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STABILITY OF $n-$ DIMENSIONAL ADDITIVE FUNCTIONAL EQUATION IN FUZZY NORMED SPACES

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ABSTRACT. In this paper, the authors investigate the generalized Hyers-Ulam-stability of new $n-$ dimensional additive functional equation

$$\sum_{j=1}^{n} f\left(-x_j + \sum_{1\le i\le n, i\ne j} x_i\right) = (n-2)\sum_{j=1}^{n} f(x_j)$$

in fuzzy normed space using direct and fixed point approach.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of S.M. Ulam [39] concerning the stability of group homomorphisms. D.H. Hyers [14] gave a first affirmative partial answer to the question of Ulam for Banach spaces. He proved the following celebrated theorem.

Theorem 1.1. *[14] Let X, Y be Banach spaces and let $f : X \to Y$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{1.1}$$

for all $x, y \in X$. Then the limit

$$a(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{1.2}$$

exists for all $x \in X$ and $a : X \to Y$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \le \varepsilon \tag{1.3}$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then the function a is linear.

Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [34] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [34] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional

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equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [13] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, J.M. Rassias [31] followed the innovative approach of the Th.M. Rassias theorem [34] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$ with $p+q=1$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [36] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 11, 15, 19]).

A.K. Katsaras [21] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [12, 23, 40]. In particular, T. Bag and S.K. Samanta [7], following S.C. Cheng and J.N. Mordeson [9], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [22]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [8].

One of the most famous functional equation is the additive functional equation

$$f(x+y) = f(x) + f(y) \tag{1.4}$$

which is additive functional equation having solution $f(x) = cx$.

We use the definition of fuzzy normed spaces given in [7] and [26–29].

Definition 1.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(F1) $N(x,c) = 0$ *for* $c \leq 0$;

(F2) $x = 0$ *if and only if* $N(x,c) = 1$ *for all* $c > 0$;

(F3) $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

(F4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;

(F5) $N(x,\cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x,t) = 1$;

(F6) *for* $x \neq 0, N(x,\cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X,N) is called a fuzzy normed linear space. One may regard $N(X,t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 1.1. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x,t) = \begin{cases} \dfrac{t}{t+||x||}, & t > 0,\ x \in X, \\ 0, & t \leq 0,\ x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.2. *Let (X,N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim\limits_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim\limits_{n\to\infty} x_n = x$.*

Definition 1.3. *A sequence x_n in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

Definition 1.4. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

Definition 1.5. *A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ covering to x_0 in X, the sequence $f\{x_n\}$ converges to $f(x_0)$. If f is continuous at each point of $x_0 \in X$ then f is said to be continuous on X.*

In this paper, author investigate the generalized Hyers-Ulam-Aoki-Rassias stability of generalized $n-$ dimensional additive functional equation

$$\sum_{j=1}^{n} f\left(-x_j + \sum_{1\leq i\leq n, i\neq j} x_i\right) = (n-2)\sum_{j=1}^{n} f(x_j) \tag{1.5}$$

in the fuzzy normed vector space setting.

2. General Solution

In this section, the general solution of the functional equation (1.5) is given.

Theorem 2.1. *Let X and Y be real vector spaces. The mapping $f : X \to Y$ satisfies the additive functional equation (1.4) for all $x,y \in X$ if and only if $f : X \to Y$ satisfying the functional equation (1.5) for all $x_1, x_2, \ldots, x_n \in X$.*

Proof. Let $f : X \to Y$ satisfies the functional equation (1.5). Setting $x_1 = x_2 = x_3 = \ldots = x_n = 0$ in (1.5), we get $f(0) = 0$. Letting $x_2 = x_3 = \ldots = x_n = 0$ in (1.5), we obtain $f(-x_1) = -f(x_1)$ for all $x_1 \in X$. Therefore f is an odd function. Replacing $(x_1, x_2, x_3, x_4, \ldots, x_n)$ by $(x, y, 0, 0, \ldots, 0)$ in (1.5), we get (1.4) as desired.

