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SPACES OF ULTRADISTRIBUTIONS OF BEURLING TYPE OVER $\mathbb{R}^d_+$ THROUGH LAGUERRE EXPANSIONS

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Dedicated to the memory of Professor Marc Krasner

Abstract. In this paper we define the spaces of ultradistributions of Beurling type over $\mathbb{R}^d_+$ and their dual spaces. Characterization of the spaces is given by the Laguerre series expansions and the growth rate of the corresponding Fourier-Laguerre coefficients. As a consequence of these characterizations, we obtain a deep insight into their topological structure and we prove the Schwartz kernel theorems.

1. Introduction

The Hankel-Clifford transform is analogous to the Fourier transform for $\mathbb{R}^d_+$. The procedure for generalization this classical transform consists in constructing the test function space where this transform turns out to be a self-inverse topological isomorphism. There are two approaches: firstly, we define a family of (F)-spaces then the test space $G^\alpha_\alpha(\mathbb{R}^d_+)$ is defined as an inductive limit of the (F)-spaces which leads to the space of ultradistributions of Roumier type over $\mathbb{R}^d_+$ or the test space $g^\alpha_\alpha(\mathbb{R}^d_+)$ is defined as a projective limit of the (F)-spaces which leads to the space of ultradistributions of Beurling type over $\mathbb{R}^d_+$. When the spaces are non-trivial we have dense and continuous inclusion:

$$g^\alpha_\alpha(\mathbb{R}^d_+) \hookrightarrow G^\alpha_\alpha(\mathbb{R}^d_+).$$

The spaces of ultradistributions of Roumier type over $\mathbb{R}^d_+$ were observed in [6] and [7]. This paper can be considered as a companion to [6]. Although

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the basic definitions and results are similar to those given in [6], this paper contains some new results that might be used in order to define a new symbol class for the Weyl pseudo-differential operators. More precisely, Theorem 7.6 in [6] can be formulated for symbols from the spaces of ultradistributions of Beurling type over $\mathbb{R}^d_+$.

We briefly describe the content of this paper. Firstly, we introduce the notation in subsection 1.1. Secondly, we give the definitions of the basic sequence and test spaces in subsections 1.2 and 1.4. Then we define the partial Hankel-Clifford transform in subcestion 1.4. In Section 2 we define the test spaces $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and their dual spaces. In Section 3 we study characterization of the spaces $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ in terms of the Fourier-Laguerre coefficients. The main difference between the spaces $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ is stated in Proposition 3.1. In Section 4 we analyze the topological structure of the spaces $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ and their dual spaces. We prove that the spaces $g^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ are (FN)-spaces ($G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ are (DFN)-spaces) and their dual spaces are (DFN)-spaces. Also we state the Schwartz kernel theorems.

1.1. **Notations.** We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, integers, real and complex numbers, respectively; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, \infty)$, $\mathbb{R}^d_+ = (0, \infty)^d$ and $\overline{\mathbb{R}^d_+} = [0, \infty)^d$.

We use the standard multi-index notation: for a multi-index $\alpha \in \mathbb{N}^d_0$ and $x \in \mathbb{R}^d$ (or $x \in \overline{\mathbb{R}^d_+}$), $x^{\alpha} = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $D^{\alpha} = \partial^{\alpha_1}/\partial x_1^{\alpha_1} \dots \partial^{\alpha_d}/\partial x_d^{\alpha_d}$. Furthermore, if $x, \gamma \in \overline{\mathbb{R}^d_+}$ we also use $x^{\gamma} = x_1^{\gamma_1} \dots x_d^{\gamma_d}$ and if $x_j = 0$ and $\gamma_j = 0$, we use the convention $0^0 = 1$.

By $\|\cdot\|_2$ we denote the norm in a Hilbert space (abbreviated as an (H)-space) $L^2(\mathbb{R}^d_+)$.

1.2. **Sequence spaces.** By s we denote the space of rapidly decreasing sequences; a complex sequence $\{a_n\}_{n \in \mathbb{N}^d_0} \in s$ if for each $j \in \mathbb{N}$, $\sup_{n \in \mathbb{N}^d_0} |a_n|(|n| + 1)^j < \infty$. With these seminorms, s becomes a nuclear Fréchet space; from now on abbreviated as an (FN)-space (see [11, p. 527]). By s' we denote its strong dual, the space of slowly increasing sequences i.e. s' consists of all complex valued sequences $\{b_n\}_{n \in \mathbb{N}^d_0}$ such that $\sup_{n \in \mathbb{N}^d_0} |b_n|(|n| + 1)^{-j} < \infty$ for some $j \in \mathbb{N}$. It is a (DFN)-space.

Let $\alpha \geq 1$ and $a > 1$. By $s^{\alpha,a}$ we denote the space of all complex sequences $\{a_n\}_{n \in \mathbb{N}^d_0}$ for which $\|\{a_n\}_{n \in \mathbb{N}^d_0}\|_{s^{\alpha,a}} = \sup_{n \in \mathbb{N}^d_0} |a_n| a^{|n|^{1/\alpha}} < \infty$. Equipped with this norm $s^{\alpha,a}$ becomes a Banach space (abbreviated as a (B)-space). $s^{\alpha,a}$ is continuously injected into $s^{\alpha,b}$ when $a > b > 1$. We define, as a locally convex space (abbreviated as an l.c.s.), $s^{(\alpha)} = \varprojlim_{a \to \infty} s^{\alpha,a}$; this projective limit is indeed a Hausdorff l.c.s. By the use of [6, Proposition

2.1 and Proposition 2.2] for the sequence spaces of Roumier type, we obtain the following two assertions.

Proposition 1.1. *For $a > b > 1$, the canonical inclusion $s^{\alpha,a} \to s^{\alpha,b}$ is nuclear. In particular, $s^{(\alpha)}$ is a nuclear (FS)-space (i.e. an (FN)-space) and its strong dual $(s^{(\alpha)})'$ is a (DFN)-space.*

Proposition 1.2. *The strong dual $(s^{(\alpha)})'$ of $s^{(\alpha)}$ is a (DFN)-space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}_0^d}$ such that $\|\{b_n\}_{n\in\mathbb{N}_0^d}\|_{(s^\alpha)',a} = \sum_{n\in\mathbb{N}_0^d}$ $|b_n|a^{-|n|^{1/\alpha}} < \infty$ for some $a > 1$. Its topology is generated by the system of seminorms $\|\cdot\|_{(s^\alpha)',a}$.*

1.3. **Space $\mathcal{S}(\mathbb{R}_+^d)$.** We racall that the (F)-space $\mathcal{S}(\mathbb{R}_+^d)$ consists of all $f \in \mathcal{C}^\infty(\mathbb{R}_+^d)$ such that all derivatives $D^p f$, $p \in \mathbb{N}_0^d$ extend to continuous functions on $\overline{\mathbb{R}_+^d}$ and $\sup_{x\in\mathbb{R}_+^d} x^k|D^p f(x)| < \infty$, $\forall k, p \in \mathbb{N}_0^d$. By $(\mathcal{S}(\mathbb{R}_+^d))'$, we denote its strong dual (see, e.g., [1], [5], [10] and [13]).

