SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.12.3.12
Vol.12 (25), No.2, (2016), 401–417, Suppl.

GRADED STRUCTURES AND DIFFERENTIAL OPERATORS ON NEARLY HOLOMORPHIC AND QUASIMODULAR FORMS ON CLASSICAL GROUPS

ALEXEI PANCHISHKIN

Dedicated to the memory of Professor Marc Krasner

Abstract. We wish to use Krasner graded [20] and Krasner-Vuković paragraded structures [22], [34] on differential operators and quasimodular forms on classical groups and show that these structures provide a tool to construct p-adic measures and p-adic L-functions on the corresponding non-archimedean weight spaces.

An approach to constructions of automorphic L-functions on unitary groups and their p-adic analogues is presented. For an algebraic group G over a number field K these L functions are certain Euler products $L(s, \pi, r, \chi)$.

We present a method using arithmetic nearly-holomorphic forms and general quasi-modular forms, related to algebraic automorphic forms. It gives a technique of constructing p-adic zeta-functions via quasi-modular forms and their Fourier coefficients.

Introduction

Let p be a prime number. Our purpose is to indicate a link of Krasner graded [20] and Krasner-Vuković paragraded structures [22], [34] and constructions of p-adic L-functions via distributions and quasi-modular forms on classical groups.

2010 *Mathematics Subject Classification.* 11F67, 11F85, 11F33 (See also 14G20, 22E50), 16W50, 16E45.

Key words and phrases. Graded structures, automorphic forms, classical groups, p-adic L-functions, differential operators, non-archimedean weight spaces, quasi-modular forms, Fourier coefficients.

This paper was presented at the International Scientific Conference *Graded structures in algebra and their applications*, dedicated to the memory of Prof. Marc Krasner, IUC-Dubrovnik, Cratia, September, 22-24, 2016.

Krasner's graded structures are flexible and well adapted to various applications, e.g. the rings and modules of differential operators on classical groups and non-archimedean weight spaces.

Nearly holomorphic modular forms (in the sense of Shimura) arise naturally when one wants to study L-values for arithmetic automorphic forms by using some kind of Rankin-Selberg method (i.e. using an integral representation involving some Eisenstein series; this includes the doubling method). The constant terms of such nearly holomorphic modular forms are called quasi-modular forms. Typically these quasi-modular forms are simpler, their Fourier expansion is easier to understand (but they do not satisfy simple transformation properties). The aim of the paper is to describe, how one can use the quasi-modular forms for p-adic interpolation, using recent results of Ichikawa [13], [14].

The essential part of the paper is in Section 3, the other sections describe the general background.

1. Automorphic L-functions and their p-adic analogues

Our main objects in this paper are automorphic L-functions and their p-adic analogues.

1.1. Krasner grade components for proving Kummer-type congruences for L and zeta-values.

Graded groups in the sense of Krasner.

Definition 1.1. Let G be a multiplicative group with the neutral element 1. A graduation of G is a mapping $\gamma : \Delta \to Sg(G), \delta \in \Delta$ of a set Δ "the set of grades of γ ") to the set $Sg(G)$ of subgroups of G such that G is the direct decomposition

$$G = \bigoplus_{\delta\in\Delta} G_\delta, g = (g_\delta)_{\delta\in\Delta}$$

The group G endowed with such graduation is called graded group, and g_δ are Krasner grade components of g with grade δ.

Graded rings and modules.

Moreover, M. Krasner introduced useful notions of graded rings and modules.

Let $(A; x + y, xy)$ be a ring (not necessarily associative) and let $\gamma : \Delta \to Sg(A; x + y)$ be a graduation of its addivive group. The graduation γ is called a graduation of a ring $(A; x + y, xy) = A$ if, for all $\xi, \eta \in \Delta$ there exists a $\zeta \in \Delta$ such that $A_\xi A_\eta \subset A_\zeta$.

1.2. **Examples for p-adic groups $\mathcal{X}$, and group rings.** Use the Tate field $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$, the completion of $\bar{\mathbb{Q}}_p$, which is a fundamental object in p-adic analysis, and thanks to Krasner we know that $\mathbb{C}_p$ is algebraically closed, see [1], Thorme 2.7.1 ("Lemme de Krasner"), [19]. This famous result allows to develop analytic functions and analytic spaces over $\mathbb{C}_p$ ([19], Tate, Berkovich...). and we embed $incl_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

1) Algebraically, a p-adic measure μ on $\mathcal{X}$ is an element of the completed group ring $A[[\mathcal{X}]]$, A any p-adic subring of $\mathbb{C}_p$.

2) The p-adic L-function of μ is given by the evaluation $\mathcal{L}_\mu(y) := y(\mu)$ on the group $\mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*)$ of $\mathbb{C}_p^*$-valued characters of $\mathcal{X}$. *The values $\mathcal{L}_\mu(y_j)$ on algebraic characters $y_j \in \mathcal{Y}^{alg}$ determine $\mathcal{L}_\mu$ iff they satisfy Kummer-type congruences.*

3) Our setting: a p-adic torus $T = \mathcal{X}$ of a unitary group G attached to a CM field K over $\mathbb{Q}$, a quadratic extension of a totally real field F, and an n-dimensional hermitian K-vector space V. Elements of $\mathcal{Y}^{alg}$ are identified with some algebraic characters of the torus T of the unitary group.

1.3. **An extension problem.** From a subset $J = \mathcal{Y}^{alg}$ of classical weights in $\mathcal{Y} = \mathrm{Hom}_{cont}(\mathcal{X}, \mathbb{C}_p^*)$, via the A-module $\mathcal{QM}$ of quasimodular forms, we wish to extend continuously a given mapping $\mathcal{L}$ to the group ring $A[\mathcal{Y}]$:

$$\mathcal{L} : \mathcal{Y}^{alg} \longrightarrow \begin{matrix} \curvearrowright \mathcal{D}_A \\ \mathcal{QM} \\ \circlearrowleft \mathcal{H}_A \end{matrix} \longrightarrow \mathbb{C}_p,\ y_j \overset{\ell}{\mapsto} \mathcal{L}(y_j), y_j \in \mathcal{Y}^{alg}$$

(where $\mathcal{H}_A$ a Hecke algebra, $\mathcal{D}_A$ a ring of differential operators over A) in such a way that the values $\mathcal{L}(y_j)$ on $y_j \in \mathcal{Y}^{alg}$ are given by certain algebraic L-values under the embedding $incl_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$.

