SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.12.3.13
Vol.12 (25), No.2, (2016), 419–428, Suppl.

QUASIMODULAR SIEGEL MODULAR FORMS AS P-ADIC MODULAR FORMS

SIEGFRIED BÖCHERER

Dedicated to the memory of Professor Marc Krasner

Abstract. There is a sophisticated theory of nearly holomorphic Siegel modular forms by Shimura. Using previous results by Nagaoka and myself on Rankin-Cohen operators and theta-operators we will present a proof that quasimodular forms (defined as constant terms or as holomorphic part of a nearly holomorphic Siegel modular form) are always p-adic.

Introduction

Quasimodular forms are closely related to derivatives of modular forms. We combine our previous work (joint with S.Nagaoka) on p-adic properties of certain theta-operators with results of Shimura on nearly holomorphic modular forms to show that quasimodular Siegel forms are p-adic modular forms. Our method works in the framework of (vector-valued) Siegel modular forms, but it can be generalized to more general types of holomorphic modular forms on tube domains (in particular for Hermitian or quaternion modular forms). We mention that there is also an approach to these questions in a geometric setting by T. Ichikawa [10] (for levels bigger or equal to three).

Our results should be of intrinsic interest within the realm of p-adic modular forms; for interesting applications to p-adic interpolation of L-functions for arithmetic modular forms on symplectic or unitary groups (using the

2010 *Mathematics Subject Classification.* 11F33, 11F46, 11F60.

Key words and phrases. Siegel modular forms, quasimodular forms, differential operators, Rankin-Cohen brackets, p-adic modular forms.

This paper was presented at the International Scientific Conference *Graded structures in algebra and their applications*, dedicated to the memory of Prof. Marc Krasner, IUC-Dubrovnik, Croatia, September, 22-24, 2016.

doubling method as in [1] and in [5]) we refer to a recent preprint of A. Pantchichkin [13].

The main technical part of this paper has nothing to do with modular forms, but with a space of (vector-valued) functions on Siegel's upper half space, on which the real symplectic group acts by an automorphy factor. We have to review Shimura's theory of nearly holomorphic functions: It allows us to obtain all nearly holomorphic functions from holomorphic functions by applying certain differential operators (under a certain condition on the weight of the action of $Sp(n,\mathbb{R})$). Note that even if we are only interested in scalar-valued functions we need for this description the full setting of vector-valued functions. This will be done in Section 1. We transfer the results to the case of nearly holomorphic modular forms in Section 2 and finally in Section 3 we show that quasimodular forms are p-adic modular forms.

1. Nearly holomorphic functions

1.1. **Generalities.** We recall here some results of Shimura in a language appropriate for us:

Let $\mathbb{H}_n$ be the Siegel upper half space of degree n with the usual action of the group $Sp(n,\mathbb{R})$, given by $(M,Z) \longmapsto M < Z > := (AZ+B)(CZ+D)^{-1}$. For a polynomial representation $\rho : GL(n,\mathbb{C}) \longrightarrow Aut(V)$ on a finite-dimensional vector space $V = V_\rho$ we define an action of $Sp(n,\mathbb{R})$ on V-valued functions on $\mathbb{H}_n$ by

$$(f,M) \longmapsto (f \mid_\rho M)(Z) = \rho(CZ+D)^{-1} f(M < Z >).$$

We choose the smallest nonnegative integer k such that $\rho = \det^k \otimes \rho_0$ with ρ_0 is still polynomial and we call this k the weight of ρ; if ρ itself is scalar-valued, we often write k instead of $\det^k$.

We denote by $\mathcal{N}_\rho^\nu$ the space of all V-valued nearly holomorphic functions on $\mathbb{H}_n$ of degree $\leq \nu$; these are the functions given as polynomials in the entries of Y^{-1} of total degree smaller or equal to ν with holomorphic V-valued functions as coefficients. The subscript ρ indicates that we equip this space with the $|_\rho$ - action of $Sp(n,\mathbb{R})$. Note that for $\nu = 0$ we just get the space $\mathcal{H}ol(\mathbb{H}_n)_\rho := \mathcal{N}_\rho^0$ of all holomorphic V_ρ-valued functions on $\mathbb{H}_n$.