For $j \in \mathbb{N}_0$ and $\gamma \geq 0$, the j-th Laguerre polynomial of order γ is defined by

$$L_j^\gamma(t) = \frac{t^{-\gamma}e^t}{j!}\frac{d^j}{dt^j}(e^{-t}t^{\gamma+j}),\ t \geq 0.$$

The j-th Laguerre function of order γ is defined by

$$\mathcal{L}_j^\gamma(t) = (j!/\Gamma(j+\gamma+1))^{1/2}\, L_j^\gamma(t)e^{-t/2}.$$

For $n \in \mathbb{N}_0^d$ and $\gamma \in \overline{\mathbb{R}_+^d}$, $L_n^\gamma(t) = \prod_{l=1}^d L_{n_l}^{\gamma_l}(t_l)$ and $\mathcal{L}_n^\gamma(t) = \prod_{l=1}^d \mathcal{L}_{n_l}^{\gamma_l}(t_l)$ are the d-dimensional Laguerre polynomials and Laguerre functions of order γ, respectively. In the case $\gamma = 0$, we write L_n and $\mathcal{L}_n$ instead of L_n^0 and $\mathcal{L}_n^0$, respectively.

Let $\gamma \in \overline{\mathbb{R}_+^d}$. By $L^2(\mathbb{R}_+^d, t^\gamma dt)$ we denote the (B)-space of all measurable functions on $\mathbb{R}_+^d$ such that $\int_{\mathbb{R}_+^d} |f(t)|^2 t^\gamma dt < \infty$; its norm is defined by the square root of the last quantity. Moreover, $\{\mathcal{L}_n^\gamma\}_{n\in\mathbb{N}_0^d}$ is an orthonormal basis for $L^2(\mathbb{R}_+^d, t^\gamma dt)$ [12, Proposition 20.1].

1.4. **Partial Hankel-Clifford transform.** Let $\Lambda' = \{\lambda'_1, \dots, \lambda'_{d'}\} \subseteq \{1, \dots, d\}$ and $\Lambda'' = \{\lambda''_1, \dots, \lambda''_{d''}\} = \{1, \dots, d\}\backslash\Lambda'$. We define the partial Hankel-Clifford transform with respect to $x_{\Lambda'} = (x_{\lambda'_1}, \dots, x_{\lambda'_{d'}})$ by:

$$\mathcal{H}_0^{(\Lambda')}f(t) = 2^{-d'}\int_{\mathbb{R}_+^{d'}} f(x_{\Lambda'}, t_{\Lambda''})\prod_{l=1}^{d'} J_0\Big(\sqrt{x_{\lambda'_l}t_{\lambda'_l}}\Big)dx_{\Lambda'},$$

where J_0 is the Bessel function of the first kind. $\mathcal{H}_0^{(\Lambda')}$ is a self-inverse topological isomorphism on $g_\alpha^\alpha(\mathbb{R}_+^d)$ (see [6, Corollary 4.7]. Furthermore, in

[6, Corollary 4.5] it was proved that

$$\left\| t^{(p+k)/2} D_t^p f(t) \right\|_2 = \left(\prod_{l=1}^{d'} 2^{-p_{\lambda'_l}+k_{\lambda'_l}} \right) \left\| t^{(p+k)/2} D_{t_{\Lambda'}}^{k_{\Lambda'}} D_{t_{\Lambda''}}^{p_{\Lambda''}} \mathcal{H}_0^{(\Lambda')} f(t) \right\|_2 . \quad (1)$$

2. Definition and basic properties of spaces $g_\alpha^\beta(\mathbb{R}_+^d)$, $\alpha, \beta \geq 0$

For the definition of the test spaces for the spaces of ultradistributions of Roumier type over $\mathbb{R}_+$ see [2, Definition 2.1]; d-dimensional case is considered in [6]. We define the test spaces for the spaces of ultradistributions of Beurling type over $\mathbb{R}_+^d$:

Let $A > 0$. By $G_{\alpha,A}^{\beta,A}(\mathbb{R}_+^d)$, we denote the space of all $f \in \mathcal{S}(\mathbb{R}_+^d)$ such that

$$\sup_{p,k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|p+k|} k^{(\alpha/2)k} p^{(\beta/2)p}} < \infty.$$

With the following seminorms

$$\sigma_{A,j}(f) = \sup_{p,k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{|p+k|} k^{(\alpha/2)k} p^{(\beta/2)p}} + \sup_{\substack{|p| \leq j \\ |k| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^k D^p f(t)|, \ j \in \mathbb{N}_0,$$

one easily verifies that it becomes an (F)-space. Clearly, if $A_1 < A_2$, $G_{\alpha,A_1}^{\beta,A_1}(\mathbb{R}_+^d)$ is continuously injected into $G_{\alpha,A_2}^{\beta,A_2}(\mathbb{R}_+^d)$.

Spaces $g_\alpha^\beta(\mathbb{R}_+^d)$ are projective limits of the spaces $G_{\beta,A}^{\alpha,A}(\mathbb{R}_+^d)$ with respect to A:

$$g_\alpha^\beta(\mathbb{R}_+^d) = \varprojlim_{A \to 0} G_{\alpha,A}^{\beta,A}(\mathbb{R}_+^d).$$

Thus $g_\alpha^\beta(\mathbb{R}_+^d)$ is an (F)-space and it is continuously injected into $\mathcal{S}(\mathbb{R}_+^d)$.

For $A > 0$ we define $G_{\alpha,A}(\mathbb{R}_+^d)$ to be the space of all $f \in \mathcal{S}(\mathbb{R}_+^d)$ such that

$$\sup_{k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|k|} k^{(\alpha/2)k}} < \infty, \ \forall p \in \mathbb{N}_0^d$$

and similarly, $G^{\beta,A}(\mathbb{R}_+^d)$ to be the space of all $f \in \mathcal{S}(\mathbb{R}_+^d)$ such that

$$\sup_{p \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_2}{A^{|p|} p^{(\beta/2)p}} < \infty, \ \forall k \in \mathbb{N}_0^d.$$

Equipped with the following seminorms

$$\sigma'_{A,j}(f) = \sup_{|p| \leq j} \sup_{k \in \mathbb{N}_0^d} \frac{\|t^{(p+k)/2} D^p f(t)\|_{L^2(\mathbb{R}_+^d)}}{A^{|k|} k^{(\alpha/2)k}} + \sup_{\substack{|p| \leq j \\ |k| \leq j}} \sup_{t \in \mathbb{R}_+^d} |t^k D^p f(t)|, \ j \in \mathbb{N}_0,$$

$G_{\alpha,A}(\mathbb{R}^d_+)$ becomes an (F)-space. Analogously, by equipping $G^{\beta,A}(\mathbb{R}^d_+)$ with the following seminorms

$$\sigma''_{A,j}(f) = \sup_{|k|\le j}\sup_{p\in\mathbb{N}_0^d}\frac{\|t^{(p+k)/2}D^p f(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{|p|}p^{(\beta/2)p}} + \sup_{\substack{|p|\le j\\ |k|\le j}}\sup_{t\in\mathbb{R}^d_+}|t^k D^p f(t)|,\ j\in\mathbb{N}_0,$$

it also becomes an (F)-space. Similarly as above, we define $g_\alpha(\mathbb{R}^d_+) = \varprojlim_{A\to 0} G_{\alpha,A}(\mathbb{R}^d_+)$ and $g^\beta(\mathbb{R}^d_+) = \varprojlim_{A\to 0} G^{\beta,A}(\mathbb{R}^d_+)$. Thus, $g_\alpha(\mathbb{R}^d_+)$ and $g^\beta(\mathbb{R}^d_+)$ are (F)-spaces and they are continuously injected into $\mathcal{S}(\mathbb{R}^d_+)$.