In the favorable (ordinary) case one extends $\mathcal{L}$ to all continuous functions $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$, or just to locally-analytic functions $\mathcal{C}^{loc-an}(\mathcal{X}, \mathbb{C}_p)$ (in the admissible case).

Advantages of the A-module $\mathcal{QM}$:

1) simpler Fourier expansions (q-expansions);

2) action of D and of the ring of differential operators $\mathcal{D}_A = A[D]$

3) action of the Hecke algebra $\mathcal{H}_A$;

4) projection $\pi_\alpha : \mathcal{QM}^\alpha \to \mathcal{QM}^\alpha$ to finite rank component ("generalized eigenvectors of Atkin's U-operator") for any non-zero Hecke eigenvalue α of level p ; ℓ goes through $\mathcal{QM}^\alpha$ if $U^*(\ell) = \alpha^*\ell$.

Solution (extension of $\mathcal{L}$ to $\mathcal{C}(\mathcal{X}, \mathbb{C}_p)$) is given in the ordinary case by the abstract Kummer-type congruences:

$$\forall x \in \mathcal{X},\ \sum_j \beta_j y_j(x) \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \ell(y_j) \equiv 0 \mod p^N (\beta_j \in A).$$

Such congruences imply the *p-adic analytic continuation of the Riemann zeta function.*

For more general L-function $L(f,s)$, of an automorphic form f one can prove certain Kummer-type congruences using various Krasner grade components, with respect to *weights, Hecke-Dirichlet characters, eigenvalues of Hecke operators acting on spaces automorphic forms (including Atkin-type U_p-operators), and the classical Fourier coefficients of quasi-modular forms.*

It turns out that certain critical L-values $L(f,s)$, expressed through Petersson-type product $\langle f, g_s\rangle$, reduces to $\langle \pi_\alpha(f), g_s\rangle$, where g_s is an explicit arithmetical automorphic form, $\alpha \neq 0$ is a eigenvalue attached to f, and $\pi_\alpha(f)$ is the component given by the α-characteristic projection, known to be in a fixed finite dimensional space (known for Siegel modular case, and extends to the unitary case).

For an algebraic group G over a number field K these L functions are defined as certain Euler products. More precisely, we apply our constructions to the L-functions studied in Shimura's book [31].

1.4. **Constructions of p-adic analogues.** In the general case of an irreducible automorphic representation of the adelic group $G(\mathbb{A}_K)$ there is an L-function

$$L(s,\pi,r,\chi) = \prod_{\mathfrak{p}_v\, primes\, in\, K}\prod_{j=1}^{m}(1-\beta_{j,\mathfrak{p}_v}N\mathfrak{p}_v^{-s})^{-1}$$

where

$$\prod_{j=1}^{m}(1-\beta_{j,\mathfrak{p}}X) = \det(1_m - r(\mathrm{diag}(\alpha_{i,\mathfrak{p}})_i X)),$$

$\alpha_{i,\mathfrak{p}}$ are the Satake parameters of $\pi = \bigotimes_v \pi_v$ $v \in \Sigma_K$ (places in K),$\mathfrak{p} = \mathfrak{p}_v$. Here $h_v = \mathrm{diag}(\alpha_{i,\mathfrak{p}})_i$ live in the Langlands group ${}^LG(\mathbb{C})$, $r : {}^LG(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ denotes a finite dimensional representation, and $\chi : \mathbb{A}_K^*/K^* \to \mathbb{C}^*$ is a character of finite order. Constructions admit extention to rather general automorphic representations on Shimura varieties via the following tools:

- Modular symbols and their higher analogues (linear forms on cohomology spaces related to automorphic forms)
- Petersson products with a fixed automorphic form, or
- linear forms coming from the Fourier coefficients (or Whittaker functions), or throught the
- CM-values (special points on Shimura varieties),

2. Automorphic L-functions attached to symplectic and unitary groups

Let us briefly describe the L-functions attached to symplectic and unitary groups as certain Euler products in Chapter 5 of [31], with critical values computed in Chapter 7, Theorem 28.8 using general nearly holomorphic arithmetical automorphic forms for the group

$$G = G(\varphi) = \{\alpha \in \mathrm{GL}_m(K) \mid \alpha\varphi^t\alpha^\rho = \nu(\alpha)\varphi\}, \nu(\alpha) \in F^*,$$

where $\varphi = \eta_n = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$ or $\varphi = \left(\begin{smallmatrix} 1_n & 0 \\ 0 & 1_m \end{smallmatrix}\right)$, see also Ch.Skinner and E.Urban [32] and Shimura G., [31].

2.1. The groups and automorphic forms studied by Shimura in [31]. Let F be a totally real algebraic number field, K be a totally imaginary quadratic extension of F and ρ be the generator of $\mathrm{Gal}(K/F)$. Take $\eta_n = \left(\begin{smallmatrix} 0 & 1_n \\ -1_n & 0 \end{smallmatrix}\right)$ and define

$$\begin{aligned} &G = \mathrm{Sp}(n, F) && \text{(Case Sp)} \\ &G = \{\alpha \in \mathrm{GL}_{2n}(K) | \alpha\eta_n\alpha^* = \eta_n\} && \text{(Case UT = unitary tube)} \\ &G = \{\alpha \in \mathrm{GL}_{2n}(K) | \alpha T\alpha^* = T\} && \text{(Case UB = unitary ball)} \end{aligned}$$

according to three cases. Assume $F = \mathbb{Q}$ for a while. The group of the real points G_∞ acts on the associated domain

$$\mathcal{H} = \begin{cases} \{z \in M(n,n,\mathbb{C}) \mid {}^tz = z, \mathrm{Im}\,(z) > 0\} & \text{(Case Sp)} \\ \{z \in M(n,n,\mathbb{C}) \mid i(z^* - z) > 0\} & \text{(Case UT)} \\ \{z \in M(p,q,\mathbb{C}) \mid 1_q - z^*z > 0\} & \text{(Case UB),} \end{cases}$$

with $(p,q), p+q = n$, being the signature of iT. Here $z^* = {}^t\bar{z}$ and ¿ means that a hermitian matrix is positive definite. In Case UB, there is the standard automorphic factor $M(g,z)$, $g \in G_\infty$, $z \in \mathcal{H}$ taking values in $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C})$.