The "constant term" of a nearly holomorphic function f (free of entries of Y^{-1}) will always be denoted by f^0.

Examples of nearly holomorphic functions are obtained by applying certain differential operators to holomorphic (or nearly holomorphic) functions. These operators D are polynomials (with coefficients in $\mathbb{Q}$) in the holomorphic derivatives ∂_{ij} with coefficients depending on Y^{-1}; they act on V_ρ-valued functions and map them to $V_{\rho'}$ -valued functions and they are equivariant w.r.t. the action of $Sp(n,\mathbb{R})$, i.e.

$$D(f\mid_\rho M) = D(f)\mid_{\rho'} M, \quad (M\in Sp(,\mathbb{R}))\,.$$

We call these operators Maaß-Shimura differential operators. Sometimes we write $D = D(\rho,\rho')$ to indicate the change of the automorphy factor.

A version of Shimura's structure theorem tells us that under some condition, we can obtain all nearly holomorphic functions from holomorphic ones by applying such differential operators. For this, we need vector-valued functions even if we are just interested in the scalar-valued case:

Theorem 1.1. *For given degree ν and a polynomial representation ρ_0 (of weight 0) there exists k_0 such that for all weights $k\geq k_0$, all representations $\rho = \det^k\otimes\rho_0$ and all $f\in\mathcal{N}^\nu_\rho$ there exist polynomial representations ρ_i $(0\leq i\leq\nu)$ and Maaß-Shimura differential operators $D_i = D_i(\rho_i,\rho)$ and holomorphic V_{ρ_i}-valued functions f_i such that*

$$f = \sum_i D_i(f_i). \tag{1}$$

The differential operators D_i are of total degree i in the entries of Y^{-1}. Shimura [17] constructs them in a rather explicit way and he denotes them by

$$D_i = \theta^i_V D^i_{\rho\otimes\sigma^i}.$$

We will call them *special Shimura differential operators* in the sequel. In particular, D_0 is the constant map and we tacitly normalize it to be the identity.

We should point out that in the general case as in the theorem, a nearly holomorphic function can have many different decompositions (1) depending on the action of $Sp(n,\mathbb{R})$ imposed; if necessary, we call (1) the $\mathcal{N}^\nu_\rho$ -decomposition of f.

Remarks 1.1.
1) We may rephrase Shimura's result as follows:
There are linear maps

$$\psi_i : \mathcal{N}^0_{\rho_i} \longmapsto \mathcal{N}^\nu_\rho \qquad (\text{“}\textit{defined over } \mathbb{Q}\text{”})$$

such that

$$f_i = \psi_i(f).$$

We do not claim that the decomposition (1) is unique, it is sufficient for us that we may choose linear maps ψ_i and keep them fixed throughout.
2) Under suitable growth conditions, $f_0 = \psi_0(f)$ is a holomorphic projection of f with respect to the Petersson inner product.

1.2. **Constant term of a Maaß-Shimura differential operator as leading term in a Rankin-Cohen bracket.** We start with an example from [4]:

Example 1.1. For $0 \leq r \leq n$ we put

$$\delta^{[r]} := det(Y)^{-k+\frac{r-1}{2}} \partial^{[r]} det(Y)^{k-\frac{r-1}{2}},$$

where for any matrix A of size n we denote by $A^{[r]}$ the matrix of size $\binom{n}{r}$ consisting of the minors of A of order r. This operator maps holomorphic functions f to (vector-valued) nearly holomorphic functions and it is known [6, 11] to satisfy the transformation rule

$$\delta^{[r]}(f \mid_k \gamma) = \left(\delta^{[r]}(f)\right) \mid_{\det^k \otimes \rho_o} \gamma$$

