra L is usually denoted by $\mathrm{SkDer}(L)$.

As usually, we call a color Lie superalgebra *simple* when it has no non-trivial graded ideals.

Example 1. For the case $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a \rangle \times \langle b \rangle$ and $\beta(a, a) = \beta(b, b) = -\beta(a, b) = 1$ all finite dimensional simple color Lie algebras are classified [2]. A series of such algebras have the form of the tensor product $L = F[G] \otimes Q$ of the group algebra $F[G]$ (with canonical G-grading) with an arbitrary finite dimensional simple Lie algebra Q (with trivial grading) on which we define the product as $[a^i b^j \otimes x, a^k b^l \otimes y] = (-1)^{j+k} a^{i+k} b^{j+l} \otimes [x, y]$.

The following two results of Q. Zhang and Y. Zhang [22] illustrate well the relations between the notions introduced above:

Let L be a color Lie superalgebra and V its color module such that

(a) $\forall g \in G \setminus \{0\}\;\; H^1(L_0, V_g) = 0$ and
(b) $\forall g \neq h\;\; \mathrm{Hom}_{L_0}(L_g, V_h) = \{0\}$.

Then $\mathrm{Der}\,(L, V) = \mathrm{Der}\,(L, V)_0 + \mathrm{Inn}\,(L, V)$.

For every simple color Lie algebra L and each its nondegenerate associative bilinear form b the factor group $\mathrm{SkDer}(L)/\,\mathrm{Inn}\, L$ is color isomorphic to $H^2(L, F)$.

Corollary 3. *Let L be a simple color Lie algebra over field F, V a color L-module and $b : L \times V \to F$ an arbitrary nondegenerate homogeneous invariant bilinear form on these L-modules. Then*

(a) *for every color derivation $D : L \to L$ of degree g the map $\widetilde{D} = D + D_b$ is a color homomorphism of L-modules, where D_b satisfies $b(x, D(y)) = \beta(g, k)\beta(h, g+k)\, b(y, D_b(x))$ for all $x \in L_h, y \in L_k$;*
(b) *whenever a L-module V is not color isomorphic to L (viewed as L-module) every color derivation $D : L \to L$ is skew.*

Next we will focus on the study of some numerical invariants of polynomial identities of finite dimensional simple color Lie superalgebras over an algebraically closed field of characteristic zero.

2. PI-algebras and their invariants

It is well known that two finite dimensional simple associative algebras over an algebraically closed field F are isomorphic if and only if they satisfy the same polynomial identities (consequence of Amitsur-Levitzki theorem [1]). Similar results were later obtained for Lie algebras (by A. Koshkulei and Y. Razmyslov [11]) and for Jordan algebras (V. Drensky and M. Racine in [3]).

An alternative approach to the characterization of finite dimensional simple algebras by their identities uses some numerical invariants of identities of algebras. Namely, given an algebra A, one can associate with it a sequence of integers $\{c_n(A)\}$, called its *codimensions.*

It appears that $c_n(A) \leq d^{n+1}$ where $d = \dim A$. For associative, Lie and Jordan algebras it has also been proved that $c_n(A)$ grows asymptotically like t^n, where $t \leq d$ is a nonnegative integer (A. Regev [14], A. Giambruno, M. Zaicev in [6] and [9], respectively). Moreover, it has been proved that $t = d$ if and only if A is simple. We start with definitions of all relevant objects.

Let $F\{X\}$ be a free algebra over an arbitrary field F with a countable set of free generators $X = \{x_1, x_2, \ldots\}$. A non-associative polynomial $f = f(x_1, \ldots, x_n)$ is said to be *an identity* of the F-algebra A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. The set of all identities of A forms an ideal $\mathrm{Id}(A) \triangleleft F\{X\}$ stable under all endomorphisms of $F\{X\}$.

Denote by $P_n = P_n(x_1, \ldots, x_n)$ the subspace of algebra $F\{X\}$ consisting of all multilinear polynomials in variables $x_1, \ldots, x_n$. Then $P_n \cap \mathrm{Id}(A)$ is a subspace of all multilinear identities of algebra A in variables $x_1, \ldots, x_n$. The non-negative integer $c_n(A) = \dim P_n/(P_n \cap \mathrm{Id}(A))$ is called the n-th *codimension of algebra* A.