For each $m\in\mathbb{N}_0^d$, $f(t)\mapsto t^m f(t)$ is a continuous mapping $g_\alpha(\mathbb{R}^d_+)\to g_\alpha(\mathbb{R}^d_+)$, $g^\beta(\mathbb{R}^d_+)\to g^\beta(\mathbb{R}^d_+)$ and $g^\beta_\alpha(\mathbb{R}^d_+)\to g^\beta_\alpha(\mathbb{R}^d_+)$.

By $(g^\beta(\mathbb{R}^d_+))'$, $(g_\alpha(\mathbb{R}^d_+))'$ and $(g^\beta_\alpha(\mathbb{R}^d_+))'$ we denote the strong dual spaces of $g^\beta(\mathbb{R}^d_+)$, $g_\alpha(\mathbb{R}^d_+)$ and $g^\beta_\alpha(\mathbb{R}^d_+)$, respectively.

It can easily be checked that when $\alpha,\beta>1$, $\mathcal{L}_n\in g^\beta_\alpha(\mathbb{R}^d_+)$ and hence $g^\beta_\alpha(\mathbb{R}^d_+)$ is dense in $\mathcal{S}(\mathbb{R}^d_+)$. In particular, for $\alpha>1$, $g_\alpha(\mathbb{R}^d_+)$, $g^\alpha(\mathbb{R}^d_+)$ and $g^\alpha_\alpha(\mathbb{R}^d_+)$ are dense in $\mathcal{S}(\mathbb{R}^d_+)$. Hence, $(\mathcal{S}(\mathbb{R}^d_+))'$ is continuously injected into $(g_\alpha(\mathbb{R}^d_+))'$, $(g^\alpha(\mathbb{R}^d_+))'$ and $(g^\alpha_\alpha(\mathbb{R}^d_+))'$.

When discussing the non-triviality we have the following:

Let $\alpha,\beta>0$. The spaces $g^\beta_\alpha(\mathbb{R}^d_+)$ are non-trivial if and only if $\alpha+\beta>2$. The non-triviality for the test spaces of the spaces of ultradistributions of Beurling type over $\mathbb{R}_+$ is given in [2, Corollary 3.9]. For the d-dimensional case it follows after considering the function $\varphi(t)=\varphi_1(t_1)\dots\varphi_d(t_d)$, where φ_j, $j=1,\dots,d$, is a non-zero element of $g^\beta_\alpha(\mathbb{R}_+)$.

3. Characterization of $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha>1$ via the Fourier-Laguerre coefficients

In this section, we characterise the spaces $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha>1$, in terms of the Fourier-Laguerre coefficients by the use of the partial Hankel-Clifford transform.

Proposition 3.1. *Let $f\in L^2(\mathbb{R}^d_+)$ and $a_n=\int_{\mathbb{R}^d_+} f(t)\mathcal{L}_n(t)dt$, $n\in\mathbb{N}_0^d$. If for every $a>1$ there exists $c>0$ such that*

$$|a_n|\le ca^{-|n|^{1/\alpha}},\ n\in\mathbb{N}_0^d, \tag{2}$$

then $f\in g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha>1$.

Proof. As $\{a_n\}_{n\in\mathbb{N}_0^d}\in s^{(\alpha)}\subseteq s$, it follows $f\in\mathcal{S}(\mathbb{R}^d_+)$ and the series $\sum_n a_n\mathcal{L}_n$ converges absolutely in $\mathcal{S}(\mathbb{R}^d_+)$ to f. For arbitrary but fixed $a>1$ there is $c>0$ such that (2) holds. Since $n_1^{1/\alpha}+\dots+n_d^{1/\alpha}\le d|n|^{1/\alpha}$, denoting

$\tilde{a} = a^{1/(2d)} > 1$, we have $a^{-|n|^{1/\alpha}} \leq \prod_{l=1}^{d} \tilde{a}^{-n_l^{1/\alpha}} \tilde{a}^{-n_l^{1/\alpha}}$.

Using the following estimates

$$\begin{aligned}
&\left| x^{(p+k)/2} \frac{d^k}{dx^k}(e^{-x/2} L_n^{\gamma}(x)) \right| \\
&\leq 2^{p+k+5}(n+1) \cdot \ldots \cdot (n + \left[\frac{p+k}{2}\right] + 2) \binom{n+k}{n}
\end{aligned}$$

and

$$\begin{aligned}
&\left| x^{(p+k)/2} \frac{d^p}{dx^p}(e^{-x/2} L_n^{\gamma}(x)) \right| \\
&\leq 2 \cdot 4^{k+2}(n+1) \cdot \ldots \cdot (n + \left[\frac{p+k}{2}\right] + 2) \binom{n+k}{n}
\end{aligned}$$

cf. the proof of [3, Lemma 2.1], for $p \in \mathbb{N}_0^d$ we have

$$\left\| t^{p/2} \mathcal{L}_n(t) \right\|_2 \leq 2^{|p|+5d} \prod_{l=1}^{d} (n_l + 1) \ldots \left(n_l + \left[\frac{p_l}{2}\right] + 2 \right), \tag{3}$$

$$\left\| t^{p/2} D^p \mathcal{L}_n(t) \right\|_2 \leq 2^{5d} \prod_{l=1}^{d} (n_l + 1) \ldots \left(n_l + \left[\frac{p_l}{2}\right] + 2 \right). \tag{4}$$

