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REMEMBERING ACADEMICIAN MAHMUT BAJRAKTAREVIĆ

It is thirty years since Mahmut Bajraktarević - Mašo, professor at the Natural Sciences and Mathematics Faculty of the University of Sarajevo and member of the Academy of Sciences and Arts of Bosnia and Herzegovina (ANUBiH) passed away. Academician Bajraktarević was one of a very small number of mathematicians who started the development of mathematical sciences in Bosnia and Herzegovina (B-H). In light of his results he was the most distinguished mathematician in B-H and occupied a very prestigious position in the world of Yugoslavian mathematics.

Mahmut Bajraktarević was born on December 22, 1909 in Sarajevo and died suddenly on April 13, 1985, in Bugojno, where he was attending the XXVII Mathematics contest for high school students - as an honored guest of the organizers.

He completed elementary school and high school in Sarajevo. He graduated from the First Male Gymnasium in 1929 and immediately started his mathematical studies at the Philosophy Faculty of the University of Belgrade, graduating in 1933, having studied mathematics, rational mechanics with physics and experimental physics. He received his PhD in mathematics at the world famous Sorbonne University in Paris in 1953 with the thesis *"Sur certaines suites itéréres"* and thus became the first doctor of mathematics in B-H.

Academician Bajraktarević started his professional career at the Šerijatski (Islamic) High School in 1934 where he worked till the end of the Second World War, first as an assistant and later as a professor of mathematics. He also, for some time,

worked at the First High School in Sarajevo. After the war he continued to work at the Šerijatska High School and the Second Male Gymnasium and then transferred to the Higher Pedagogical School in Sarajevo. From April 1950 till 1960 he worked at the Philosophy Faculty of the University of Sarajevo as a lecturer and later as an assistant and associate professor. At this time (1960) the Natural Sciences and Mathematics Department separated from the Philosophy Faculty, forming a new faculty - the Natural Sciences and Mathematics Faculty. Mahmut Bajraktarević quickly became a full professor at this newly formed faculty.

Mahmut Bajraktarević together with colleagues Vera Šnajder and Šefkija Raljević was the founder of the Mathematics Department at the new Faculty, teaching many courses in mathematics. They worked tirelessly to teach generations of mathematics' and physics' students.

He was elected as a corresponding member of the Scientific Society 1961. That society grew into the ANUBiH in 1966. First he was corresponding, and in 1967 he became a full member of ANUBiH.

Academician M. Bajraktarević published about sixty scientific works. His scientific papers acquired significant attention, from both domestic and foreign mathematicians. Many of those papers, apart from being highly reviewed in international journals, were cited in some of word's monographies, such as: J. Aczel: *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Basel-Stuttgart-Berlin, 1961; M. Kuczma: *A Survey of the Theory of Functional Equations*, Publikacije Elektrotehničkog fakulteta u Beogradu 1964; M. Kuczma: *Functional Equations in a single Variable*, Polska Akademia nauk, Monographie Mathematyczne 46, Warszawa 1968; K. Zeller i W. Beckmann, *Theorie der Limitierungsverfahren*, Springer-Verlag, 1970, Itogi nauki i tehniki, Matematicheskij analiz, Tom 12, Moskva, 1974.

The scientific interests of Academician Bajraktarević were mainly connected with two important areas of mathematical analysis: the theory of functional equations, and the theory of sequences and summability in which, in particular he studied the summability of iterated sequences with the aid of general means and the speed of convergence of these sequences. Among the first papers of Academician Bajraktarević were a series dealing with sequences obtained by iteration. These papers were connected with solutions of functional equations and systems of functional equations, in particular the Schröder and the Abel functional equations. These results improved and modified results of various authors including Morgan, Warda, Fuller, Korkin, Bennet.

These problems were the subject matter of M. Bajraktarević's PhD thesis in which he examined the convergence of sequences (x_n), constructed with the aid of continuous and especially continuous decreasing functions f, where $x_{n+1} = f(x_n)$. These are followed by more complicated sequences (x_n), defined by

$$x_n(z) = \varepsilon_0 f(\varepsilon_1 f(\ldots(\varepsilon_n f(0)))), \text{ where } \varepsilon_n = \frac{1-2d_n}{1-2d_{n-1}},$$

d_n is the digit of number $z \in [0,2]$ represented in system with base 2, for which are proven several general properties and theorems concerning convergence. These

results are illustrated by numerous interesting examples.