Conversely, let $f : X \to Y$ satisfies the functional equation (1.4). Setting $x = y = 0$ in (1.4), we obtain $f(0) = 0$. Replacing x by $-y$, we arrive

$$f(-x) = -f(x) \tag{2.1}$$

for all $x \in X$. Setting $y = x$ and $y = 2x$ in (1.4) respectively, we get

$$f(2x) = 2f(x), \; f(3x) = 3f(x) \tag{2.2}$$

for all $x \in X$. In general, for any positive integer a, we have

$$f(ax) = af(x) \tag{2.3}$$

for all $x \in X$. It is easy to verify from (1.4) that

$$f(x_1 + x_2 + \ldots + x_n) = f(x_1) + f(x_2) + \ldots + f(x_n) \tag{2.4}$$

for all $x_1, x_2, \ldots, x_n \in X$. Now, replacing x_1 by $-x_1$, x_2 by $-x_2$, $\cdots$, x_{n-1} by $-x_{n-1}$ and x_n by $-x_n$ respectively in (2.4) and using (2.1), (2.2) and (2.3), we arrive the following equations

$$f(-x_1 + x_2 + \ldots + x_{n-1} + x_n) = -f(x_1) + f(x_2) + \ldots + f(x_{n-1}) + f(x_n), \tag{2.5}$$

$$f(x_1 - x_2 + \ldots + x_{n-1} + x_n) = f(x_1) - f(x_2) + \ldots + f(x_{n-1}) + f(x_n), \tag{2.6}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f(x_1 + x_2 + \ldots - x_{n-1} + x_n) = f(x_1) + f(x_2) + \ldots - f(x_{n-1}) + f(x_n), \tag{2.7}$$

$$f(x_1 + x_2 + \ldots + x_{n-1} - x_n) = f(x_1) + f(x_2) + \ldots + f(x_{n-1}) - f(x_n). \tag{2.8}$$

for all $x_1, x_2, \ldots, x_n \in X$. Adding all the n equations, we arrive (1.5) as desired. □

3. Main Results: Direct Method

Throughout this section, assume that $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now use the following notation for a given mapping $f : X \to Y$

$$D\ f(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{n} f\left(-x_j + \sum_{1 \le i \le n, i \ne j} x_i\right) - (n-2)\sum_{j=1}^{n} f(x_j)$$

for all $x_1, x_2, \cdots, x_n \in X$.

Now, we investigate the generalized Hyers-Ulam stability of $n-$ dimensional additive functional equation (1.5).

Theorem 3.1. *Let $\beta \in \{-1, 1\}$ be fixed and $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space respectively. Let $\psi : X^n \to Z$ be a function with $0 < \left(\frac{\alpha}{2}\right)^{\beta} < 1$*

$$N'\left(\psi\left((n-2)^{\beta}x, (n-2)^{\beta}x, \ldots, (n-2)^{\beta}x\right), t\right) \ge N'\left(\alpha^{\beta}\psi(x, x, \ldots, x), t\right) \tag{3.1}$$

for all $x \in X$ and all $t > 0, \alpha > 0$ and

$$\lim_{k\to\infty} N'\left(\psi\left((n-2)^{\beta k}x_1, (n-2)^{\beta k}x_2, \cdots, (n-2)^{\beta k}x_n\right), (n-2)^{\beta k}t\right) = 1 \tag{3.2}$$

for all $x_1, x_2, \cdots, x_n \in X$ and all $t > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N(Df(x_1, x_2, \cdots, x_n), t) \ge N'(\psi(x_1, x_2, \cdots, x_n), t) \tag{3.3}$$

for all $t > 0$ and all $x_1, x_2, \cdots, x_n \in X$. Then the limit

$$A(x) = N - \lim_{k\to\infty} \frac{f((n-2)^{\beta k}x)}{(n-2)^{\beta k}} \tag{3.4}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive function such that

$$N\left(f(x) - A(x), t\right) \geq N'\left(\psi(x, x, \ldots, x), nt|(n-2) - \alpha|\right) \tag{3.5}$$

for all $x \in X$ and all $t > 0$.