If the following limit exists we shall call it *the PI-exponent of algebra* A: $\text{PI-exp}\,(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$. In 1980's S. Amitsur made a hypothesis that the PI-exponent exists and it is an integer number for all associative *PI-algebras* A (algebras satisfying some polynomial identity). This conjecture has been so far confirmed for associative and finite dimensional Lie algebras (A. Giambruno and M. Zaicev in [7] and [20], respectively). On the other hand, for general (nonassociative) algebras a series of counterexamples with a fractional value of the above limit were constructed by S. Mishchenko and M. Zaicev [21].

We examined on the existence of the PI-exponent the series of simple color Lie superalgebras from the above mentioned example where Q is a finite dimensional simple Lie algebra over an algebraically closed field F of characteristic zero, G equals $\langle a\rangle \times \langle b\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\beta : G \times G \to F^*$ is some skew-symmetric bicharacter on G. Then the direct product $L = F[G] \otimes Q$ becomes a simple color Lie superalgebra with the following multiplication: $[a^i b^j \otimes x, a^k b^l \otimes y] = (-1)^{j+k} a^{i+k} b^{j+l} \otimes [x, y]$.

Defining a linear transformation $\mathrm{ad}\, x : L \to L$ as the right multiplication: $\mathrm{ad}\, x\,(y) = [y, x]$ we arrive to the Killing form $\rho(x, y) = \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$ which is a symmetric non-degenerate bilinear form on our algebra L. For the given simple color Lie superalgebras L of dimension $d = 4 \dim Q$ we are going to construct some alternating polynomials which are not the identities of these algebras.

A polynomial $f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is said to be *alternating* in variables $x_1, \ldots, x_m$ if it changes sign whenever we switch any two of the variables x_i, x_j leaving all the y_k fixed. Given a set of unknowns $Y = \{y_1, \ldots, y_n\}$, we then denote by Alt_Y the so called *alternation on* Y: for an arbitrary multilinear polynomial $f = f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ in variables $x_1, \ldots, y_n$ this equals to $\mathrm{Alt}_Y(f) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) f(x_1, \ldots, x_m, y_{\sigma(1)}, \ldots, y_{\sigma(n)})$.

Lemma 1. *Let Q be a finite dimensional simple Lie algebra of dimension q. Then there exists a left-normed monomial $f = [x_1^{(1)}, \ldots, x_{t_1}^{(1)}, y_1, x_1^{(2)}, \ldots, x_{t_2}^{(2)}, y_2, \ldots, x_1^{(q)}, \ldots, x_{t_q}^{(q)}, y_q, x_1^{(q+1)}]$ with $t_1, \ldots, t_q \in \mathbb{N}$ such that $Alt_Y(f)$ is not an identity of Q.*

Next we can construct the first alternating polynomial for our simple color Lie superalgebra $L = F[G] \otimes Q$.

Lemma 2. *There exists a multilinear polynomial $f = f(x_1, \ldots, x_q, y_1, \ldots, y_k)$ which is not vanishing on L and is alternating in variables $x_1, \ldots, x_q$.*

The following statement is useful for extending the number of alternating sets of variables.

Lemma 3. *Let $f = f(x_1, \ldots, x_m, y_1, \ldots, y_k)$ be a multilinear polynomial alternating in variables $x_1, \ldots, x_m$. Then for any variables z, w from the set X the polynomial $g = \sum_{i=1}^{m} f(x_1, \ldots, x_{i-1}, [x_i, z, w], x_{i+1}, \ldots, x_m, y_1, \ldots, y_k)$ is also alternating in $x_1, \ldots, x_m$.*

In what follows, we for an arbitrary set of variables $Y = \{y_1, \ldots, y_n\}$ simplify the expression $f(x_1, \ldots, x_m, y_1, \ldots, y_n)$ to the form $f(x_1, \ldots, x_m, Y)$.

Lemma 4. *Let $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_r \subseteq X$ be a disjoint union with $r \geq 0$, where Y_0 may be an empty set. Let $f(x_1, \ldots, x_q, Y)$ be a multilinear polynomial alternating on each Y_i, $1 \leq i \leq r$, and in $x_1, \ldots, x_q$. Then for any $k \geq 1$ and $\forall z_1, w_1, \ldots, z_k, w_k \in X$ there exists a multilinear polynomial $g(x_1, \ldots, x_q, z_1, w_1, \ldots, z_k, w_k, Y)$ such that for any evaluation $\varphi : X \to L$, $\varphi(x_i) = \bar{x}_i$, $1 \leq i \leq q$, $\varphi(z_j) = \bar{z}_j$, $\varphi(w_j) = \bar{w}_j$, $1 \leq j \leq k$, $\varphi(y) = \bar{y}$, where $y \in Y$, we have*