These studies were extended, considering a sequence of two place functions $x_n(z,t)$, for which among other things, necessary and sufficient conditions are obtained so that the limit function is continuous with respect to z. Then, the function f is introduced in connection with solving the functional equation $f(g(2z)) = g(z)$ $(0 \leq z \leq 2)$ which, under certain conditions, represents a form of Schröder's respectively Abel's equation. Furthermore, necessary and sufficient conditions for the convergence of the above mentioned sequence modify results of Morgan, Ward and Fuller which are concerned with continuous iteration. These results examine one solution of the system of functional equations $f_n(g_{n+1}(2z)) = g_n(z)$ obtained by iteration. Continuing work in this area, M. Bajraktarević, using a strictly increasing continuous function $F(x)$, introduced continuous iteration $F_y(x)$, generalizing the work of Morgan and Fuller and determined necessary and sufficient conditions for $F_y(x)$, to be a continuous function in x.

Examining the properties of iterative sequences, whose limit function $f_n(z,t)$ represents the solutions of the functional equations of the form $f_n(z,t) = \varepsilon_0 f_n(f_{n+1}$ $((z-d_0)p,t)$, M. Bajraktarević generalized some results from his doctoral thesis, in the case, when $p = 2$.

Very important are also the papers of Academician Bajraktarević in which he examines and extends results on mean values. That is, starting with the works of Bojanić and Rothe, in which there are given necessary and sufficient conditions under which a function is a ξ- function, a similar theorem is proved about mean values and then that result is modified to use in dealing with certain systems of two functional equations whose solution ξ, together with the corresponding functions f and φ, satisfy an extended theorem concerning mean values.

A large number of papers of M. Bajraktarević dealt with functional equations or were connected to this subject. In this group of his papers, connected to functional equations, fall those dealing with integral-functional equations. These works depend heavily on a general fixed point theorem for single integral-functional equations. M. Bajraktarević proved the existence of a class of continuous solutions and under strict conditions proved the existence and uniqueness of solutions.

M. Bajraktarević deepened his study of integral-functional equations, in which he generalized earlier results of Kuczma, respectively Kordylewski - Kuczma. He determined various necessary and sufficient conditions for the existence and uniqueness of continuous solutions of functional equations by proving that, under certain conditions, solutions could be written in the form of an integral equation.

In series of papers, some of which are his most important, he deals with various types of means, in particular quasi-arithmetic means which expand the concept of means with weights in which, in the first place are quasi-linear means formed with constant weights, which were studied by world class authors such as Hardy, Littlewood, Polya, Aczel and others. The concept of a mean is a broad concept which includes many elementary means, such as the antiharmonic mean which is also a generalization of the arithmetic mean.

Up until 1958 the problem of the mean was studied by numerous authors (Beckenbah, Azcel, Daroczi, Fenye, Anghelut, Ghermanisku, Pompeiu and others), but

only in special cases. For means the questions of existence and uniqueness and characterizations are fundamental. These questions for quasi-linear means were studied by Knopp, Jensen, Nagumo, Finetti and Kolmogorov. M. Bajraktarević studied the uniqueness problem for means of the type

$$M_r(x_i, f) = \varphi^{-1}\Big(\sum_{i=1}^{n} f(x_i)\varphi(x_i)\Big/\sum_{i=1}^{n} f(x_i)\Big), \tag{1}$$

which consists in finding the conditions under which the equality

$$M_\varphi(x_i, f) = M_\psi(x_i, g), \quad \text{for every} \quad x_1, x_2, \ldots, x_n\,, \tag{2}$$

and he solved the problem for all natural numbers $n > 2$, under the hypothesis that all functions in (2) have derivatives of the second order. In the case, when $n \neq 2$, he obtained an entire class of solutions and he considered the special case when $f = const > 0$ and $n = 2$, and then $n > 2$.

If we assume that (2) holds, not only for one fixed n, but for all $n > 1$, then the class of solutions of this problem will eventually be smaller. Problem (2) of the type (1) observed this way was completely solved by Aczel and Daroczy without the hypothesis about the differentiability of the functions involved as well as the question of homogenity of these means. Immediately after this M. Bajraktarević generalized the mean of the type (1) introducing a new mean of the form

$$M_\varphi(t, f, p_i) = \varphi^{-1}\Big(\sum_{i=1}^{n} p_i f(t_i)\varphi(t_i)\Big/\sum_{i=1}^{n} p_i f(t_i)\Big),$$

and for this mean, without the differentiability hypothesis, he succeeded in solving all three fundamental questions for means of this type. His proof is much simpler than the proof of Aczel and Daroczy, as well as is for mean of type (1).

Some what latter (1969), M. Bajraktarević returned to means of type (1) and examined in place of (2), a certain inequality with additional hypotheses and obtained necessary conditions under which, that inequality holds for all fixed $n \geq 2$.