Let $\Lambda = \{\lambda_1, \ldots, \lambda_{d'}\} \subseteq \{1, \ldots, d\}$. Since $\mathcal{H}_0^{(\Lambda)} \mathcal{L}_n = (-1)^{n_{\lambda_1} + \ldots + n_{\lambda_{d'}}} \mathcal{L}_n$ (see [6, (6)]), (3) implies

$$\begin{aligned}
\left\| t^{p/2} \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 &\leq \sum_{n \in \mathbb{N}_0^d} |a_n| \left\| t^{p/2} \mathcal{L}_n(t) \right\|_2 \\
&\leq c 2^{|p|+5d} \sum_{n \in \mathbb{N}_0^d} \prod_{l=1}^{d} \tilde{a}^{-n_l^{1/\alpha}} \tilde{a}^{-n_l^{1/\alpha}} (n_l + 1) \ldots \left(n_l + \left[\frac{p_l}{2}\right] + 2 \right) \\
&\leq c 2^{|p|+5d} \sum_{n \in \mathbb{N}_0^d} \prod_{l=1}^{d} \tilde{a}^{-n_l^{1/\alpha}} \tilde{a}^{-n_l^{1/\alpha}} \left(n_l + \left[\frac{p_l}{2}\right] + 2 \right)^{\left[\frac{p_l}{2}\right]+2}.
\end{aligned} \tag{5}$$

For $u \geq 1$, consider the function $g(x) = \tilde{a}^{-x^{1/\alpha}} (u + x)^u$, $x \geq 0$. When $1 \leq u \leq x$, observe that $g(x) = \tilde{a}^{-x^{1/\alpha}} x^u (1 + u/x)^u \leq 2^u \tilde{a}^{-x^{1/\alpha}} x^u$. Since the function $x \mapsto \tilde{a}^{-x^{1/\alpha}} x^u$, $x \geq 0$, attains its maximum value at $x = (\alpha u / \ln \tilde{a})^{\alpha}$, we have $g(x) \leq (2\alpha / \ln \tilde{a})^{\alpha u} u^{\alpha u}$, when $1 \leq u \leq x$. When $x \leq u$, we have

$$g(x) \leq u^u (1 + x/u)^u \leq 2^u u^u = (2u^{(1-\alpha)})^u u^{\alpha u}.$$

If we apply these estimates on g with $u = [p_l/2] + 2$ in (5), keeping in mind that $a > 1$ can be chosen arbitrary large, we can conclude that for every $A > 0$ there exist $C > 0$ such that

$$\left\| t^{p/2} \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C A^{|p|} p^{(\alpha/2)p}, \text{ for all } p \in \mathbb{N}_0^d, \ \Lambda \subseteq \{1, \dots, d\}. \tag{6}$$

Similarly, by using (4), for every $A > 0$ there exists $C > 0$ such that

$$\left\| t^{p/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C A^{|p|} p^{(\alpha/2)p}, \text{ for all } p \in \mathbb{N}_0^d, \ \Lambda \subseteq \{1, \dots, d\}. \tag{7}$$

Denote by $(\cdot, \cdot)$ the inner product in $L^2(\mathbb{R}_+^d)$. Since $\mathcal{H}_0^{(\Lambda)} f \in \mathcal{S}(\mathbb{R}_+^d)$, by integration by parts one easily verifies that

$$\begin{aligned}
&\left(t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t), t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right) \\
&\qquad\qquad = \left| \left(D^p \left(t^{p+k} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right), \mathcal{H}_0^{(\Lambda)} f(t) \right) \right|.
\end{aligned}$$

Hence, for all $k, p \in \mathbb{N}_0^d$ such that $2k \geq p$ by (6) and (7), we obtain

$$\begin{aligned}
&\left\| t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2^2 \\
&\leq \sum_{m \leq p} \binom{p}{m} \frac{(p+k)!}{(p+k-m)!} \left| \left(t^{p+k-m} D^{2p-m} \mathcal{H}_0^{(\Lambda)} f(t), \mathcal{H}_0^{(\Lambda)} f(t) \right) \right| \\
&\leq 2^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m! \left| \left(t^{(2p-m)/2} D^{2p-m} \mathcal{H}_0^{(\Lambda)} f(t), t^{(2k-m)/2} \mathcal{H}_0^{(\Lambda)} f(t) \right) \right| \\
&\leq C_1 2^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m^m A^{2p+2k-2m} (2p-m)^{(\frac{\alpha}{2})(2p-m)} (2k-m)^{(\frac{\alpha}{2})(2k-m)} \\
&\leq C' (2A^2)^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m^{(\frac{\alpha}{2})m} m^{(\frac{\alpha}{2})m} (2p-m)^{(\frac{\alpha}{2})(2p-m)} (2k-m)^{(\frac{\alpha}{2})(2k-m)} \\
&\leq C' (2A^2)^{|p|+|k|} 2^{|p|} (2p)^{\alpha p} (2k)^{\alpha k},
\end{aligned}$$

where, in the fourth inequality we used the fact $m^m \leq c' A^{2m} m^{\alpha m}$, $\forall m \in \mathbb{N}_0^d$, for some $c' > 0$. We conclude that for every $A > 0$ there exists $C > 0$ such that for all $k, p \in \mathbb{N}_0^d$ such that $2k \geq p$ and all $\Lambda \subseteq \{1, \dots, d\}$

$$\left\| t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C A^{|p+k|} p^{(\alpha/2)p} k^{(\alpha/2)k}. \tag{8}$$

Let now $p, k \in \mathbb{N}_0^d$ be arbitrary but fixed. Let $\Lambda' = \{\lambda'_1, \dots, \lambda'_{d'}\} \subseteq \{1, \dots, d\}$ be such that $k_{\lambda'_l} < p_{\lambda'_l}/2$, $l = 1, \dots, d'$ and $\Lambda'' = \{\lambda''_1, \dots, \lambda''_{d''}\} = \{1, \dots, d\} \backslash \Lambda'$

be such that $k_{\lambda''_l} \geq p_{\lambda''_l}/2$, $l = 1, \dots, d''$. Then (1) and (8) imply

$$\begin{aligned}&\left\| t^{(p+k)/2} D_t^p f(t) \right\|_2 \\ &\qquad \leq 2^{|k|} \left\| t^{(p+k)/2} D_{t_{\Lambda'}}^{k_{\Lambda'}} D_{t_{\Lambda''}}^{p_{\Lambda''}} \mathcal{H}_0^{(\Lambda')} f(t) \right\|_2 \leq C(2A)^{|p+k|} p^{(\alpha/2)p} k^{(\alpha/2)k}.\end{aligned}$$

Since $A > 0$ is arbitrary, we conclude that $f \in g_\alpha^\alpha(\mathbb{R}_+^d)$. $\square$

Our next goal is to prove that if $f \in g_\alpha^\alpha(\mathbb{R}_+^d)$ holds (2). The Fourier-Laplace type operator defined by

$$\mathcal{F}_{\mathbf{D}} u(w) = \left\langle u(t), \prod_{l=1}^{d} e^{-\frac{1}{2}\frac{1+w_l}{1-w_l} t_l} \right\rangle, \ \mathbf{D} \to \mathbb{C},$$

where $\mathbf{D} = D_1 \times \cdots \times D_d$, $D_l = \{w_l \in \mathbb{C} |\, |w_l| < 1\}$, $l = 1, \dots, d$, plays a fundamental role.