$$\begin{aligned} \varphi(g) &= g(\bar{x}_1, \ldots, \bar{x}_q, \bar{z}_1, \bar{w}_1, \ldots, \bar{z}_k, \bar{w}_k, \bar{Y}) \\ &= tr(ad\, z_1\ ad\, w_1) \ldots tr(ad\, z_k\ ad\, w_k) f(\bar{x}_1, \ldots, \bar{x}_q, \bar{Y}). \end{aligned}$$

Moreover, g is alternating on each set Y_i and on the set $\{x_1, \ldots, x_q\}$.

Detailed proofs of the above technical propositions are available in [12]. Finally we are ready to construct the required multialternating polynomial for our simple color Lie superalgebra L, $\dim L = d$, over an algebraically closed field F of characteristic 0.

Theorem 1. *For any $k \geq 0$ there exists a multilinear polynomial*

$$g_k = g_k(x_1^{(1)}, \dots, x_q^{(1)}, \dots, x_1^{(2k+1)}, \dots, x_q^{(2k+1)}, y_1, \dots, y_n)$$

satisfying the following conditions:

1. *g_k is alternating on each set $\{x_1^{(i)}, \dots, x_q^{(i)}\}$, $1 \leq i \leq 2k+1$;*
2. *g_k is not an identity of the algebra L and*
3. *the integer n does not depend on k.*

Proof. Let $f = f(x_1, \dots, x_q, y_1, \dots, y_m)$ be the multilinear polynomial from Lemma 2. Hence f is alternating on $x_1, \dots, x_q$ and does not vanish on L. If we suppose that $k = 1$ and write $Y = \{y_1, \dots, y_m\}$ then by Lemma 4 there exists such multilinear polynomial $g = g(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y)$ that $g(\bar{x}_1, \dots, \bar{x}_q, \bar{v}_1^{(1)}, \bar{z}_1^{(1)}, \dots, \bar{v}_q^{(1)}, \bar{z}_q^{(1)}, \bar{Y}) = \mathrm{tr}(\mathrm{ad}\, \bar{v}_1^{(1)} \cdot \mathrm{ad}\, \bar{z}_1^{(1)}) \cdots \mathrm{tr}(\mathrm{ad}\, \bar{v}_q^{(1)} \cdot \mathrm{ad}\, \bar{z}_q^{(1)})\, f\,(\bar{x}_1, \dots, \bar{x}_q, \bar{Y})$.
So defining for any $\sigma, \tau \in S_q$ the polynomial $g_{\sigma,\tau} = g_{\sigma,\tau}(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y) = g(x_1, \dots, x_q, v_{\sigma(1)}^{(1)}, z_{\tau(1)}^{(1)}, \dots, z_{\tau(q)}^{(1)}, Y)$, we obtain that $g_1(x_1, \dots, x_q, v_1^{(1)},\ z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y) = \frac{1}{q!} \sum_{\sigma,\tau \in S_q} (\mathrm{sgn}\,\sigma)(\mathrm{sgn}\,\tau) g_{\sigma,\tau}$ is a polynomial, alternating on each of the sets $\{x_1, \dots x_q\}$, $\{v_1^{(1)}, \dots, v_q^{(1)}\}$ and $\{z_1^{(1)}, \dots, z_q^{(1)}\}$. It remains to show that for any evaluation φ, we have $\varphi(g_1) = \det \bar{\rho}_1 \cdot \varphi(f)$, where

$$\bar{\rho}_1 = \begin{pmatrix} \rho(\bar{v}_1^{(1)}, \bar{z}_1^{(1)}) & \cdots & \rho(\bar{v}_1^{(1)}, \bar{z}_q^{(1)}) \\ \vdots & & \vdots \\ \rho(\bar{v}_q^{(1)}, \bar{z}_1^{(1)}) & \cdots & \rho(\bar{v}_q^{(1)}, \bar{z}_q^{(1)}) \end{pmatrix}.$$