A good number of papers of M. Bajraktarević deal with functional equations of various types. In particular it bares mentioning that several papers, in this area, consider the functional equation $\varphi(x) + \varphi(f(x)) = F(x)$, on $[a, b]$. With the same assumptions that Kuczma used M. Bajraktarević proved that: 1) A given sequence formed with the help of the functions f and F, convergences or is summable for the given equation; 2) if the series under consideration is summable to the function $\varphi(x)$, which is continuous on $[a, b]$, then that series converges to the same function and is the unique solution of that equation, and 3) a continuous solution on $[a, b]$ exists only under additional conditions on f and F.

M. Bajraktarević then considered a generalization of this equation and its solution in the form of a series which is continuous on $[a, b]$. This led to various, sufficient conditions for there to exist at most one, exactly one solution, and then gives sufficient conditions for the existence of infinitely many solutions of the equation, which differ from the earlier obtained solution only by additive constants.

He continued studying this equation and then proved that the equation has infinitely many solutions and gave sufficient conditions under which the solutions

are continuous. Besides that he obtained sufficient conditions under which the sum, respectively the T-sum is a well determined series, which represents a solution of the equation under consideration. M. Bajraktarević, further, on some sets E, considered the functional equations

$$f^N(x) = g(x) \text{ respectively } f^{N-n}(h(f^n(x))) = g(x).$$

He first constructed the general solution on the entire set E, and then for some of its subsets E' and proved that its solution on $E\backslash E'$, leads to solving a certain system of functional equations. The results obtained, then, are used in considering the functional equation

$$(gh)^k(f(x)) = f((hg)^k(x)) \text{ respectively } f((hg)^n(h(x))) = g((hg)^n(x)).$$

Generalizing work on a certain functional equation, which earlier was solved by P. M. Vasić, M. Bajraktarević obtained an equation with an unknown vector function whose components depend on several variables and have ranges in a certain field. The general solution of this equation reduces to that of P. M. Vasić as a special case in a simple way. Also, as a corollary of the main results of M. Bajraktarević, one obtains solutions of some systems of functional equations similar to types, which can be considered as special cases, of systems previously considered by R. Ž. Djoković and M. S. Mitrinović.

Among the papers of M. Bajraktarević a preeminent place, by the importance of their results, and by their connection with results, and by their connection with results with a large number of mathematicians who had worked on the functional equation $f(g(f(x))) = g(x)$, in which f is unknown, and g given as a permutation of a certain set X. In solving this equation he obtained the general solution in explicit form, which S. Lajosewicz had been unable to do, while working on the same equation, earlier arrived at a solution, but using another method. Bajraktarević's method is therefore superior and can be used to solve more general equations.

M. Bajraktarević worked on various questions in summability theory, obtaining significant results. In one of his first papers in this area he generalized some results of B. Martić in connection with $S^{\alpha,\beta,\rho}$ - summability that was introduced by V. Vučković. By a new approach to $S^{\alpha,\beta,\rho}$ - summability, which in the literature is called the Bajraktarević method, he examines the summability he introduced and among other results, he proves the regularity of his approach and determines necessary and sufficient conditions under which every series that is Riesz summable is summable by the $S^{\alpha,\beta,\rho}$ - method.

This approach, when applied on several power series in a neighborhood of a, under certain conditons, proves that the power series, which inside its circle of convergence, represents an analytic function $f(z)$, summable to that function and with a, inside of a Borel polygon, and finally he proved $E^\lambda \subseteq S^{\alpha,\beta,\rho}$, for $\lambda > 0$, and $\alpha > 0$, where E^λ denotes the Euler summability method.

M. Bajraktarević studied the problem of speed of convergence of sequence of complex numbers. In these works he determines necessary and sufficient conditions for the existence of a sequence which converges faster, respectively slower to every sequence in a given class of sequences A and proved that under certain conditions,

that the class $\varphi \setminus A$ has cardinality of the continuum, where φ denotes the set of all null sequence of complex members, all of whose terms are different from zero.

Besides these results, under various conditions, he proves the existence of matrix summability methods which, for a class of null sequences, speeds up, respectively slow down or does not change the speed of convergence of all sequences in that class. In the end, he proved that there exists a null sequence, whose convergence is speed up by every matrix summability method T in a certain class of matrix summability methods, and given, a regular matrix summability method T, there exists a class of null sequences, all of whose members speed of convergence is increased, i.e, speeded up, by T. These results are very interesting and are among M. Bajraktarević's most important.

What, shortly, can we say about the results and works of Academic M. Bajraktarević? His works about iterative sequences, general means and later works on speeds of convergence have led to important, meaningful results, over a long period by authors from the ex-Yugoslavia, Poland, Hungary, Romania and elsewhere, having a lasting effect on mathematics.