Let $\mathbf{\Pi} = \Pi_1 \times \cdots \times \Pi_d$, where $\Pi_l = \{z_l \in \mathbb{C} |\, \operatorname{Im} z_l < 0\}$, $l = 1, \dots, d$. It can easily be checked that for each $z = x + iy \in \mathbf{\Pi}$, the function $t \mapsto e^{-2\pi i z t}$, $\mathbb{R}_+^d \to \mathbb{C}$, is in $g_\alpha^\alpha(\mathbb{R}_+^d)$ (also in $\mathcal{S}(\mathbb{R}_+^d)$). Also, for $z = x + iy \in \mathbf{\Pi}$, the functions $t \mapsto D_{x_l} e^{-2\pi i (x+iy)t} = -2\pi i t_l e^{-2\pi i (x+iy)t}$, $\mathbb{R}_+^d \to \mathbb{C}$ and $t \mapsto D_{y_l} e^{-2\pi i (x+iy)t} = 2\pi t_l e^{-2\pi i (x+iy)t}$, $\mathbb{R}_+^d \to \mathbb{C}$ are in $g_\alpha^\alpha(\mathbb{R}_+^d)$ (also in $\mathcal{S}(\mathbb{R}_+^d)$) for $l = 1, \dots, d$. Denote by e_l, $l = 1, \dots, d$, the standard orthonormal basis elements in $\mathbb{R}^d$, i.e. all coordinates of e_l are 0 except the l-th coordinate which is equal to 1. By standard arguments, one proves that for the fixed $x^{(0)} = (x_1^{(0)}, \dots, x_d^{(0)}) \in \mathbb{R}^d$ and $y^{(0)} = (y_1^{(0)}, \dots, y_d^{(0)}) \in \mathbb{R}^d$ with $y_l^{(0)} < 0$, $l = 1, \dots, d$ (i.e. $z^{(0)} = x^{(0)} + iy^{(0)} \in \mathbf{\Pi}$) we have

$$\left(e^{-2\pi i (x^{(0)} + x_l e_l + iy^{(0)})t} - e^{-2\pi i (x^{(0)} + iy^{(0)})t} \right) / x_l \to -2\pi i t_l e^{-2\pi i (x^{(0)} + iy^{(0)})t},$$

as $x_l \to 0$ and

$$\left(e^{-2\pi i (x^{(0)} + i(y^{(0)} + y_l e_l))t} - e^{-2\pi i (x^{(0)} + iy^{(0)})t} \right) / y_l \to 2\pi t_l e^{-2\pi i (x^{(0)} + iy^{(0)})t},$$

as $y_l \to 0$ in $G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$ for some $A > 0$ and consequently in $g_\alpha^\alpha(\mathbb{R}_+^d)$ and $\mathcal{S}(\mathbb{R}_+^d)$. Moreover,

$$\begin{aligned}&-2\pi i t_l e^{-2\pi i (x+iy)t} \to -2\pi i t_l e^{-2\pi i (x^{(0)} + iy^{(0)})t} \text{ and} \\ &\qquad\qquad\qquad\qquad 2\pi t_l e^{-2\pi i (x+iy)t} \to 2\pi t_l e^{-2\pi i (x^{(0)} + iy^{(0)})t}\end{aligned}$$

as $(x, y) \to (x^{(0)}, y^{(0)})$ in $G_{\alpha,A}^{\alpha,A}(\mathbb{R}_+^d)$ for some $A > 0$. Hence, the same holds in $g_\alpha^\alpha(\mathbb{R}_+^d)$ and $\mathcal{S}(\mathbb{R}_+^d)$. It follows that for each $u \in (g_\alpha^\alpha(\mathbb{R}_+^d))'$ or $u \in (\mathcal{S}(\mathbb{R}_+^d))'$, the function $z \mapsto \mathcal{F}_{\mathbf{\Pi}} u(z) = \langle u(t), e^{-2\pi i z t} \rangle$, $\mathbf{\Pi} \to \mathbb{C}$, is of the class $\mathcal{C}^1$; $D_{x_l} \mathcal{F}_{\mathbf{\Pi}} u(x + iy) = \langle u(t), D_{x_l} e^{-2\pi i (x+iy)t} \rangle$ and $D_{y_l} \mathcal{F}_{\mathbf{\Pi}} u(x + iy) =$

$\langle u(t), D_{y_l} e^{-2\pi i(x+iy)t}\rangle$. Since the Cauchy-Riemann equations hold for $\mathcal{F}_{\mathbf{\Pi}}u$, it is analytic on $\mathbf{\Pi}$. Observe that the mapping

$$w \mapsto \Omega(w) = ((1+w_1)/(4\pi i(1-w_1)), \ldots, (1+w_d)/(4\pi i(1-w_d)))$$

is a biholomorphic mapping from $\mathbf{D}$ onto $\mathbf{\Pi}$ with an inverse

$$z \mapsto \Omega^{-1}(z) = ((4\pi i z_1 - 1)/(4\pi i z_1 + 1), \ldots, (4\pi i z_d - 1)/(4\pi i z_d + 1)).$$

Hence, we have the following result.

Lemma 3.2. *For each $u \in (g^{\alpha}_{\alpha}(\mathbb{R}^d_+))'$ or $u \in (\mathcal{S}(\mathbb{R}^d_+))'$, the function*

$$\mathcal{F}_{\mathbf{D}}u(w) = \mathcal{F}_{\mathbf{\Pi}}u(\Omega(w)) = \left\langle u(t), \prod_{l=1}^{d} e^{-\frac{1}{2}\frac{1+w_l}{1-w_l}t_l} \right\rangle, \ \mathbf{D} \to \mathbb{C},$$

is analytic on $\mathbf{D}$, i.e. $\mathcal{F}_{\mathbf{D}}u \in \mathcal{O}(\mathbf{D})$.

Given a functional $u \in (\mathcal{S}(\mathbb{R}^d_+))'$, there exists a close relationship between the function $\mathcal{F}_{\mathbf{D}}(u)$ and the Fourier-Laguerre coefficients of u. The Fourier-Laguerre coefficients of u are almost the Taylor coefficients of the analytic function $\mathcal{F}_{\mathbf{D}}(u)$.

Proposition 3.3. ([6, Proposition 5.3]) *Let $u \in (\mathcal{S}(\mathbb{R}^d_+))'$ and $a_n = \langle u, \mathcal{L}_n\rangle$, $n \in \mathbb{N}^d_0$. Then,*

$$\mathcal{F}_{\mathbf{D}}(u)(w) = \prod_{j=1}^{d}(1-w_j) \sum_{n\in\mathbb{N}^d_0} a_n w^n, \ w \in \mathbf{D}. \tag{9}$$

In particular, if $\mathcal{F}_{\mathbf{D}}u = 0$ then $u = 0$.

We denote by $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$. Let $z \in \mathbb{C}^d$ then $z^{\mathbf{1}}$ stands for $z_1 \cdot \ldots \cdot z_d$.