Proof. Assume $\beta = 1$. Replacing $(x_1, x_2, \cdots, x_n)$ by $(x, x, \ldots, x)$ in (3.3), we get

$$N\left(nf((n-2)x) - n(n-2)f(x), t\right) \geq N'\left(\psi(x, x, \ldots, x), t\right) \tag{3.6}$$

for all $x \in X$ and all $t > 0$. Replacing x by $(n-2)^k x$ in (3.6), we get

$$N\left(\frac{f((n-2)^{k+1}x)}{n-2} - f((n-2)^k x), \frac{t}{n(n-2)}\right) \geq N'\left(\psi((n-2)^k x, (n-2)^k x, \ldots, (n-2)^k x), t\right) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. Using (3.1), $(F3)$ in (3.7), we get

$$N\left(\frac{f((n-2)^{k+1}x)}{n-2} - f((n-2)^k x), \frac{t}{n(n-2)}\right) \geq N'\left(\psi(x, x, \ldots, x), \frac{t}{\alpha^k}\right) \tag{3.8}$$

for all $x \in X$ and all $t > 0$. It is simple to check from (3.8), that

$$N\left(\frac{f((n-2)^{k+1}x)}{(n-2)^{k+1}} - \frac{f((n-2)^k x)}{(n-2)^k}, \frac{t}{n(n-2)^{k+1}}\right) \geq N'\left(\psi(x, x, \ldots, x), \frac{t}{\alpha^k}\right) \tag{3.9}$$

holds for all $x \in X$ and all $t > 0$. Replacing t by $\alpha^k t$ in (3.9), we get

$$N\left(\frac{f((n-2)^{k+1}x)}{(n-2)^{k+1}} - \frac{f((n-2)^k x)}{(n-2)^k}, \frac{\alpha^k t}{n(n-2)^{k+1}}\right) \geq N'\left(\psi(x, x, \ldots, x), t\right) \tag{3.10}$$

for all $x \in X$ and all $t > 0$. It is easy to see that

$$\frac{f((n-2)^k x)}{(n-2)^k} - f(x) = \sum_{i=0}^{k-1}\left[\frac{f((n-2)^{i+1}x)}{(n-2)^{i+1}} - \frac{f((n-2)^i x)}{(n-2)^i}\right] \tag{3.11}$$

for all $x \in X$. From equations (3.10) and (3.11), we have

$$\begin{aligned} N&\left(\frac{f((n-2)^k x)}{(n-2)^k} - f(x), \sum_{i=0}^{k-1}\frac{\alpha^i\, t}{n(n-2)^{i+1}}\right) \\ &\geq min \bigcup_{i=0}^{k-1}\left\{\frac{f((n-2)^{i+1}x)}{(n-2)^{i+1}} - \frac{f((n-2)^i x)}{(n-2)^i}, \frac{\alpha^i\, t}{n(n-2)^{i+1}}\right\} \\ &\geq min \bigcup_{i=0}^{k-1}\left\{N'\left(\psi(x, x, \ldots, x), t\right)\right\} \\ &\geq N'\left(\psi(x, x, \ldots, x), t\right) \end{aligned} \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Replacing x by $(n-2)^m x$ in (3.12) and using (3.1), $(F3)$, we arrive

$$N\left(\frac{f((n-2)^{k+m}x)}{(n-2)^{k+m}}-\frac{f((n-2)^m x)}{(n-2)^m},\sum_{i=0}^{k-1}\frac{\alpha^i\, t}{n(n-2)^{m+i+1}}\right)\geq N'\left(\psi(x,x,\ldots,x,\frac{t}{\alpha^m}\right) \tag{3.13}$$

for all $x\in X$ and all $t>0$ and all $m,k\geq 0$. Replacing t by $\alpha^m t$ in (3.13), we get

$$N\left(\frac{f((n-2)^{k+m}x)}{(n-2)^{k+m}}-\frac{f((n-2)^m x)}{(n-2)^m},\sum_{i=m}^{m+k-1}\frac{\alpha^i\, t}{n(n-2)^{i+1}}\right)\geq N'\left(\psi(x,x,\ldots,x,t\right) \tag{3.14}$$

for all $x\in X$ and all $t>0$ and all $m,k\geq 0$. Using $(F3)$ in (3.14), we obtain

$$N\left(\frac{f((n-2)^{k+m}x)}{(n-2)^{k+m}}-\frac{f((n-2)^m x)}{(n-2)^m},t\right)\geq N'\left(\psi(x,x,\ldots,x),\frac{t}{\sum_{i=m}^{m+k-1}\frac{\alpha^i}{n(n-2)^{i+1}}}\right) \tag{3.15}$$

for all $x\in X$ and all $t>0$ and all $m,n\geq 0$.