for all $\gamma \in Sp(n, \mathbb{R})$, where ρ_r denotes the irreducible representation of $GL(n, \mathbb{C})$ of highest weight $(2, \dots, 2, 0, \dots, 0)$. Obviously,

$$\left(\delta^{[r]} f\right)^0 = \partial^{[r]}(f).$$

On the other hand we have shown in [4, prop.3] that there is a bilinear holomorphic differential operator

$$[\,,\,]_{k_1,k_2,\rho_r} : \mathcal{H}ol(\mathbb{H}_n)_{k_1} \times \mathcal{H}ol(\mathbb{H}_n)_{k_2} \longmapsto \mathcal{H}ol(\mathbb{H}_n)_{\det^{k_1+k_2} \otimes \rho_r}$$

compatible with the action of $Sp(n, \mathbb{R})$; such operators are usually called Rankin-Cohen bracket operators, for generalities on them we refer to [9]. This Rankin-Cohen bracket is of the form

$$[f, g]_{k_1,k_2,\rho_r} = (\partial^{[r]} f) \cdot g + \dots,$$

where $\dots$ consists of summands involving only nontrivial derivatives of g (not g itself!). This is true at least for k_1 outside a finite set and k_2 sufficiently large.

This means that *the "constant term" of $\delta^{[r]} f$ and the "leading term" of $[f, g]_{k_1,k_2,\rho^{[r]}}$ are the same (up to g).*

A weak version of this is true more generally by applying Shimura's theorem to a nearly holomorphic function of type $D(f) \cdot g$:

Proposition 1.1. *Let D be a Maaß-Shimura differential operator of degree ν, changing an automorphy factor ρ to ρ'; furthermore, let f, g be arbitrary holomorphic functions on $\mathbb{H}_n$, V_ρ-valued and scalar-valued respectively. Then in the $\mathcal{N}^\nu_{\rho' \otimes \det^l}$- decomposition with l large*

$$(Df) \cdot g = \sum D_i \left((Df) \cdot g)_i\right)$$

the holomorphic functions $((Df)\cdot g)_i = \psi_i(Df \cdot g)$ *are given by Rankin-Cohen brackets* $\mathcal{L}_i(f,g)$*, more precisely, if* D *is of degree* ν*, then*

$$D(f)\cdot g = \mathcal{L}_0(f,g) + \left(\sum_{i=1}^{\nu-1} D_i(\mathcal{L}_i(f,g))\right) + D_\nu(f\cdot g). \tag{2}$$

Example 1.2. The simplest case of the proposition above is the degree one Maaß-Shimura differential operator:

$$\delta_k := \frac{k}{2iy} + \frac{\partial}{\partial z}.$$

In this case we can write down an identity for all weights k, l:

$$(k+l)\cdot\delta_k(f)\cdot g = [f,g]_{k,l} + k\cdot\delta_{k+l}(f\cdot g).$$

We may apply this proposition for the function $g = 1$ and obtain as a (trivial)

Observation. Under the same conditions as in the proposition,

$$D(f)^0 = \mathcal{L}_0(f,1) + \sum_{i=1}^{\nu-1} D_i(\mathcal{L}_i(f,1)))^0 + D_\nu(f)^0. \tag{3}$$

In the next sections we would like to prove some properties of these differential operators by using induction over the degree ν in (2) and (3). To do so, we have to overcome the problem that summands of degree ν appear on both sides of these identities. Such a procedure is possible, if $D = D(\rho\otimes\det^k, \rho'\otimes\det^k)$ is a *special* Shimura differential operator: Then such an operator decomposes in the form

$$D = R_\partial + r(k)R_Y + \mathcal{R}$$

where R_∂ is the part of D free of Y and R_Y is free of ∂ and consists of monomials of exact degree ν in the entries of Y^{-1}. The remaining unspecified terms are collected in $\mathcal{R}$. The important property here is that R_∂ and R_Y do not depend on k at all and $r(k)$ is a *nonconstant* polynomial in k.