2.2. Shimura's arithmeticity in the theory of automorphic forms [31], p-adic zeta functions and nearly-holomorphic forms on classical groups. *Automorphic L-functions via general quasi-modular forms.* Automorphic L-functions ans their p-adic versions can be obtained for quite general automorphic representations on Shimura varieties by constructing p-adic distributions out of algebraic numbers attached to automorphic forms. These numbers satisfy certain Kummer-type congruences established in different ways component-wise for various Krasner-type components.

In order to describe both algebraicity and congruences of the critical values of the zeta functions of automorphic forms on unitary and symplectic groups, we follow the review by H.Yoshida [35] of Shimura's book "Arithmeticity in the theory of automorphic forms" [31].

Also unitary Shimura varieties have recently attracted much interest (in particular by Ch. Skinner and E. Urban), see [32], in relation with the proof of the The Iwasawa Main Conjecture for GL(2).

2.3. **Integral representations and critical values of the zeta functions.** Automorphic forms are assumed scalar valued in this part. For Cases Sp and UT, Eisenstein series $E(z, s)$ associated to the maximal parabolic subgroup of G of Siegel type is introduced. Its analytic behaviour and those values of $\sigma \in 2^{-1}\mathbb{Z}$ at which $E(z, \sigma)$ is nearly holomorphic and arithmetic, are studied in [31]. This is achieved by proving a relation giving passage from s to $s - 1$ for $E(z, s)$, involving a differential operator, then examining Fourier coefficients of Eisenstein series using the theory of confluent hypergeometric functions on tube domains.

For a Hecke eigenform f on $G_{\mathbb{A}}$ and an algebraic Hecke character χ on the idele group of K (in Case Sp, $K = F$), the zeta function $\mathcal{Z}(s, f, \chi)$ is defined. Viewing it as an Euler product extended over prime ideals of F, the degree of the Euler factor is $2n + 1$ in Case Sp, $4n$ in Case UT, and $2n$ in Case UB, except for finitely many prime ideals, see Chapter 5 of [31].

This zeta function is almost the same as the so called standard L-function attached to f twisted by χ but it turns out to be more general in the unitary case, see also [10].

Main results on critical values of the L-functions studied in Shimura's book [31] is stated in Theorem 28.5, 28.8 (Cases Sp, UT), and in Theorem 29.5 in Case UB.

Theorem 2.1 (algebraicity of critical values in Cases Sp and UT)**.**
Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ *be a non zero arithmetical automorphic form of type Sp or UT. Let* χ *be a Hecke character of* K *such that* $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$ *with* $\ell \in \mathbb{Z}^{\mathbf{a}}$*, and let* $\sigma_0 \in 2^{-1}\mathbb{Z}$*. Assume, in the notations of Chapter* 7 *of* [31] *on the weights* k_v, μ_v, ℓ_v*, that*

$$\begin{array}{ll}
\text{Case Sp} & 2n+1-k_v+\mu_v \le 2\sigma_0 \le k_v-\mu_v, \\
& \textit{where } \mu_v = 0 \textit{ if } [k_v]-l_v \in 2\mathbb{Z} \\
& \textit{and } \mu_v = 1 \textit{ if } [k_v]-l_v \notin 2\mathbb{Z};\ \sigma_0 - k_v + \mu_v \\
& \textit{for every } v \in \mathbf{a} \textit{ if } \sigma_0 > n \textit{ and} \\
& \sigma_0 - 1 - k_v + \mu_v \in 2\mathbb{Z} \textit{ for every } v \in \mathbf{a} \textit{ if } \sigma_0 \le n. \\
\text{Case UT} & 4n - (2k_{v\rho}+\ell_v) \le 2\sigma_0 \le m_v - |k_v - k_{v\rho} - \ell_v| \\
& \textit{and } 2\sigma_0 - \ell_v \in 2\mathbb{Z} \textit{ for every } v \in \mathbf{a}.
\end{array}$$

Further exclude the following cases

(A) Case Sp $\sigma_0 = n+1, F = \mathbb{Q}$ *and* $\chi^2 = 1$;

(B) Case Sp $\sigma_0 = n + (3/2), F = \mathbb{Q}; \chi^2 = 1$ *and* $[k] - \ell \in 2\mathbb{Z}$

(C) Case Sp $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ *and* $\chi = 1$;

(D) Case Sp $0 < \sigma_0 \leq n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ *and the conductor of* χ *is* $\mathfrak{g}$;

(E) Case UT $2\sigma_0 = 2n+1, F = \mathbb{Q}, \chi_1 = \theta,$ *and* $k_v - k_{v\rho} = \ell_v$;

(F) Case UT $0 \leq 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ *and the conductor of* χ *is* $\mathfrak{r}$

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi)/\langle \mathbf{f}, \mathbf{f} \rangle \in \pi^{n|m|+d\varepsilon}\bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathbf{a}} m_v$, *and*

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \textit{Case Sp}, k \in \mathbb{Z}^{\mathbf{a}}, \ \textit{and } \sigma_0 > n_0), \\ n\sigma_0 - n^2, & \textit{Case Sp}, k \notin \mathbb{Z}^{\mathbf{a}}, \textit{or} \sigma_0 \leq n_0), \\ 2n\sigma_0 - 2n^2 + n & \textit{Case UT} \end{cases}$$

Notice that $\pi^{n|m|+d\varepsilon} \in \mathbb{Z}$ in all cases; if $k \notin \mathbb{Z}^{\mathbf{a}}$, the above parity condition on σ_0 shows that $\sigma_0 + k_v \in \mathbb{Z}$, so that $n|m| + d\varepsilon \in \mathbb{Z}$.