Since $0<\alpha<n-2$ and $\sum_{i=0}^{k}\left(\frac{\alpha}{n-2}\right)^i<\infty$, the cauchy criterion for convergence and $(F5)$ gives that $\left\{\frac{f((n-2)^k x)}{(n-2)^k}\right\}$ is a Cauchy sequence in (Y,N). Since (Y,N) is a fuzzy Banach space, this sequence converges to some point $A(x)\in Y$. Define $A:X\to Y$ by

$$A(x)=N-\lim_{k\to\infty}\frac{f((n-2)^k x)}{(n-2)^k}$$

for all $x\in X$.

Taking $m=0$ in (3.15), we get

$$N\left(\frac{f((n-2)^k x)}{(n-2)^k}-f(x),t\right)\geq N'\left(\psi(x,x,\ldots,x),\frac{t}{\sum_{i=0}^{k-1}\frac{\alpha^i}{n(n-2)^{i+1}}}\right) \tag{3.16}$$

for all $x\in X$ and all $t>0$. Allowing $k\to\infty$ in (3.16) and using $(F6)$, we get

$$N\left(f(x)-A(x),t\right)\geq N'\left(\psi(x,x,\ldots,x),n(n-2-\alpha)t\right)$$

for all $x\in X$ and all $t>0$.

To prove A satisfies the functional equation (1.5), replacing $(x_1,\cdots,x_n)$ by $((n-2)^k x_1,(n-2)^k x_2,\cdots,(n-2)^k x_n)$ in (3.3) respectively, we get

$$\begin{aligned}N\left(\frac{1}{(n-2)^k}Df\left((n-2)^k x_1,(n-2)^k x_2,\cdots,(n-2)^k x_n\right),t\right)\\ \geq N'\left(\psi\left((n-2)^k x_1,(n-2)^k x_2,\cdots,(n-2)^k x_n\right),(n-2)^k t\right)\end{aligned} \tag{3.17}$$

for all $t>0$ and all $x_1,\cdots,x_n\in X$. Now,

$$N\left(\sum_{j=1}^{n} A(-x_j+\sum_{1\le i\le n, i\ne j} x_i)-(n-2)\sum_{j=1}^{n} A(x_j),t\right)$$
$$\ge min\left\{N\left(\sum_{j=1}^{n} A(-x_j+\sum_{1\le i\le n, i\ne j} x_i)-\frac{1}{(n-2)^k}\sum_{j=1}^{n} f(-x_j+\sum_{1\le i\le n, i\ne j} x_i),\frac{t}{3}\right),\right.$$
$$N\left(-(n-2)\sum_{j=1}^{n} A(x_j)+\frac{(n-2)}{(n-2)^k}\sum_{j=1}^{n} f(x_j),\frac{t}{3}\right),$$
$$\left.N\left(\frac{1}{(n-2)^k}\sum_{j=1}^{n} f(-x_j+\sum_{1\le i\le n, i\ne j} x_i)-\frac{(n-2)}{(n-2)^k}\sum_{j=1}^{n} f(x_j),\frac{r}{3}\right)\right\} \tag{3.18}$$

for all $x_1,\cdots,x_n\in X$ and all $t>0$. Using (3.17) and (3.18) we get

$$N\left(\sum_{j=1}^{n} A\left(-x_j+\sum_{1\le i\le n, i\ne j} x_i\right)-(n-2)\sum_{j=1}^{n} A(x_j),t\right)$$
$$\ge min\left\{1,1,N'\left(\psi\left((n-2)^k x_1,\cdots,(n-2)^k x_n\right),(n-2)^k t\right)\right\}$$
$$\ge N'\left(\psi\left((n-2)^k x_1,\cdots,(n-2)^k x_n\right),(n-2)^k t\right) \tag{3.19}$$

for all $x_1,\cdots,x_n\in X$ and all $t>0$. Letting $k\to\infty$ in (3.19) and using (3.2), we get