But Lemma 4 yields that $\varphi(g_1) = \gamma \varphi(f)$ for all evaluations $\varphi : X \to L$, with $\gamma = \frac{1}{q!} \sum_{\sigma,\tau \in S_q} (\mathrm{sgn}\,\sigma)(\mathrm{sgn}\,\tau) \rho(\bar{v}_{\sigma(1)}^{(1)}, \bar{z}_{\tau(1)}^{(1)}) \cdots \rho(\bar{v}_{\sigma(q)}^{(1)}, \bar{z}_{\tau(q)}^{(1)})$. Fixing $\sigma \in S_q$ and computing $\gamma_\sigma = \sum_{\tau \in S_q} (\mathrm{sgn}\,\tau) \rho(\bar{v}_{\sigma(1)}^{(1)}, \bar{z}_{\tau(1)}^{(1)}) \cdots \rho(\bar{v}_{\sigma(q)}^{(1)}, \bar{z}_{\tau(q)}^{(1)})$ we obtain

$$\gamma_\sigma = \sum_{\tau \in S_q} (\mathrm{sgn}\,\tau) \rho(a_1, b_{\tau(1)}) \cdots \rho(a_q, b_{\tau(q)}) = \det \begin{pmatrix} \rho(a_1, b_1) & \cdots & \rho(a_1, b_q) \\ \vdots & & \vdots \\ \rho(a_q, b_1) & \cdots & \rho(a_q, b_q) \end{pmatrix}$$

$$= (\operatorname{sgn}\sigma)\det\begin{pmatrix} \rho(a_{\sigma^{-1}(1)}, b_1) & \cdots & \rho(a_{\sigma^{-1}(1)}, b_q) \\ \vdots & & \vdots \\ \rho(a_{\sigma^{-1}(q)}, b_1) & \cdots & \rho(a_{\sigma^{-1}(q)}, b_q) \end{pmatrix} = (\operatorname{sgn}\sigma)\det\bar{\rho}_1,$$

where we have put $\bar{v}^{(1)}_{\sigma(i)} = a_i$, $\bar{z}^{(1)}_i = b_i$, for $i = 1, \ldots, q$.
Hence, $\gamma = \frac{1}{q!}\sum_{\sigma\in S_q}(\operatorname{sgn}\sigma)\gamma_\sigma = \det\bar{\rho}_1$ and $\varphi(g_1) = \det\bar{\rho}_1 \cdot \varphi(f)$. As ρ is a non-degenerate form, g_1 does not vanish in L what completes the proof in this case.

For $k > 1$ we use the inductive hypothesis of the existence of a required multilinear polynomial $g_{k-1}(x_1, \ldots, x_q, v^{(1)}_1, z^{(1)}_1, \ldots, v^{(1)}_q, z^{(1)}_q, \ldots, v^{(k-1)}_1, z^{(k-1)}_1, \ldots, z^{(k-1)}_q, Y)$. Starting with $g_{k-1} = g_{k-1}(x_1, \ldots, x_q, Y')$, where $Y' = Y \cup \{v^{(1)}_1, z^{(1)}_1, \ldots, z^{(1)}_q, \ldots, v^{(k-1)}_1, z^{(k-1)}_1, \ldots, z^{(k-1)}_q\}$ we can once more apply Lemma 4 and the already established property to the polynomial g_{k-1}. So we will arrive to a polynomial g_k and for any evaluation φ we will obtain

$$\varphi(g_k) = \det\bar{\rho}_k\ \varphi(g_{k-1}) = \det\bar{\rho}_1 \ldots \det\bar{\rho}_k\ \varphi(f), \text{ with}$$

$$\bar{\rho}_s = \begin{pmatrix} \rho(\bar{v}^{(s)}_1, \bar{z}^{(s)}_1) & \cdots & \rho(\bar{v}^{(s)}_1, \bar{z}^{(s)}_q) \\ \vdots & & \vdots \\ \rho(\bar{v}^{(s)}_q, \bar{z}^{(s)}_1) & \cdots & \rho(\bar{v}^{(s)}_q, \bar{z}^{(s)}_q) \end{pmatrix}$$

for all $1 \le s \le k$, completing in this way the proof of the theorem. □

For computing PI-exponents of simple color Lie algebras we also need a reasonable lower bound of codimension growth, which is obtained in the next:

Theorem 2. *Let L be as above. Then for all $n \in \mathbb{N}$ there exist a constant $C > 0$ and such $t \in \mathbb{Z}$ that $Cn^t d^n \le c_n(L)$, where $d = \dim L$.*