These properties can be read off from the reasoning on page 109 in [17]. For the examples $\delta^{[r]}$ and δ_k from above, they are obviously satisfied.

Then we can reformulate (2) (if f carries a $|_{\rho\otimes\det^k}$-action and g carries a $|_{k'}$-action with $l = k + k'$) as

$$\begin{aligned}D(\rho\otimes\det^k, \rho'\otimes\det^k)(f)\cdot g - \frac{r(k)}{r(l)}D(\rho\otimes\det^l, \rho'\otimes\det^l)(f\cdot g)\\ = R_\partial(f)\cdot g + \dots\\ = \mathcal{L}_0(f,g) + \sum_{i=1}^{\nu-1} D_i(\mathcal{L}_i(f,g))\end{aligned}$$

where ... consists only of monomials of positive degree in the derivatives of g and the entries of Y^{-1}.

We can apply this to the constant function $g = 1$ and obtain

Corollary 1.1. *If l is sufficiently large, then the constant term $D(f)^o$ of $D(f)$ is proportional to the leading term in the Rankin-Cohen operator defined by $\mathcal{L}_0(f, g)$ modulo the sum of constant terms of the $D_i(\mathcal{L}_i(f, 1))$ for $0 \leq i < \nu$.*

Remark 1.1. In some cases one can show (by the same kind of argument as in [4] that the constant terms $D_i(\mathcal{L}_i(f, 1)$ for $i > 0$ are proportional to $D(f)^o$.

2. Nearly holomorphic modular forms and quasimodular forms

For a congruence subgroup Γ of $Sp(n)$ we may now define the nearly holomorphic modular forms of weight ρ for Γ by

$$\mathcal{N}_\rho^\nu(\Gamma) := \{f \in \mathcal{N}_\rho^\nu \mid \forall\gamma \in \Gamma : f \mid_\rho \gamma = f\},$$

where for $n = 1$ we have to impose the usual additional conditions in the cusps.

Note that for $\nu = 0$ we get the usual holomorphic Siegel modular forms of degree n, and automorphy factor ρ for Γ; we denote them by $M_\rho(\Gamma)$.

For the results of Section 1, there are (almost evident) versions in the context of modular forms (one has to consider the functions which are invariant under Γ for the appropriate slash-action). As an example, the theorem of the previous section becomes

Theorem 2.1. *For given degree ν and a polynomial representation ρ_0 (of weight 0) there exists k_0 such that for all weights $k \geq k_0$, all representations $\rho = \det^k \otimes \rho_0$ and all $f \in \mathcal{N}_\rho^\nu(\Gamma)$ there exist polynomial representations ρ_i ($0 \leq i \leq \nu$) and Maaß-Shimura differential operators $D_i = D_i(\rho_i, \rho)$ and holomorphic modular forms $f_i \in M_{\rho_i}(\Gamma)$ such that*

$$f = \sum_i D_i(f_i) \tag{4}$$

In the same way, there is an analogue for the proposition of Section 1.

Remarks 2.1.

1) In degree 1, Hecke's classical Eisenstein series of weight two [8] is an example of a nearly holomorphic modular form, for which the statement above does not hold. Also, for degree one, the condition on weights and degrees is explicit: we need $k > 2\nu$, see [15]. In some sense, Hecke's example is the only nearly holomorphic modular form of degree one not obtained by differential operators from holomorphic ones [19].

2) Under the condition of the theorem above, there is no problem about rationality or bounded denominators for Fourier expansions of nearly holomorphic modular forms: They inherit such properties from corresponding statements about the holomorphic modular forms f_i.
3) Nearly holomorphic modular forms arise naturally in the theory of Eisenstein series. They have important appliciations in the arithmetic of modular forms, see [15, 16, 17, 12].

Quasimodular forms have so far been mainly considered for degree 1 (see e.g. [18]), where several equivalent characterizations are given.) A convenient definition for arbitrary degree is

Definition 2.1. A quasimodular form (with automorphy factor ρ for Γ) is a holomorphic V_ρ-valued function g, which appears as the "constant term" in a nearly holomorphic modular form of weight ρ for Γ, i.e. $g = f^0$ for some nearly holomorphic modular form f.