$$N\left(\sum_{j=1}^{n} A\left(-x_j+\sum_{1\le i\le n, i\ne j} x_i\right)-(n-2)\sum_{j=1}^{n} A(x_j),t\right)=1 \tag{3.20}$$

for all $x_1,\cdots,x_n\in X$ and all $t>0$. Using (F2) in (3.20),we obtain

$$\sum_{j=1}^{n} A\left(-x_j+\sum_{1\le i\le n, i\ne j} x_i\right)=(n-2)\sum_{j=1}^{n} A(x_j)$$

for all $x_1,\cdots,x_n\in X$.
Hence A satisfies the additive functional equation (1.5).
To prove the uniqueness of A, Suppose that there exists a another additive function $A':X\to Y$ satisfying (1.5) and (3.5). We have

$$N(A(x)-A'(x),t)=N\left(\frac{A((n-2)^k x)}{(n-2)^k}-\frac{A'((n-2)^k x)}{(n-2)^k},t\right)$$
$$\ge min\left\{N\left(\frac{A((n-2)^k x)}{(n-2)^k}-\frac{f((n-2)^k x)}{(n-2)^k},\frac{t}{2}\right),\right.$$
$$\left.N\left(\frac{f((n-2)^k x)}{(n-2)^k}-\frac{A'((n-2)^k x)}{(n-2)^k},\frac{t}{2}\right)\right\}$$
$$\ge N'\left(\psi((n-2)^k x,\ldots,(n-2)^k x),\frac{nt(n-2)^k((n-2)-\alpha)}{2}\right)$$
$$\ge N'\left(\psi(x,x,\ldots,x),\frac{nt(n-2)^k((n-2)-\alpha)}{2\alpha^k}\right)$$

for all $x\in X$ and all $t>0$.

Since
$$\lim_{k\to\infty}\frac{nt(n-2)^k((n-2)-\alpha)}{2\alpha^k}=\infty,$$
we obtain
$$\lim_{k\to\infty}N'\left(\psi(x,x,\ldots,x),\frac{nt(n-2)^k((n-2)-\alpha)}{2\alpha^k}\right)=1.$$
Thus
$$N(A(x)-A'(x),t)=1$$
for all $x\in X$ and all $t>0$. Hence $A(x)=A'(x)$. Therefore $A(x)$ is unique.

For $\beta=-1$, we can prove the stability result by a similar manner. This completes the proof of the theorem. □

The following corollary is an immediate outcome of Theorem 3.1 concerning the Hyers-Ulam-Rassias and JMRassias stabilities for the functional equation (1.5).

Corollary 3.1. *Suppose that a function $f:X\to Y$ satisfies the inequality*
$$N(Df(x_1,\cdots,x_n),t)\geq\begin{cases}N'(\varepsilon,t)\\ N'\left(\varepsilon\sum_{i=1}^n||x_i||^s,t\right), & s\neq 1;\\ N'\left(\varepsilon\prod_{i=1}^n||x_i||^s,t\right), & s\neq \frac{1}{n};\\ N'\left(\varepsilon\prod_{i=1}^n||x_i||^s+\sum_{i=1}^n||x_i||^{ns},t\right), & s\neq\frac{1}{n};\end{cases}\tag{3.21}$$
for all $t>0$ and all $x_1,\cdots,x_n\in X$, where ε,s are constants with $\varepsilon>0$. Then there exists a unique additive function $A:X\to Y$ such that
$$N(f(x)-A(x),t)\geq\begin{cases}N'(\varepsilon,n|n-3|t),\\ N'(n\varepsilon||x||^s,n|(n-2)-(n-2)^s|t),\\ N'(\varepsilon||x||^{ns},n|(n-2)-(n-2)^{ns}|t),\\ N'((1+n)\varepsilon||x||^{ns},n|(n-2)-(n-2)^{ns}|t),\end{cases}\tag{3.22}$$
for all $x\in X$ and all $t>0$.