The main tool for proving the above inequality is the representation theory of symmetric groups. We refer reader to [10] for details of this theory and to [12] for the complete proof of the above statement. Combining the last inequality with the well known inequality $c_n(A) \le d^{n+1}$ we obtain the main result:

Theorem 3. *Let F be an algebraically closed field of characteristic 0 and $L = F[G] \otimes Q$ a $4q$-dimensional simple color Lie superalgebra over F, with $G = \langle a\rangle \times \langle b\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, a skew-symmetric bicharacter β: $\beta(a,a) = \beta(b,b) = 1, \beta(a,b) = -1$ and a simple Lie algebra Q of dimension $q < \infty$ with a trivial G-grading. Then the PI-exponent of L exists and equals to dim $L = 4q$.*

Remark 1. If we take $L = F[G] \otimes Q$ and $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the trivial bicharacter β (that is, $\beta \equiv 1$), then PI-exp$(L) = \dim Q = \frac{1}{4}\dim L$. This example shows that the behavior of the codimensions, in general, strongly depends on the used bicharacter β.

3. Graded identities

When L is a G-graded algebra it makes sense to study only its graded identities. Suppose that $(G, +)$ is an arbitrary abelian group and A some G-algebra. Let us introduce a G-grading on an infinite set of generators Y by splitting it into a disjoint union $Y = \bigcup_{g\in G} Y_g$ where $\deg y_g = g \Leftrightarrow y_g \in Y_g$. Then on the free algebra $F\{Y\}$ an induced grading will be defined if we set $\deg(y_{g_1} \cdots y_{g_m}) = \deg y_{g_1} + \cdots + \deg y_{g_m} = g_1 + \cdots + g_m$ for any arrangement of the brackets. The polynomial $f(y_{g_1}, \ldots, y_{g_n})$ is called *a graded identity of the algebra A* if $f(u_1, \ldots, u_n) = 0$, as soon as $\deg u_1 = g_1, \ldots, \deg u_n = g_n$.

We denote in our case for any four nonnegative integers $k_1, \ldots, k_4$, $k_1 + \cdots + k_4 = n \in \mathbb{N} \cup \{0\}$ by $P_{k_1,\ldots,k_4} \subset P_n$ the subspace of all multilinear polynomials of degree n on four groups of variables $y_1, \ldots, y_{k_1}$, $y'_1, \ldots, y'_{k_2}$, $y''_1, \ldots, y''_{k_3}$, $y'''_1, \ldots, y'''_{k_4}$ from different grades given by group G and define the n^{th} graded codimension of algebra L as $c_n^{gr}(L) = \sum_{k_1+k_2+k_3+k_4=n} \binom{n}{k_1\ k_2\ k_3\ k_4} \dim \frac{P_{k_1,k_2,k_3,k_4}}{P_{k_1,k_2,k_3,k_4} \cap \mathrm{Id}(L)}$.

If $\mathrm{Lie}(X)$ is a free Lie algebra on a countable set of generators and L is an arbitrary Lie algebra then $P_n \,/\, (P_n \cap\ \mathrm{Id}(L)) \cong V_n \,/\, (V_n \cap\ \mathrm{Id}^{\mathrm{Lie}}(L))$ where $V_n \subset \mathrm{Lie}(X)$ is a subspace of all multilinear Lie polynomials in variables $x_1, \ldots, x_n$ and $\mathrm{Id}^{\mathrm{Lie}}(L) \lhd \mathrm{Lie}\ (X)$ is the ideal of Lie identities of algebra L.

Theorem 4. *Let F be an algebraically closed field of characteristic 0. Suppose $L = F[G] \otimes Q$ is a finite dimensional color Lie superalgebra over F with $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, arbitrary skew-symmetric bicharacter β and a finite dimensional simple Lie algebra Q (with a trivial G-grading). Then the graded PI-exponent of algebra L, defined as PI-exp$^{gr}(L) = \lim_{n\to\infty} \sqrt[n]{c_n^{gr}(L)}$, exists and is equal to $|G| \dim Q = \dim L$.*

Proof. Any multilinear Lie polynomial in $x_1, \ldots, x_n$ can be written as a linear combination of left-normed monomials $m_\sigma = [x_1, x_{\sigma(2)}, \ldots, x_{\sigma(n)}]$, where σ is a permutation of $2, \ldots, n$. If $w_1, \ldots, w_n$ are homogeneous elements of the color Lie superalgebra $L = F[G] \otimes Q$ then any multilinear polynomial expression in $w_1, \ldots, w_n$ is also a linear combination of left-normed products $[w_1, w_{\sigma(2)}, \ldots,\ w_{\sigma(n)}]$. Since we are interested in graded identities of L it is sufficient to consider only left-normed monomials and their linear combinations.