Remark of the referee: Some of the cases quoted from Shimura are concerned with half-integral weight, the metaplectic case however does not appear in the present paper.

We establish a p-adic analogue of Theorem 28.8 (in Cases Sp and UT) representing algebraic parts of critical values as values of certain p-adic analytic zeta functions.

3. Constructing p-adic zeta-functios via quasi-modular forms

We present here a new method of constructing p-adic zeta-functios based on the use of general quasi-modular forms on classical groups.

This method uses only algebraic numbers coming from holomorphic and nearly holomorphic modular forms, and quasi-modular forms.

A new method of constructing p-adic zeta-functios uses general quasi-modular forms and their Fourier coefficients. The symmetric space

$$\mathcal{H} = G(\mathbb{R})/((\text{maximal-compact subgroup})\mathcal{K} \times \mathit{Center})$$

parametrizes certain families of abelian varieties A_z ($z \in \mathcal{H}$) so that $F \subset \mathrm{End}(A_z) \otimes \mathbb{Q}$. The CM-points z correspond to a maximal multiplication ring $\mathrm{End}(A_z)$.

For the group GL(2), N.Katz [15] used arithmetical elements (real-analytic and p-adic) instead of holomorphic forms. These elements correspond also to quasi-modular forms coming from derivatives which can be defined in general using Shimura's arithmeticity and the Maass-Shimura operators. A relation real-analytic $\leftrightarrow$ p-adic modular forms comes from the notion of p-adic modular forms invented by J.-P.Serre [30] as p-adic limits of q-expansions of modular forms with rational coefficients for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The present method of constructing p-adic automorphic L-functions uses general quasi-modular forms, and their link to algebraic p-adic modular forms.

Real-analytic and p-adic modular forms. In Serre's case for $\Gamma{=}SL_2(\mathbb{Z})$, the ring M_p of p-adic modular forms contains $\mathcal{M}=\oplus_{k\geq 0}\mathcal{M}_k(\Gamma,\mathbb{Z})=\mathbb{Z}[E_4,E_6]$, and it contains $E_2 = 1-24\sum_{n\geq 1}\sigma_1(n)q^n$ as element with the q-expansion. On the other hand,

$$\tilde{E}_2 = -\frac{3}{\pi y} + E_2 = -12S + E_2, \text{ where } S = \frac{1}{4\pi y},$$

is a nearly holomorphic modular form (its coefficients are polynomials of S over $\mathbb{Q}$). Let $\mathcal{N}$ be the ring of such forms. Then $\tilde{E}_2|_{S=0} = E_2$ and it was proved by J.-P. Serre that E_2 is a p-adic modular form. Elements of the ring $QMc = \mathcal{N}|_{S=0} = 0$ will be called general quasi-modular forms. These phenomena are quite general and can be used in computations and proofs. In 2014, S. Böcherer extended these results to the Siegel modular case. A preprint by S. Böcherer is available (for proceedings of this conference).

3.1. **Injecting nearly-holomorphic forms into p-adic modular forms.** A recent discovery by Takashi Ichikawa (Saga University), [13], [14], allows to inject nearly-holomorphic arithmetical (vector valued) Siegel modular forms into p-adic modular forms. Via the Fourier expansions, the image of this injection is represented by certain quasi-modular holomorphic forms like $E_2 = 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n$, with algebraic Fourier expansions. This description provides many advantages, both computational and theoretical, in the study of algebraic parts of Petersson products and L-values, which we would like to develop here. In fact, the realization of nearly holomorphic forms as p-adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [33], Chapter 10.

Urban only treats the elliptic modular case in that paper, but I believe he and Skinner are working on applications of a more general theory. This work is related to a recent preprint [4] by S. Böcherer and Shoyu Nagaoka where it is shown that Siegel modular forms of level $\Gamma_0(p^m)$ are p-adic modular forms. Moreover they show that derivatives of such Siegel modular forms are p-adic. Parts of these results are also valid for vector-valued modular forms.

3.2. **Arithmetical nearly-holomorphic Siegel modular forms.** Nearly-holomorphic Siegel modular forms over a subfield k of $\mathbb{C}$ are certain $\mathbb{C}^d$

-valued smooth functions f of $Z = X + \sqrt{-1}Y \in \mathbb{H}_n$ given by the following expression $f(Z) = \sum_T P_T(S)q^T$ where T runs through the set B_n of all half-integral semi-positive matricies, $S = (4\pi Y)^{-1}$ a symmetric matrix, $q^T = \exp(2\pi\sqrt{-1}\mathrm{tr}(TZ)), P_T(S)$ are vectors of degree d whose entries are polynomials of a degree uniformly bounded by $r = r(f)$ of the entries of S with coefficients over k.

In other words, $f(Z) = P_f(S)$ is a vector whore entries are polynomials over k of a degree $r \le r(f)$ of the entries of S, with vectors coefficients in the ring $(k[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}, q_{ij}^{\pm 1}]_{i,j=1,\ldots,n})^{d\ *}$.