Proposition 3.4. ([6, Proposition 5.4], Roumier case) *Let $\alpha > 1$ and $\{a_n\}_{n\in\mathbb{N}^d_0}$ be a sequence of complex numbers such that $a_n \to 0$ as $|n| \to \infty$. Then*

$$F(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_{n\in\mathbb{N}^d_0} a_n w^n, \ w \in \mathbf{D},$$

belongs to $\mathcal{O}(\mathbf{D})$. The following conditions are equivalent:

(i) For every $A > 0$ there exists $C > 0$ such that

$$|D^p F(w)| \leq C A^{|p|} p^{\alpha p}, \quad p \in \mathbb{N}^d_0, w \in \mathbf{D}. \tag{10}$$

(ii) For every $a > 1$ there exists $c > 0$ such that $|a_n| \leq c a^{-|n|^{1/\alpha}}$, $n \in \mathbb{N}^d_0$.

Proof. Clearly $F \in \mathcal{O}(\mathbf{D})$. Let $\sum_{n\in\mathbb{N}_0^d} b_n w^n$ be the power series expansion of F at 0. Then, for $n \in \mathbb{N}_0^d$ we have

$$b_n = \frac{D^n F(0)}{n!} = \frac{1}{n!}\sum_{\substack{k\leq n\\ k\leq \mathbf{1}}}\binom{n}{k}(-1)^{|k|}\left(\sum_{m\geq n-k}\frac{m!}{(m-n+k)!}a_m w^{m-n+k}\right)\Bigg|_{w=0}$$

$$= \sum_{k\leq n,\, k\leq \mathbf{1}}(-1)^{|k|}a_{n-k}. \tag{11}$$

Thus, for $n, m \in \mathbb{N}_0^d$,

$$\sum_{p\leq m} b_{n+\mathbf{1}+p} = \sum_{p\leq m}\sum_{k\leq \mathbf{1}}(-1)^{|k|}a_{n+p+\mathbf{1}-k}. \tag{12}$$

Firstly, assume that $d \geq 2$. It follows from the proof of [6, Proposition 5.4] that

$$\sum_{p\leq m} b_{n+\mathbf{1}+p} = \sum_{k\leq \mathbf{1}}(-1)^{|k|}a_{n+m+\mathbf{1}-m^{(k)}},\ \forall n, m \in \mathbb{N}_0^d, \tag{13}$$

where we used the following notations: for $k \in \mathbb{N}_0^d$ with $k \leq \mathbf{1}$ denote by $m^{(k)}$ the multi-index that satisfies $m_l^{(k)} = 0$ if $k_l = 0$ and $m_l^{(k)} = m_l + 1$ if $k_l = 1$, $l = 1, \dots, d$; when k varies through the multi-indexes that are $\leq \mathbf{1}$, $n + m + \mathbf{1} - m^{(k)}$ varies through the vertices of

$$Q_{n,m} = \{x \in \mathbb{R}^d |\, n_l \leq x_l \leq n_l + m_l + 1,\, l = 1, \dots, d\}.$$

Clearly, for $d = 1$ (12) and (13) are equal.

Assume that (i) holds. Since $D^p F(w) = \sum_{n\geq p}(n!/(n-p)!)\, b_n w^{n-p}$, the hypothesis in (i) and the Cauchy formula yield $(n!/(n-p)!)\,|b_n| \leq CA^{|p|}p^{\alpha p}$, for all $n, p \in \mathbb{N}_0^d$, $n \geq p$. As $n!/(n-p)! \geq e^{-|p|}n^p$, for $n \geq p$, we have

$$|b_n| \leq C\prod_{j=1}^d \inf_{p_j\leq n_j}\frac{(eA)^{p_j}p_j^{\alpha p_j}}{n_j^{p_j}},\ n \in \mathbb{N}_0^d. \tag{14}$$

We can assume that $A \geq 1$. Then, if $p_j \geq n_j$,

$$(eA)^{p_j}p_j^{\alpha p_j}/n_j^{p_j} \geq (eA)^{n_j}n_j^{\alpha n_j}/n_j^{n_j},$$

and so the infimum in (14) can be taken varying on $p_j \geq 0$, $j = 1, \dots, d$. Thus, [4, (2) and (3), p. 169-170] imply, with suitable $c' > 0$ and $a' > 1$,

$$|b_n| \leq C\prod_{j=1}^d \inf_{p_j\geq 0}\frac{(eA)^{p_j}p_j^{\alpha p_j}}{n_j^{p_j}} \leq c' a'^{-|n|^{1/\alpha}},\ n \in \mathbb{N}_0^d.$$

Observe that for $p, n \in \mathbb{N}_0^d$ with $p \geq n$, we have $|p|^{1/\alpha} \geq (|p-n|^{1/\alpha} + |n|^{1/\alpha})/2$. Thus, if we put $a = \sqrt{a'} > 1$ we have $a'^{-|p|^{1/\alpha}} \leq a^{-|p-n|^{1/\alpha}}a^{-|n|^{1/\alpha}}$ for all

$p \geq n$. The above estimate for $|b_n|$ together with (13) implies that for all $n, m \in \mathbb{N}_0^d$

$$\begin{aligned}|a_n| &\leq \sum_{p\leq m} |b_{n+\mathbf{1}+p}| + \sum_{k\leq \mathbf{1},\, k\neq \mathbf{1}} |a_{n+m+\mathbf{1}-m^{(k)}}| \\ &\leq c' a^{-|n|^{1/\alpha}} \sum_{p\in\mathbb{N}_0^d} a^{-|p|^{1/\alpha}} + \sum_{k\leq \mathbf{1},\, k\neq \mathbf{1}} |a_{n+m+\mathbf{1}-m^{(k)}}| \\ &= c a^{-|n|^{1/\alpha}} + \sum_{k\leq \mathbf{1},\, k\neq \mathbf{1}} |a_{n+m+\mathbf{1}-m^{(k)}}|.\end{aligned}$$

The last sum has exactly $2^d - 1$ terms and since $k \neq \mathbf{1}$, $|n+m+\mathbf{1}-m^{(k)}| \geq |n| + \min\{m_l|\, l = 1,\ldots,d\}$. Let $n \in \mathbb{N}_0^d$ be arbitrary but fixed. Since the above estimate for $|a_n|$ holds for arbitrary $m \in \mathbb{N}_0^d$ and since $a_n \to 0$ as $|n| \to \infty$ (by hypothesis), this implies $|a_n| \leq ca^{-|n|^{1/\alpha}}$.

Assume now that (ii) holds. Then (11) implies that for every $a > 1$ there exists $c > 0$ such that $|b_n| \leq ca^{-|n|^{1/\alpha}}$, $\forall n \in \mathbb{N}_0^d$. Observe that $n_1^{1/\alpha} + \ldots + n_d^{1/\alpha} \leq d|n|^{1/\alpha}$. Hence, by putting $a' = a^{1/d}$, we have $a^{-|n|^{1/\alpha}} \leq \prod_{j=1}^d a'^{-n_j^{1/\alpha}}$. Now, for $p \in \mathbb{N}_0^d$ and $w \in \mathbf{D}$ we obtain

$$|D^p F(w)| \leq \sum_{n\geq p} \frac{n!}{(n-p)!}|b_n| \leq c \sum_{n\in\mathbb{N}_0^d} \prod_{j=1}^d n_j^{p_j} a'^{-n_j^{1/\alpha}}.$$

Since $\rho(x) = x^p u^{-x^{1/\alpha}}$, $x \geq 0$ ($u > 1$, $p \in \mathbb{N}_0$) attains its maximum value at $x = (\alpha p/\ln u)^\alpha$, we proved (10). $\square$

We omit the proofs of the next two propositions since they coincide with the proofs in [6].