4. MAIN RESULTS: FIXED POINT METHOD

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (1.5) in Fuzzy normed space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X,d) be a complete metric space and consider a mapping $T:X\to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx,Ty)\leq Ld(x,y)$ for some (Lipschitz constant) $L<1$. Then,

(i) The mapping T has one and only fixed point $x^*=T(x^*)$;

(ii) The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty}T^nx=x^*$, for any starting point $x\in X$;

(iii) One has the following estimation inequalities:

(A3) $d(T^nx,x^*)\leq\frac{1}{1-L}d(T^nx,T^{n+1}x),\forall\ n\geq 0,\forall\ x\in X$;

(A4) $d(x,x^*)\leq\frac{1}{1-L}d(x,x^*),\forall\ x\in X$.

Theorem 4.2. *[24] (The alternative of fixed point) Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\, n \geq 0,$$

or

$(B2)$ *there exists a natural number n_0 such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$ *;*

(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L}\, d(y, Ty)$ *for all* $y \in Y$.

For to prove the stability result we define the following:
μ_i is a constant such that

$$\mu_i = \begin{cases} n-2 & if \quad i = 0, \\ \dfrac{1}{n-2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{p \mid p : X \to Y, p(0) = 0\}.$$

Theorem 4.3. *Let $f : X \to Y$ be a mapping for which there exist a function $\psi : X^n \to Z$ with*

$$\lim_{k\to\infty} N'\left(\psi\left(\mu_i^k x_1, \mu_i^k x_2, \cdots, \mu_i^k x_n\right), \mu_i^k t\right) = 1 \;\; \forall\, x_1, x_2, \cdots, x_n \in X,\; t > 0 \tag{4.1}$$

and satisfies

$$N(D\,f(x_1,x_2,\cdots,x_n),t) \geq N'(\psi(x_1,x_2,\cdots,x_n),t)\; \forall x_1,x_2,\cdots,x_n \in X,\; t > 0. \tag{4.2}$$

If there exists $L = L(i)$ such that the function

$$x \to \alpha(x) = \frac{1}{n}\psi\left(\frac{x}{n-2}, \frac{x}{n-2}, \cdots, \frac{x}{n-2}\right),$$

has the property

$$N'\left(L\frac{1}{\mu_i}\alpha(\mu_i x), t\right) = N'\left(\alpha(x), t\right),\; \forall\, x \in X,\; t > 0. \tag{4.3}$$

Then there exists unique additive mapping $A : X \to Y$ satisfying (1.5) and

$$N\left(f(x) - A(x), t\right) \geq N'\left(\alpha(x)\frac{L^{1-i}}{1-L}, t\right),\; \forall\, x \in X,\; t > 0. \tag{4.4}$$

Proof. Let d be a general metric on Ω, such that

$$d(p,q) = inf\left\{K \in (0,\infty) | N\left(p(x) - q(x), t\right) \geq N'\left(K\alpha(x), t\right), x \in X,\; t > 0\right\}.$$

It is easy to verify that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tp(x) = \dfrac{1}{\mu_i} p(\mu_i x)$, for all $x \in X$. For $p, q \in \Omega$, we have $d(p,q) \leq K$

$$\begin{array}{rrcl} \Rightarrow & N\left(p(x)-q(x),t\right) & \geq & N'\left(\alpha(x),Kt\right) \\ \Rightarrow & N\left(\frac{p(\mu_i x)}{\mu_i}-\frac{q(\mu_i x)}{\mu_i},t\right) & \geq & N'\left(\alpha(\mu_i x),K\mu_i t\right) \\ \Rightarrow & N\left(Tp(x)-Tq(x),t\right) & \geq & N'\left(\alpha(x),KLt\right) \\ \Rightarrow & d\left(Tp(x),Tq(x)\right) & \leq & KL \\ \Rightarrow & d\left(Tp,Tq\right) & \leq & Ld(p,q) \end{array}$$

for all $p,q \in \Omega$. There fore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing $(x_1,x_2,x_3,\cdots,x_n)$ by $(x,x,\cdots,x)$ in (4.2), we get

$$N\left(nf((n-2)x)-n(n-2)f(x),t\right) \geq N'\left(\psi(x,x,\cdots,x),t\right). \tag{4.5}$$

for all $x \in X, t>0$. Using (F3) in (4.5), we get

$$N\left(\frac{f((n-2)x)}{(n-2)}-f(x),t\right) \geq N'\left(\frac{1}{n(n-2)}\psi(x,x,\cdots,x),t\right) \tag{4.6}$$

for all $x \in X, t>0$ with the help of (4.3) when $i=0$ and it follows from (4.6), we arrive

$$\begin{aligned} &\Rightarrow N\left(\frac{f((n-2)x)}{(n-2)}-f(x),t\right) \geq N'\left(\alpha(x),Lt\right) \\ &\Rightarrow d(Tf,f) \leq L = L^1 = L^{1-i} \end{aligned} \tag{4.7}$$