3. Quasimodular forms as p-adic modular forms

Up to now, the congruence subgroup was arbitrary; from now on we fix a prime p and consider only congruence subgroups

$$\Gamma_0(p^t) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid C \equiv 0 \bmod p^t \right\}.$$

For most of our considerations, the level p^t can be arbitrary, therefore we use the somewhat unusual notation Γ_p for a congruence subgroups of type $\Gamma_0(p^t)$; note however that t may possibly vary within a statement.

Main Theorem. *All quasimodular forms for $\Gamma = \Gamma_p$ with coefficients in $\mathbb{Q}$ are p-adic.*

We start from a quasimodular form h^0 with $h \in \mathcal{N}_\rho^\nu(\Gamma_p)$ with Fourier coefficients in $\mathbb{Q}$. Furthermore we fix a power p^m and we have to prove that h^0 is congruent modulo p^m to a holomorphic modular form for Γ_p in the sense of simultaneous congruence mod p^m for all Fourier coefficients. Due to the results in [4] (and their -not at all straightforward- generalization to vector-valued situations (see [2], which is inspired by the ideas in [7])) we automatically also get a congruence mod p^m to a modular form of level one. In the case of vector-valued forms the Fourier coefficients are vectors and by congruences we mean simultaneous congruence of all its components after realizing the vectors as column vectors. After multiplication by a holomorphic modular form a quasimodular form is still quasimodular, therefore, by multiplying h by a modular form F for $\Gamma_0(p)$ of sufficiently large weight and satisfying

$$F \equiv 1 \bmod p^l, \quad (l \geq m)$$

we may assume from the beginning that the degree ν of the nearly holomorphic modular form h, is small enough compared with the weight of ρ to allow the application of Shimura's theorem for h. Note that the existence of F is assured by [3].

In view of Shimura's theorem it is then enough to prove

Proposition 3.1. *For any $f \in M_{\rho_i}(\Gamma_p)$ and any special Shimura differential operator D, the "constant term" $(Df)^0$ is p-adic.*

We cannot use the corollary of the previous section directly, because (unlike the other statements of the previous section) there is no straightforward analogue for modular forms. We may however use F as above as g in the proposition of the previous section to obtain a congruence (with a suitable constant c)

$$\begin{aligned} c\,(D(f))^o &\equiv c\,(D(f)\cdot F)^o \bmod p^t \\ &\equiv \mathcal{L}_0(f,F) + \sum_{i=1}^{\nu-1} (D_i(\mathcal{L}_i(f,F)))^o \end{aligned}$$

The first summand on the right hand side is then a modular form for Γ_p and the remaining terms carry special Shimura differential operators D_i, whose degree is smaller than the degree of D; by induction on that degree we may then assume that the $(D_i(\mathcal{L}_i(f,F)))^o$ are congruent mod p^t to modular forms for Γ_p; by our results from [4], the leading term in the Rankin-Cohen operator $\mathcal{L}_0(f,F)$ is also a p-adic modular form.

Remark 3.1. Note that (after choosing F appropriately as in [4]), the term $\mathcal{L}_0(f,F)$ is congruent to $\Theta(f)$ for a suitable "theta operator", given by holomorphic derivatives of f.

Example 3.1. We consider the weight 2 Eisenstein series defined by

$$E_2(z) = \frac{-3}{\pi y} + 1 + \sum \sigma_1(n) exp(2\pi i n z).$$

To show that its constant term is a p-adic modular form one may use Kummer congruences and congruences for sums of type $\sigma_k(n)$. A proof along the lines of our method is somewhat different from the usual one:

For simplicity we only consider primes $p > 3$ and for any $m \geq 0$ we use Eisenstein series $E_{(p-1)p^m}$; by Clausen-von Staudt these Eisenstein series are congruent to 1 mod p^m.