3.3. **Algebraic Fourier expansion.** can be defined algebraically using an algebraic test object over the ring $\mathcal{R}_n = \mathbb{Z}[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}, q_{ij}^{\pm 1}]_{i,j=1,\ldots n}$; where q_{ij} $(1 \le i, j \le n)$ are variables with symmetry $q_{ij} = q_{ji}$. Mumford constructs in [Mu72] an object represented over $\mathcal{R}_n$ as

$$(\mathcal{R}_{n,N} \otimes (\mathbb{G}_m)^n / \langle (q_{ij})_{i=1,\ldots,n} \ | 1 \le j \le n \rangle,$$
$$(\mathbb{G}_m)^n = \mathrm{Spec}(\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$$

For the level N, at each 0-dimensional cusp c on $\mathcal{H}^*_{n,N}$ (Satake's minimal compactification of $\mathcal{H}_{n,N}$), this construction gives an abelian variety over the formal power series ring

$$\mathcal{R}_{n,N} = \ Z[1/N, \zeta_N][\![q_{11}^{1/N}, \ldots, q_{nn}^{1/N}]\!][q_{ij}^{\pm 1}]_{i,j=1,\ldots,n}$$

with a symplectic level N structure, and $\omega_i = dx_i/x_i (1 \le i \le n)$ form a basis of regular 1-forms. We may view algebraically Siegel modular forms as certain sections of vector bundles over $\mathcal{H}_{n,N}$. Using the morphism $\mathrm{Spec}(\mathcal{R}_{n,N}) \to \mathcal{H}_{n,N}$, $\mathbb{E}$ becomes $(\mathcal{R}_{n,N} \otimes R)^n$ in the basis $\omega_i = dx_i/x_i, (1 \le i \le n)$ of regular 1-forms.

3.4. **Fourier expansion map and q-expansion principle.** For an algebraic representation $\rho : \mathrm{GL}_n \to \mathrm{GL}_d$, $\mathbb{E}_\rho$ becomes in the above basis ω_i

$$\mathbb{E}_\rho \times_{\mathcal{H}_{n,N} \otimes R} \mathrm{Spec}(\mathcal{R}_{n,N} \otimes R) = (\mathcal{R}_{n,N} \otimes R)^d.$$

For an R-module M, the space of Siegel modular forms with coeffcients in M of weight ρ is defined as

$$\mathcal{M}_\rho(M) = H^0(\mathcal{H}_{n,N} \otimes R, \mathbb{E}_\rho \otimes_R M).$$

Then the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : \mathcal{M}_\rho(M) \to (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N, \zeta_N]} M)^d$$

which is called the Fourier expansion map associated with c. According to [14], Theorem 2, F_c satisfies the following **q-expansion principle**:

*thanks to referee's remark!

If M' is a sub R-module of M and $f \in \mathcal{M}_\rho(M)$ satisfies that

$$F_c(f) \in (\mathcal{R}_{n,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M')^d,$$

then $f \in \mathcal{M}_\rho(M')$.

For the q-expansion principle in the unitary case, see [9], [10].

3.5. **Algebraic nearly holomorphic forms as formal Fourier expansions over a commutative ring A.** Algebraically we use the notation

$$q^T = \prod_{i=1}^n q_{ii}^{T_{ii}} \prod_{i<j} q_{ij}^{2T_{ij}} \in A[\![q_{11}, \ldots, q_{nn}]\!][q_{ij}, q_{ij}^{\pm 1}]_{i,j=1,\ldots,n}$$

(with $q^T = \exp(2\pi i \mathrm{tr}(TZ)), q_{ij} = \exp(2\pi(\sqrt{-1}Z_{ij}))$ for $A = \mathbb{C}$). The elements q^T form a multiplicative semi-group so that $q^{T_1} \cdot q^{T_2} = q^{T_1+T_2}$, and one may consider f as a formal q-expansion over an arbitrary ring A via elements of the semi-group algebra $A[\![q^{B_n}]\!]$.

Algebraic definition of arithmetical nearly holomorphic forms, see [31] $f \in S_e(\mathrm{Sym}^2(A^n), A[\![q^{B_n}]\!]^d)$, where S_e denotes the A-polynomial mappings of degree e on symmetric matricies $S \in \mathrm{Sym}^2(A^n)$ of order n with vector values in $A[\![q_{B_n}]\!]^d$.

Notation: $f = \sum_T a_T(S) q^T \in \mathcal{N}(A)$.

General quasi-modular forms. For all $f = \sum_T a_T(S) q^T \in \mathcal{N}(A)$ define general quasi-modular forms as elements of the form

$$\kappa(f) = \sum_T a_T(0) q^T = f|_{S=0}.$$

Notation: $\kappa(f) \in \mathcal{QM}(A)$.

3.6. **Computing the Petersson products.** The Petersson product of a given modular form $f(Z) = \sum_T a_T q^T \in \mathcal{M}_\rho(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_T a_T q^T \in \mathcal{M}_{\rho^*}(\bar{\mathbb{Q}})$ produces a linear form

$$\ell_f : h \mapsto \frac{\langle f, h\rangle}{\langle f, f\rangle}$$

defined over a subfield $k \subset \mathbb{Q}$. Thus ℓ_f can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients: $\ell_{T_i} : h \mapsto b_{T_i} (i = 1, \ldots, n)$. It follows that $\ell_f(h) = \sum_i \gamma_i b_{T_i}$, where $\gamma_i \in k$.

We show in 3.9 below that this construction extends for two *quasi-modular forms*

$$f(Z) = \kappa\left(\sum_T a_T(S)q^T\right) \in \mathcal{QM}_\rho(\bar{\mathbb{Q}}) \text{ and}$$

$$h(Z) = \kappa\left(\sum_T b_T q^T(S)\right) \in \mathcal{QM}_{\rho^*}(\bar{\mathbb{Q}}),$$

when the Petersson product $\langle f, h\rangle_{\mathcal{QM}}$ is defined as $\langle \tilde{f}, \tilde{h}\rangle$, where

$$\tilde{f} = \sum_T a_T(S)q^T, \quad \tilde{h} = \sum_T b_T(S)q^T, f = \sum_T a_T(0)q^T, h = \sum_T b_T(0)q^T.$$

Thanks to the Ichikawa's injection $\tilde{f} \mapsto f$, the linear form $\ell_f : h \mapsto \dfrac{\langle f, h\rangle}{\langle f, f\rangle}$ is again defined over a subfield $k \subset \mathbb{Q}$ and can be expressed through the Fourier coefficients $b_T = b_T(0)$ of h under appropriate assumptions of finiteness of dimensions.

For the adelic automorphic forms, a semi-adelic setting as in [29] can be used component-wise.