Replacing x by $\frac{x}{(n-2)}$ in (4.5), we get

$$N\left(f(x)-(n-2)f\left(\frac{x}{(n-2)}\right),t\right) \geq N'\left(\psi\left(\frac{1}{n}\frac{x}{(n-2)},\frac{x}{(n-2)},\cdots,\frac{x}{(n-2)}\right),t\right) \tag{4.8}$$

for all $x \in X, t>0$ using (4.3) when $i=1$, it follows from (4.8) we obtain

$$\begin{aligned} &\Rightarrow N\left(f(x)-(n-2)f\left(\frac{x}{(n-2)}\right),t\right) \geq N'\left(\alpha(x),t\right) \\ &\Rightarrow d(f,Tf) \leq 1 = L^0 = L^{1-i} \end{aligned} \tag{4.9}$$

Then from (4.7) and (4.9) we can conclude,

$$d(f,Tf) \leq L^{1-i} < \infty$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = N-\lim_{k\to\infty}\frac{f((n-2)^k x)}{(n-2)^k}, \qquad \forall x \in X, t>0. \tag{4.10}$$

Replacing $(x_1,x_2,\cdots,x_n)$ by $(\mu_i x_1,\mu_i x_2,\cdots,\mu_i x_n)$ in (4.2), we arrive

$$N\left(\frac{1}{\mu_i^k}Df(\mu_i x_1,\mu_i x_2,\cdots,\mu_i x_n),t\right) \geq N'\left(\psi(\mu_i x_1,\mu_i x_2,\cdots,\mu_i x_n),\mu_i^k t\right) \tag{4.11}$$

for all $t>0$ and all $x_1,x_2,\cdots,x_n \in X$

By following the same manner as in the Theorem 3.1, we can prove the function, $A : X \to Y$ satisfies (1.5).

By fixed point alternative, since A is unique fixed point of T in the set

$$\Delta = \{ f \in \Omega | d(f,A) < \infty \},$$

therefore A is a uniqe function such that

$$N(f(x) - A(x),t) \geq N'(\alpha(x),Kt) \tag{4.12}$$

for all $x \in X, t > 0$ and $K > 0$. Again using the fixed point alternative, we get

$$\begin{aligned} d(f,A) &\leq \frac{1}{1-L} d(f,Tf) \\ \Rightarrow d(f,A) &\leq \frac{L^{1-i}}{1-L} \\ \Rightarrow N(f(x) - A(x),t) &\geq N'\left(\alpha(x)\frac{L^{1-i}}{1-L},t\right), \end{aligned} \tag{4.13}$$

for all $x \in X$ and $t > 0$. This completes the proof of the theorem. □

From Theorem 4.3, we obtain the following corollary concerning the stability for the functional equation (1.5).

Corollary 4.1. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N(D\, f(x_1,x_2,\cdots,x_n),t) \geq \begin{cases} N'(\varepsilon,t) & \\ N'(\varepsilon \sum_{i=1}^{n} ||x_i||^s,t), & s \neq 1; \\ N'(\varepsilon \prod_{i=1}^{n} ||x_i||^s,t), & s \neq \frac{1}{n}; \\ N'(\varepsilon \prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns},t), & s \neq \frac{1}{n}; \end{cases} \tag{4.14}$$

for all $x_1,x_2,\cdots,x_n \in X$ and $t > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x),t) \geq \begin{cases} N'(\varepsilon, n|n-3|t), \\ N'(n\varepsilon||x||^s, n|(n-2)-(n-2)^s|t), \\ N'(\varepsilon||x||^{ns}, n|(n-2)-(n-2)^{ns}|t), \\ N'((n-1)\varepsilon||x||^{ns}, n|(n-2)-(n-2)^{ns}|t), \end{cases} \tag{4.15}$$

for all $x \in X$ and all $t > 0$.

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