Proposition 3.5. ([6, Proposition 5.5], Roumier case) *Let $f \in g_\alpha(\mathbb{R}_+^d)$, $\alpha > 1$. Then for every $A > 0$ there exists $C > 0$ such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq CA^{|p|}p^{\alpha p},\ p \in \mathbb{N}_0^d,\ w \in \mathbf{D},\ \operatorname{Re} w_l \leq 0,\ l = 1,\ldots,d.$$

Proposition 3.6. ([6, Proposition 5.6], Roumier case) *Let $f \in g_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha > 1$. Then for every $A > 0$ there exists $C > 0$ such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq CA^{|p|}p^{\alpha p},\ p \in \mathbb{N}_0^d,\ w \in \mathbf{D} \tag{15}$$

and $\lim_{w\to\mathbf{1}} \mathcal{F}_{\mathbf{D}}(f)(w) = 0$.

Now, Proposition 3.1, Proposition 3.6, Proposition 3.3 and Proposition 3.4 give the main result of this section:

Theorem 3.7. ([3, Theorem 3.6], Roumieu case for d=1) *Let $\alpha > 1$. For $f \in L^2(\mathbb{R}^d_+)$ let $a_n = \int_{\mathbb{R}^d_+} f(t)\mathcal{L}_n(t)dt$, $n \in \mathbb{N}^d_0$. The following conditions are equivalent:*

- (i) *For every $a > 1$ there exists $c > 0$ such that $|a_n| \leq ca^{-|n|^{1/\alpha}}$ for $n \in \mathbb{N}^d_0$.*
- (ii) *$f \in g^\alpha_\alpha(\mathbb{R}^d_+)$.*
- (iii) *For every $A > 0$ there exist $C > 0$ such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq CA^{|p|}p^{\alpha p} \text{ for } p \in \mathbb{N}^d_0 \text{ and } w \in \mathbf{D}$$

and $\lim_{w\to\mathbf{1}} \mathcal{F}_{\mathbf{D}}(f)(w) = 0$.

Conversely, given a sequence $\{a_n\}_{n\in\mathbb{N}^d_0}$ satisfying condition (i) or given $F \in \mathcal{O}(\mathbf{D})$ of the form $F(w) = (\mathbf{1}-w)^{\mathbf{1}} \sum_n a_n w^n$ with $a_n \to 0$ as $|n| \to \infty$ which satisfies (10)*, there exists $f \in g^\alpha_\alpha(\mathbb{R}^d_+)$ such that $a_n = \int_{\mathbb{R}^d_+} f(t)\mathcal{L}_n(t)dt$ and $\mathcal{F}_{\mathbf{D}}(f)(w) = F(w)$ for $w \in \mathbf{D}$.*

4. Topological properties of $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha > 1$. The Kernel theorems

Theorem 3.7 will allow us to gain deep insights into the topological structure of $g^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha > 1$. Let $\iota : g^\alpha_\alpha(\mathbb{R}^d_+) \to s^\alpha$, $\iota(f) = \{\langle f, \mathcal{L}_n\rangle\}_{n\in\mathbb{N}^d_0}$. Theorem 3.7 proves that ι is a well defined bijection.

Theorem 4.1. *Let $\alpha > 1$. The mapping $\iota : g^\alpha_\alpha(\mathbb{R}^d_+) \to s^\alpha$, $\iota(f) = \{\langle f, \mathcal{L}_n\rangle\}_{n\in\mathbb{N}^d_0}$, is a topological isomorphism between $g^\alpha_\alpha(\mathbb{R}^d_+)$ and s^α. In particular, $g^\alpha_\alpha(\mathbb{R}^d_+)$ is an (FN)-space and $(g^\alpha_\alpha(\mathbb{R}^d_+))'$ is a (DFN)-space.*

For each $f \in g^\alpha_\alpha(\mathbb{R}^d_+)$, $\sum_{n\in\mathbb{N}^d_0} \langle f, \mathcal{L}_n\rangle \mathcal{L}_n$ is summable to f in $g^\alpha_\alpha(\mathbb{R}^d_+)$.

Proof. If we consider ι as a linear mapping from $g^\alpha_\alpha(\mathbb{R}^d_+)$ into s ($s^{(\alpha)}$ is canonically injected into s) then ι is continuous since it decomposes as $g^\alpha_\alpha(\mathbb{R}^d_+) \longrightarrow \mathcal{S}(\mathbb{R}^d_+) \xrightarrow{f\mapsto\{\langle f,\mathcal{L}_n\rangle\}_n} s$, where the first mapping is the canonical inclusion. Hence, ι has a closed graph in $g^\alpha_\alpha(\mathbb{R}^d_+) \times s$. Since the range of ι is in $s^{(\alpha)}$ and $s^{(\alpha)}$ is continuously injected into s, the graph of ι is closed in $g^\alpha_\alpha(\mathbb{R}^d_+) \times s^{(\alpha)}$. $g^\alpha_\alpha(\mathbb{R}^d_+)$ is a reduced projective limit of (F)-spaces. For this reason, $g^\alpha_\alpha(\mathbb{R}^d_+)$ is ultrabornological (cf. [8, §28, 3.(4), p. 383 and 4.(4), p. 384], [9, p.43]; every (F)-space is ultrabornological). Moreover, $s^{(\alpha)}$ is a webbed space of De Wilde (see [9, §35, 4.(11), p. 64]). Hence, the closed graph theorem of De Wilde (see [9, §35, 2.(2), p. 57]) implies that $\iota : g^\alpha_\alpha(\mathbb{R}^d_+) \to s^{(\alpha)}$ is continuous. Also, $s^{(\alpha)}$ is ultrabornological since it is bornological and complete and $g^\alpha_\alpha(\mathbb{R}^d_+)$ is a webbed space of De Wilde (cf. [9, §35, 4.(7), p. 63]; every (F)-space is a webbed space of De Wilde).

The mapping $\iota^{-1} : s^{(\alpha)} \to g_\alpha^\alpha(\mathbb{R}_+^d)$, which has a closed graph, is continuous by the De Wilde closed graph theorem (see [9, §35, 2.(2), p. 57]). Now, Proposition 1.1 implies that $g_\alpha^\alpha(\mathbb{R}_+^d)$ is an (FN)-space and $(g_\alpha^\alpha(\mathbb{R}_+^d))'$ is a (DFN)-space.