We study the nearly holomorphic function

$$E_2 \cdot E_{p^m(p-1)}.$$

We may write it as

$$E_2 \cdot E_{p^m(p-1)} = \alpha_p(m) \cdot \delta_{p^m(p-1)}(E_{p^m(p-1)}) + f_0$$

where f_0 is a holomorphic modular form and

$$\alpha_p(m) = \frac{3}{\pi p^m (p-1)}$$

The right hand side is a sum of a modular form f_0 and an image of a modular form under a Maass-Shimura differential operator. The constant term $(\delta_{p^m(p-1)}(E_{p^m(p-1)}))^0$ and also f_0 are both p-adic, therefore the constant term of the left hand side is also p-adic. The constant term of $E_2 \cdot E_{p^m(p-1)}$ is congruent mod p^{m+1} to $1 + \sum \sigma_1(n) exp(2\pi inz)$ and we are done.

Acknowledgement. This work was initiated by a discussion with A. Panchishkin. I wish to thank him for his encouragement and interest.

References

[1] S. Böcherer and C.-G. Schmidt, *p-adic measures attached to Siegel modular forms*, Ann. Inst. Fourier, 50, (2000), 1375–1443.

[2] S. Böcherer, *Vector-valued Siegel modular forms of degree n arising from scalar valued forms of higher degree*, in preparation.

[3] S. Böcherer and S. Nagaoka, *On mod p properties of Siegel modular forms*, Math. Ann., 338 (2007), 421–433.

[4] S. Böcherer and S. Nagaoka, *On p-adic properties of Siegel modular forms*, in: *Automorphic Forms, Research in Number theory from Oman*, Springer 2014.

[5] E. Eischen, M. Harris, J.-S. Li and C. Skinner, *p-adic L-functions for unitary groups*, Arxiv: 1602.01776v3.

[6] E. Freitag, *Siegelsche Modulfunktionen*, Springer 1983.

[7] P. B. Garrett, *On the arithmetic of Hilbert-Siegel cusp forms: Petersson inner products and Fourier coefficients*, Inv. Math., 107 (1992), 453–481.

[8] E. Hecke, *Theorie der Eisensteinschen Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik*, Mathematische Werke No. 24.

[9] T. Ibukiyama, *On differential operators on automorphic forms and invariant pluriharmonic polynomials*, Comment. Math. Univ. St. Pauli, 48 (1999), 103–118.

[10] T. Ichikawa, *Arithmeticity of vector-valued Siegel modular forms in analytic and p-adic cases*, Arxiv: 1508.03138v2.

[11] H. Maaß, *Siegel's modular forms and Dirichlet series*, Lect. Notes Math., 216 (1971).

[12] S. Mizumoto, *Nearly holomorphic Eisenstein liftings*, Abh. Math. Semin. Univ. Hamb., 67 (1997), 173–194.

[13] A. A. Panchishkin, *Graded structures and differential operators on nearly holomorphic and quasimodular forms on classical groups*, this issue 401–417.

[14] J.-P. Serre, *Formes modulaires et fonctions zeta p-adique*, in: *Modular Forms of One Variable III*, Lect. Notes Math., 350 (1973).

[15] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Commun. Pure Appl. Math., 29 (1976), 783–804

[16] G. Shimura, *Arithmeticity of the special values of various zeta functions and the periods of abelian integral*, Sugaku Expositions, 8 (1995), 17–38.

[17] G. Shimura, *Arithmeticity in the Theory of automorphic Forms*, AMS 2000.

[18] D. Zagier, *Elliptic modular forms and their applications*, in: *The* 1-2-3 *of modular forms*, Springer 2008.

[19] S. Zemel, *On quasimodular forms, almost holomorphic modular forms, and the vector-valued modular forms of Shimura*, ArXiv:1307.1997 (2013).

(Received: May 26, 2016)
(Revised: September 16, 2016)

Mathematisches Institut
Universität Mannheim
68131 Mannheim
Germany
boecherer@math.uni-mannheim.de