Let $(g_1, \ldots, g_n)$ be an n-tuple of such elements of group G that $(g_1, \ldots, g_n) = (\underbrace{e, \ldots, e}_{k_1}, \underbrace{a, \ldots, a}_{k_2}, \underbrace{b, \ldots, b}_{k_3}, \underbrace{ab, \ldots, ab}_{k_4})$. Then $m_\sigma(g_1 \otimes x_1, \ldots, g_n \otimes x_n) =$

$g_1 \cdots\ g_n \otimes \lambda_\sigma m_\sigma(x_1,\dots,x_n)$ in $F[G]\otimes F\{X\}$ where $\lambda_\sigma = \pm 1$ for given $g_1,\dots,g_n$ depends only on σ. Taking a multilinear polynomial $f = f(x_1,\dots,x_n) = \sum_{\sigma\in S_{n-1}} \alpha_\sigma m_\sigma$ of $F\{X\}$ we denote by $\widetilde{f}$ the element $\widetilde{f}(x_1,\dots,x_n) = \sum_{\sigma\in S_{n-1}} \lambda_\sigma\alpha_\sigma m_\sigma$. Then for any $w_1^1,\dots,w_{k_1}^1,\dots,w_1^4,\dots,w_{k_4}^4 \in Q$ we have $f(e\otimes w_1^1,\dots,e\otimes w_{k_1}^1,\dots,ab\otimes w_1^4,\dots,ab\otimes w_{k_4}^4) = a^{k_2}b^{k_3}(ab)^{k_4}\otimes\widetilde{f}(w_1^1,\dots,w_{k_1}^1,\dots,w_1^4,\dots,w_{k_4}^4)$. In particular, f is a graded identity of L, $f\in P_{k_1,k_2,k_3,k_4}\cap \mathrm{Id}(L)$ if and only if f is an identity of the Lie algebra Q. Now, if $c_n(Q) = N$ and $m_{\sigma_1},\dots,m_{\sigma_N}$ is a basis of V_n in $\mathrm{Lie}(X)$ modulo $\mathrm{Id}^{\,\mathrm{Lie}}(Q)$ then also $m_{\sigma_1},\dots,m_{\sigma_N}$ is a basis of $P_{k_1,\dots,k_4}$ modulo $\mathrm{Id}(L)$ in $F\{X\}$ and we have proved the required relation. Hence, we obtain

$$\begin{aligned} c_n^{gr}(L) &= \sum_{\substack{k_1\ge 0,k_2\ge 0,k_3\ge 0,k_4\ge 0\\ k_1+k_2+k_3+k_4=n}} \binom{n}{k_1\ k_2\ k_3\ k_4} \dim \frac{P_{k_1,k_2,k_3,k_4}}{P_{k_1,k_2,k_3,k_4}\cap\ \mathrm{Id}(L)} \\ &= c_n(Q) \sum_{\substack{k_1\ge 0,k_2\ge 0,k_3\ge 0,k_4\ge 0\\ k_1+k_2+k_3+k_4=n}} \binom{n}{k_1\ k_2\ k_3\ k_4} = 4^n c_n(Q) \end{aligned}$$

completing our proof. □

Some additional results on graded identities and graded codimensions of certain Lie superalgebras were recently obtained by D. Repovš and M. Zaicev in [15, 16]. They studied $\mathbb{Z}_2$-graded identities of Lie superalgebras of the type $b(t),\ t\ge 2$, over a field of characteristic 0. These algebras contain all matrices of the form $\begin{pmatrix} A & B \\ C & -A^T\end{pmatrix}$, where B and C are a symmetric and antisymmetric $t\times t$ matrix, respectively, while tr $A=0$. The 0-grade consists of all block diagonal matrices of this form, while 1-grade contains the matrices with blocks on the skew diagonal and the commutator of two homogenous matrices is given as $[M_i,N_j] = M_iN_j - (-1)^{ij}N_jM_i$. It appears that in this case the n-th codimension of the given Lie superalgebra is asymptotically strictly less than $(\dim\, b(t))^n$. As a consequence, an upper bound for ordinary (non-graded) PI-exponent for each simple Lie superalgebra of type $b(t),\ t\ge 3$, can be given.

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