3.7. **Applications to constructions of p-adic L-functions.** We recall here some methods of construction of p-adic L-functions.

There exist two kinds of L-functions

- Complex L-functions (Euler products) on $\mathbb{C} = \mathrm{Hom}(\mathbb{R}_+^*; \mathbb{C}^*)$.
- p-adic L-functions on the $\mathbb{C}_p$-analytic group $\mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ (Mellin transforms $\mathcal{L}_\mu$ of p-adic measures μ on $\mathbb{Z}_p^*$).

Both are used in order to obtain a number (L-value) from an automorphic form. Such a number can be algebraic (after normalization) via the embeddings,

$$\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$$

and we may compare the complex and p-adic L-values at many points.

How to define and to compute p-adic L-functions? The Mellin transform of a p-adic distribution μ on $\mathbb{Z}_p^*$ gives an analytic function on the group of p-adic characters

$$x \mapsto \mathcal{L}_\mu(x) = \int_{\mathbb{Z}_p^*} x(y)d\mu(y), \ x \in X_{\mathbb{Z}_p^*} = \mathrm{Hom}_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$$

A general idea is to construct p-adic measures directly from Fourier coefficients of modular forms proving Kummer-type congruences for L-values. We present a new method to construct p-adic L-functions via quasimodulat forms as follows

3.8. **Proving Kummer-type congruences using the Fourier coefficients.** Suppose that we are given some L-function $L_f^*(s,\chi)$ attached to a Siegel modular form f and assume that for infinitely many "critical pairs" $(s_j;\chi_j)$ one has an integral representation $L_f^*(s,\chi) = \langle f, h_j\rangle$ with all $h_j = \sum_T b_{j,T}q^T \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing f and defined over $\bar{\mathbb{Q}}$. We want to prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*,\ \sum_j \beta_j\chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \frac{L_f^*(s,\chi)}{\langle f,f\rangle} \equiv 0 \mod p^N,$$

where

$$\beta_j \in \bar{\mathbb{Q}}, k_j = \begin{cases} s_j - s_0 & \text{if } s_0 = \min_j s_j \text{ or} \\ k_j = s_0 - s_j & \text{if } s_0 = \max_j s_j. \end{cases}$$

Using the above expression for $\ell_f(h_j) = \sum_i \gamma i, j b_{j,T_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j}\beta_j b_{j,T_i} \equiv 0 \mod p^N.$$

The last congruence is done by elementary check on Fourier coefficients b_{j,T_i}.

Using *Krasner graded structures* in general, the abstract Kummer congruences are checked for a family of test elements (e.g. certain p-adic Dirichlet characters with values in $\bar{\mathbb{Q}}_p^*$, viewed as homogeneous elements of grade $j \in J \subset \mathcal{Y}$, with above $J = \mathcal{Y}^{alg}$.

3.9. **From holomorphic to nearly holomorphic and p-adic modular forms.** Next we explain, how to treat the functions h_j which belong to a certain infine dimensional $\mathbb{Q}$-vector space $\mathcal{N} \subset \mathcal{N}_\rho(\bar{\mathbb{Q}})$ (of nearly holomorphic modular forms).

Usually, h_j can be expressed through the functions $\delta_{k^j}(\varphi_0(\chi_j))$ for a certain non-negative power k_j of the Maass-Shimura-type diffential operator applied to a holomorphic form $\varphi_0(\chi_j)$. Then the idea is to proceed in two steps:

1) A passage from the infinite dimensional $\bar{\mathbb{Q}}$-vector space $\mathcal{N} = \mathcal{N}(\mathbb{Q})$ of nearly holomorphic modular forms,

$$\mathcal{N}(\mathbb{Q}) := \bigcup_{m\geq 0} \mathcal{N}_{k,r}(\mathbb{Q})(Np^m, Q)$$

(of the depth r): to a fixed finite dimensional characteristic subspace $\mathcal{N}^\alpha \subset \mathcal{N}(Np)$ of U_p in the same way as for the holomorphic forms. This step controls Petersson products using conjugate f^0 of an eigenfunction f_0 of $U(p)$:

$$\langle f^0, h\rangle = \alpha^{-m}\langle f^0, h|U(p)^m\rangle = \langle f^0, \pi^\alpha(h)\rangle.$$

2) The use of Ichikawa's injecgtion $\iota_p : \mathcal{N}(Np) \to \mathcal{M}_p(Np)$ to a certain space $\mathcal{M}_p(Np)$ of p-adic Siegel modular forms. Notice also that the realization of nearly holomorphic forms as p-adic modular forms has been studied by Eric Urban, who calls them "Nearly overconvergent modular forms" [33], Chapter 10.

Let us assume algebraically,

$$h_j = \sum_T b_{j,T}(S) q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0) q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computations become much easier, as the action of δ^{k_j} becomes simply a k_j -power of the Ramanujan Θ-operator

$$\Theta : \sum_T b_T q^T \mapsto \sum_T \det(T) b_T(0) q^T$$

in the scalar-valued case. In the vector-valued case such operators were studied in [4].

After this step, proving the Kummer-type congruences reduces to those for the Fourier coefficients of the quasi-modular forms $\kappa(h_j(\chi_j))$ which can be explicitely evaluated using the Θ-operator.

3.10. **Computing with Siegel modular forms over a ring A.** There are several types of Siegel modular forms (vector-valued, nearly-holomorphic, quasi-modular, p-adic). Consider modular forms over a ring $A = \mathbb{C}, \mathbb{C}_p, \Lambda = \mathbb{Z}_p[\![T]\!], \ldots$ as certain formal Fourier expansions over A. Let us fix the congruence subgroup Γ of a nearly holomorphic modular form $f \in \mathcal{N}_\rho$ and its depth r as the maximal S-degree of the poynomial Fourier Fourier coefficients $a_T(S)$ of a nearly holomorphic form

$$h_j = \sum_T b_{j,T}(S) q^T \mapsto \kappa(h_j) = \sum_T b_{j,T}(0) q^T$$

which is also a certain Siegel quasi-modular form. Under this mapping, computation become much easier, as the action of δ^{k_j} becomes simply a k_j-power of the Ramanujan Θ-operator

$$\Theta : \sum_T a_T(S) q^T \in \mathcal{N}(A),$$

over R, and denote by $\mathcal{N}_{\rho,r}(\Gamma, A)$ the A-module of all such forms. This module is often locally-free of finite rank, that is, it becomes a finite-dimensional F-vector space over the fraction field $F = Frac(A)$.