The last statement can be obtained as in the proof of [6, Theorem 6.1]. $\square$

Theorem 4.2. *Let $\alpha > 1$. The mapping $\tilde{\iota} : (g_\alpha^\alpha(\mathbb{R}_+^d))' \to (s^{(\alpha)})'$, $\tilde{\iota}(T) = \{\langle T, \mathcal{L}_n\rangle\}_{n\in\mathbb{N}_0^d}$, is a topological isomorphism. Moreover, $\sum_{n\in\mathbb{N}_0^d}\langle T, \mathcal{L}_n\rangle\mathcal{L}_n$ is summable to T in $(g_\alpha^\alpha(\mathbb{R}_+^d))'$.*

Proof. By Theorem 4.1, both the transpose of ι, ${}^t\iota : (s^\alpha)' \to (g_\alpha^\alpha(\mathbb{R}_+^d))'$, and its inverse $({}^t\iota)^{-1} : (g_\alpha^\alpha(\mathbb{R}_+^d))' \to (s^{(\alpha)})'$ are topological isomorphisms. For $T \in (g_\alpha^\alpha(\mathbb{R}_+^d))'$, let $\{b_n\}_n = ({}^t\iota)^{-1}(T)$. Then

$$\langle T, \mathcal{L}_n\rangle = \langle {}^t\iota(\{b_n\}_n), \mathcal{L}_n\rangle = \langle \{b_n\}_n, \iota(\mathcal{L}_n)\rangle = b_n.$$

Thus $\{\langle T, \mathcal{L}_n\rangle\}_n = \{b_n\}_n = ({}^t\iota)^{-1}(T) \in (s^{(\alpha)})'$. Hence $\tilde{\iota}$ is in fact a topological isomorphism $({}^t\iota)^{-1} : (g_\alpha^\alpha(\mathbb{R}_+^d))' \to (s^{(\alpha)})'$. Now, by the similar approach as in [6, Theorem 6.2], one proves that $\sum_{n\in\mathbb{N}_0^d}\langle T, \mathcal{L}_n\rangle\mathcal{L}_n$ is summable to T in $(g_\alpha^\alpha(\mathbb{R}_+^d))'$. $\square$

For $T \in (g_\alpha^\alpha(\mathbb{R}_+^d))'$, by Lemma 3.2, $\mathcal{F}_{\mathbf{D}}T \in \mathcal{O}(\mathbf{D})$. Since $\sum_n\langle T, \mathcal{L}_n\rangle\mathcal{L}_n$ is summable to T in $(g_\alpha^\alpha(\mathbb{R}_+^d))'$, by the same method as in the proof of Proposition 3.3, one proves the following result.

Proposition 4.3. *Let $T \in (g_\alpha^\alpha(\mathbb{R}_+^d))'$, $\alpha > 1$ and $b_n = \langle T, \mathcal{L}_n\rangle$, $n \in \mathbb{N}_0^d$. Then,*

$$\mathcal{F}_{\mathbf{D}}(T)(w) = \prod_{j=1}^{d}(1 - w_j)\sum_{n\in\mathbb{N}_0^d} b_n w^n,\ w \in \mathbf{D}.$$

In particular, if $\mathcal{F}_{\mathbf{D}}T = 0$ then $T = 0$.

Now, we state the kernel theorems:

Theorem 4.4. *Let $\alpha > 1$. We have the following canonical isomorphism:*

$$g_\alpha^\alpha(\mathbb{R}_+^{d_1})\hat{\otimes} g_\alpha^\alpha(\mathbb{R}_+^{d_2}) \cong g_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}),\ (g_\alpha^\alpha(\mathbb{R}_+^{d_1}))'\hat{\otimes}(g_\alpha^\alpha(\mathbb{R}_+^{d_2}))' \cong (g_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))'.$$

Proof. Let $s_{d_1}^{(\alpha)}$, $s_{d_2}^{(\alpha)}$ and $s^{(\alpha)}$ be the d_1-dimensional, the d_2-dimensional and the d-dimensional space $s^{(\alpha)}$, respectively, where $d = d_1 + d_2$. By the same technique as in the proof of [6, Theorem 6.4], we obtain $(s_{d_1}^{(\alpha)})'\hat{\otimes}(s_{d_2}^{(\alpha)})' \cong (s^{(\alpha)})'$. As all spaces in consideration are (DFN)-spaces, by duality follows $s_{d_1}^{(\alpha)}\hat{\otimes}s_{d_2}^{(\alpha)} \cong s^{(\alpha)}$. Note that the isomorphism is in fact the extension of the canonical inclusion $\kappa : s_{d_1}^{(\alpha)} \otimes s_{d_2}^{(\alpha)} \to s^{(\alpha)}$, $\kappa(\{a_n\}_n \otimes \{b_m\}_m) = \{a_n b_m\}_{(n,m)}$.

Now observe that the diagram

$$\begin{array}{ccc} s^{(\alpha)}_{d_1} \otimes s^{(\alpha)}_{d_2} & \xrightarrow{\kappa} & s^{(\alpha)} \\ {\scriptstyle \iota\otimes\iota}\Big\uparrow & & \Big\uparrow{\scriptstyle \iota} \\ g^{\alpha}_{\alpha}(\mathbb{R}^{d_1}_+) \otimes g^{\alpha}_{\alpha}(\mathbb{R}^{d_2}_+) & \longrightarrow & g^{\alpha}_{\alpha}(\mathbb{R}^{d}_+) \end{array}$$

commutes, where the bottom horizontal line is the canonical inclusion $f \otimes g(x,y) \mapsto f(x)g(y)$. Since κ extends to an isomorphism, by Theorem 4.1, it follows that the canonical inclusion $g^{\alpha}_{\alpha}(\mathbb{R}^{d_1}_+) \otimes g^{\alpha}_{\alpha}(\mathbb{R}^{d_2}_+) \to g^{\alpha}_{\alpha}(\mathbb{R}^{d}_+)$ is continuous and it extends to an isomorphism $g^{\alpha}_{\alpha}(\mathbb{R}^{d_1}_+)\hat{\otimes} g^{\alpha}_{\alpha}(\mathbb{R}^{d_2}_+) \cong g^{\alpha}_{\alpha}(\mathbb{R}^{d}_+)$. The assertion $(g^{\alpha}_{\alpha}(\mathbb{R}^{d_1}_+))'\hat{\otimes}(g^{\alpha}_{\alpha}(\mathbb{R}^{d_2}_+))' \cong (g^{\alpha}_{\alpha}(\mathbb{R}^{d}_+))'$ can be obtained by the duality of an isomorphism $g^{\alpha}_{\alpha}(\mathbb{R}^{d_1}_+)\hat{\otimes} g^{\alpha}_{\alpha}(\mathbb{R}^{d_2}_+) \cong g^{\alpha}_{\alpha}(\mathbb{R}^{d}_+)$ since $g^{\alpha}_{\alpha}(\mathbb{R}^{d}_+)$ is an (FN)-space. □

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