3.11. **Types of modular forms.**

- $\mathcal{M}_\rho$ (holomorphic vector-valued Siegel modular forms attached to an algebraic representation $\rho : GL_n \to GL_d$)
- $\mathcal{QM} = \mathcal{N}|_{S=0}$ (quasi-modular vector-valued forms attached to ρ)
- $\mathcal{N}_\rho$ (holomorphic vector-valued Siegel modular forms, algebraic p-adic vector-valued forms attached to ρ over a number field $k \subset \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$))

Definitions and interrelations:

- $\mathcal{QM}_{\rho,r} = \kappa(\mathcal{N}_{\rho,r})$, where $\kappa : f \mapsto f|_{S=0} = \sum_T P_T(0)q^T$, with the notation $\mathcal{R}_{n,\infty} = \mathbb{C}[[q_{11}, \dots q_{nn}]][q_{ij}, q_{ij}^{-1}]_{ij=1,\dots,n}$
- $\mathcal{M}^\flat_{\rho,r}(R,\Gamma))) = F_c(\iota_p(\mathcal{N}_{\rho,r}(R,\Gamma))) \subset \mathcal{R}^d_{n,p}$ where

$$\mathcal{R}_{n,p} = \mathbb{C}_p[[q_{11}, \dots q_{nn}]][q_{ij}, q_{ij}^{-1}]_{ij=1,\dots,n}$$

Let us fix the level Γ, the depth r , and a subring R of $\bar{\mathbb{Q}}$, then all the R-modules $\mathcal{M}_{\rho,r}(R,\Gamma)))$, $\mathcal{N}_{\rho,r}(R,\Gamma)))$ $\mathcal{QM}_{\rho,r}(R,\Gamma)))$, $\mathcal{M}^\flat_{\rho,r}(R,\Gamma)))$ are then *locally free of finite rank*.

In interesting cases, there is an inclusion

$$\mathcal{QM}_{\rho,r}(R,\Gamma) \hookrightarrow \mathcal{M}^\flat_{\rho,r}(R,\Gamma).$$

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $k = 2, P = E_2$ is a p-adic modular form, see [30], p.211.

4. Applications to unitary groups

Although we treat here only the Siegel modular case here, the results can be extended to the general Sp- and unitary cases (UT in Shimura's terminology).

Towards general constructions of p-adic L-functions for unitary groups. In the most recent version of the paper [10] by Ellen Eischen, Michael Harris, Jianshu Li, Christopher Skinner, a construction of p-adic L-functions for unitary group in the ordinary case was completed using p-adic modular forms. Also, the case of Hida's families of such forms is treated using a construction of Eisenstein measures.

Our present method provides a construction of algebraic-valued distributions using only $\bar{\mathbb{Q}}$-linear forms of arithmetical quasi-modular forms defined through the Garrett map ([10], p.6), Petersson products and their coefficients.

According to Proposition 8.2.10., p.124 of [10], supposing π (anti-) holomorphic and (anti-) ordinary, the Newton polygon of π_w and Hodge polygon of π_w meet at the midpoint. In motivic terms, this says that the motive obtained by restriction of scalars to $\mathbb{Q}$ of the motive attached to π satisfies an admissibility condition, see [25].

It would be interesting to consider the difference h of two polygons at the center point, and in the admissible non-ardinary case to construct h-admissibles measures of type Amis-Vélu and B. Mazur, Tate, J. Teitelbaum, [24].

If $h > 0$, such measures produce p-adic L-fonctions of logarithmic growth $\log^h$ from the sequences of agebraic valued distributions.

In particular, this construction provides Amice-Velu admissible measures through certain sequences of quasi-modular valued distributions as in [26], without the explicit use of p-adic modular forms, but only quasi-modular forms over $\bar{\mathbb{Q}}$.

More details appear elsewhere.

On the other hand, in [16] by Toshiyuki Kikuta, Shoyu Nagaoka, some important congruence relations with respect to some Hermitian modular forms are given. We wish also to interpret them as a passage from the p-adic modular forms to algebraic quasi-modular forms, simplifying constructions of p-adic L-functions mentioned above.

Acknowledgements. Many thanks to Mirjana Vuković, distinguished co-author of Professor Marc Krasner, for the invitation to participate at this conference, whom I first met when she was at Moscow State University–Lomonosov (1975-1976), where we worked with eminent Russian mathematicians.

My deep gratitude go to Khoai Ha Huy, Siegfried Boecherer and Vladimir Berkovich for fruitful discussions during a visit of the author to Vietnam in April 2015.

References

[1] Y. Amice, *Les nombres p-adiques*, 1975, PUF, Collection SUP, p.189.

[2] Y. Amice and J. Vélu, *Distributions p-adiques associées aux séries de Hecke*, Journées Arithmétiques de Bordeaux (Conf. Univ. Bordeaux, 1974), Astérisque no. 24/25, Soc. Math. France, Paris 1975, pp. 119–131.

[3] S. Böcherer, *Über die Funktionalgleichung automorpher L–Funktionen zur Siegelscher Modulgruppe.* J. Reine Angew. Math., 362 (1985), 146–168.

[4] Siegfried Boecherer, Nagaoka and Shoyu, *On p-adic properties of Siegel modular forms*, in: Automorphic Forms, Research in Number Theory from Oman, Springer Proceedings in Mathematics and Statistics 115. Springer 2014, see also arXiv:1305.0604 [math.NT].

[5] S. Böcherer and A. A. Panchishkin, *p-adic Interpolation of Triple L-Functions: Analytic Aspects.* In: Automorphic Forms and L-functions II: Local Aspects–David Ginzburg, Erez Lapid, and David Soudry, Editors, AMS, BIU, 2009, 313 pp.; pp.1-41.

[6] S Böcherer and A. A. Panchishkin, *Higher twists and higher Gauss sums*, Vietnam J. Math., 39 (3) (2011), 309–326.

[7] S. Böcherer and C. -G. Schmidt, *p-adic measures attached to Siegel modular forms*, Ann. Inst. Fourier, 50 (5) (2000), 1375–1443.

[8] M. Courtieu and A. A.Panchishkin, *Non-Archimedean L-Functions and Arithmetical Siegel Modular Forms*, Lect. Notes Math., 1471, Springer-Verlag, 2004 (2nd augmented ed.)

[9] Ellen E. Eischen, *p-adic Differential Operators on Automorphic Forms on Unitary Groups*, Ann. Inst. Fourier, 62 (1) (2012), 177–243.

[10] Ellen E. Eischen, Michael Harris, Jian-Shu Li and Christopher M. Skinner, *p-adic L-functions for unitary groups*, `arXiv:1602.01776v3 [math.NT]`.

[11] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-functions*, Academic Press, New York, 1988.

[12] S. S. Gelbart, I. I. Piatetski-Shapiro and S. Rallis, *Explicit constructions of automorphic L-functions.* Springer-Verlag, Lect. Notes Math., 1254 (1987), 152p.

[13] T. Ichikawa, *Vector-valued p-adic Siegel modular forms*, J. Reine Angew. Math., DOI 10.1515/ crelle-2012-0066.

[14] T. Ichikawa, *Arithmeticity of vector-valued Siegel modular forms in analytic and p-adic cases*, Arxiv: 1508.03138v2 [MathNT].

[15] N. M. Katz, *p-adic interpolation of real analytic Eisenstein series,* Ann. Math., 104 (1976), 459–571.

[16] Toshiyuki Kikuta and Shoyu Nagaoka, *Note on mod p property of Hermitian modular forms*, arXiv:1601.03506 [math.NT].

[17] Neal Koblitz, *p-adic Analysis. A Short Course on Recent Work*, Cambridge Univ. Press, 1980

[18] M. Krasner, *Prolongemcnt analytique uniforme et multiforme*, Collogue C.N.R.S. No. 143, Clermont-Ferrand, 1963, Paris, Ed. C.N.R.S., 1966, p. 97-141.

[19] M. Krasner, *Rapport sur le prologement analytique dans les corps values complete par la methode des e1ements analytiques quasiconnexes*, Bull. Soc. Math. France Mem., 39-40 (1974), 131–254.

[20] M. Krasner, *Anneaux gradués généraux*, Colloque d'Algébre Rennes, 1980, 209–308.

[21] M. Krasner and L. Kaloujnine, *Produit complet des groupes de permutations et problme d'extension de groupes II* , Acta Sci. Math. Szeged, 14 (1951), p. 39-66 et 69-82.

[22] M. Krasner and M. Vuković, *Structures paragraduées (groupes, anneaux, modules)*, Queen's Papers in Pure and Applied Mathematics, 77, Queen's University, Kingston, Ontario, Canada, 1987.

[23] Serge Lang, *Introduction to Modular Forms. With appendixes by D. Zagier and Walter Feit*, Springer-Verlag, Berlin, 1995.

[24] B. Mazur, J. Tate and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math., 84 (1) (1986), 1–48.

[25] A. A. Panchishkin, *Motives over totally real fields and p-adic L-functions*, Ann. Inst. Fourier (Grenoble), 44 (4) (1994), 989–1023.

[26] A. A. Panchishkin, *A new method of constructing p-adic L-functions associated with modular forms*, Mosc. Math. J., 2 (2) (2002), 1–16.

[27] A. A. Panchishkin, *Two variable p-adic L functions attached to eigenfamilies of positive slope*, Invent. Math., 154 (3) (2003), 551–615.

[28] A. A. Panchishkin, *Families of Siegel modular forms, L-functions and modularity lifting conjectures*, Israel J. Math., 185 (2011), 343–368.

[29] A. Scholl, *An introduction to Kato's Euler systems*, Galois representations in arithmetic algebraic geometry (Durham, 1996), 379–460, London Math. Soc. Lect. Note Ser., 254, Cambridge Univ. Press, Cambridge, 1998.

[30] J.–P. Serre, *Formes modulaires et fonctions zêta p-adiques*, Lect. Notes Math., 350 (1973), 191–268.

[31] G. Shimura, *Arithmeticity in the theory of automorphic forms*, Mathematical Surveys and Monographs, vol. 82 (Amer. Math. Soc., Providence, 2000).
[32] C. Skinner and E. Urban, *The Iwasawa Main Cconjecture for GL(2)*. `http://www.math.jussieu.fr/~urban/eurp/MC.pdf`.
[33] Eric Urban, *Nearly Overconvergent Modular Forms*, Iwasawa Theory 2012, Contributions in Mathematical and Computational Sciences, Volume 7, 2014, pp 401–441, `http://link.springer.com/chapter/10.1007/978-3-642-55245-8_14`.
[34] M. Vuković, *Structures graduées et paragraduées*, Prépublication de l'Institut Fourier, No. 536, (2001), https://www-fourier.ujf-grenoble.fr/sites/default/files/ref_536.pdf.
[35] H. Yoshida, *Review on Goro Shimura, Arithmeticity in the theory of automorphic forms* [31], Bull. Am. Math. Soc., New Ser., 39, (3) (2002), 441–448.
[36] S. Zemel, *On quasimodular forms, almost holomorphic modular forms, and the vector-valued modular forms of Shimura*, arXiv:1307.1997 (2013).

(Received: May 24, 2016)
(Revised: October 18, 2016)

Institut Fourier
Université Grenoble Alpes
100, rue des mathématiques
38610, Gières
France
Alexei.Pantchichkine@ujf-grenoble.fr