The results of M. Bajraktarević in field of functional equations comprise the largest and most important part of his creative work and his work on various classes of functional equations are cited in tens of articles during his life time and are included in a large number of monographs, and first rate mathematicians, world wide, have been inspired, by his work, to develop new ideas and results. Still we must mention his results connected with $S^{\alpha,\beta,\rho}$-summability which have been named - the Bajraktarević's method, which have appeared to be a crown of his research.

It is important, also, to point out that the works, of M. Bajraktarević are of pure theoretical character, but, some of them, for example, in the development of computer geometry, can be applied. That is, concretely, using maple, it is proved that the fractal curve can be obtained as the solution of a functional equation, using the so called Read-Bajraktarević operator. M. Bajraktarević presented the results of his studies at numerous domestic and international conferences. Making presentations he attended almost every congress of mathematics, physics and astronomy in Yugoslavia, international symposiums of functional equations: in Zakopan (Poland, 1967), at Oberwolfach (Germany), at the Congress of Balkan mathematicians in Istanbul (1971), and Belgrade (1974) and was invited to lecture on his results at many universities and scientific institutes, for example at the University of Debrecin, the Mathematical Institute of the Serbian Academy of Sciences in Belgrade, the Academy of Sciences of the Soviet Union in Moscow and many others.

Due to his scientific results Mahmut Bajraktarević obtained, here and abroad, a very respected reputation as a scientist and not only achieved an outstanding reputation for himself but for Bosnian-Herzegovinian mathematics as a whole.

The activity of Mahmut Bajraktarević wasn't exclusively restricted to his mathematical research, but in all his activities, especially when he battled for the better recognition of mathematics, it is sufficient to acknowledge his presence, as he was so broadly esteemed. He was associate dean of the Philosophy Faculty, long time chairman of the Mathematics Department of the Natural Sciences and Mathematics

Faculty, secretary and vice president of the Society of Mathematicians, Physicists and Astronomers of Bosnia and Herzegovina, member of the Commission of the Office for Education, Science and Culture of Bosnia and Herzegovina, and also the same Office at the national Yugoslav level, long term member of the Pedagogical Advisory Board for Bosnia and Herzegovina and member of numerous committees in the Mathematics Department, Faculty, University and ANUBiH.

Besides all of the above, he was a long time member of the Editorial Board of Publications de l'Institut de Mathématiques from Belgrade. Certainly it is very important to mention that he was the editor in chief of the journal "Radovi Odjeljenja Prirodnih i matematičkih nauka ANUBiH", which as the first scientific journal of mathematics in Bosnia and Herzegovina, greatly aided in affirming mathematics and mathematicians in Bosnia and Herzegovina, and enabled the founding of the journal "Radovi matematički" with Prof. Bajraktarević as the first editor in chief. His early death, unfortunately did not allow him to see the first edition of this journal, which today appears as the "Sarajevo Journal of Mathematics".

For his long and successful scientific, educational and public service, as for his entire activity in the field of mathematical life in Bosnia and Herzegovina, Academician Mahmut Bajraktarević received many number recognitions and many awards of which we mention only a few, most important recognitions: Society of Mathematicians, Physicists and Astronomers of Yugoslavia (1974), as well as of Bosnia and Herzegovina (1977) and in the name of the President of SFRJ Josip Broz Tito, three awards: Medal of work - Third Order (1957), Medal of work with a red star (1971) and Medal of Service to the People with silver rays (1977). However, in particular it must be mentioned that he received the 27th of July award for scientific achievements (1966) and the 6th of April award of the City of Sarajevo for the complete development of mathematics and mathematical thought in our society (1985).

One would not have a complete picture of the life and work of Mahmut Bajraktarević if we did not consider his human aspects. Every project he undertook he did so with complete commitment working tirelessly. Through his immense efforts in teaching and educational programs, Academician Bajraktarević provided a very meaningful long term effort in educational development at the elementary and high school levels, working on plans and programs for schools, referring text books, recommending the best and together with Vera Šnajder as chair of Mathematics Departments, organised the first postgraduate studies program in mathematics in 1962. His contribution and influence on education and the development of generations of young mathematicians, was not any less important then his contribution to the development of science in Bosnia and Herzegovina.

His lectures, orderly, known for clarity, written using his small hand written text and drawings, meeting the strictest pedagogical norms, demanded attention, including these who returned to the class room after the lecture and marveled at the text on the black board. One could say, these lectures could by photographed and reproduced as a text book. He sometimes even remarked that these lectures could be reproduced in book form. Unfortunately due to the large number of different courses he taught and other obligations, Mašo never wrote books, so that

his lectures remain only in the memories of his former students and especially among the people who had the honour to be, as both of us, his assistants.

During our long term work of over 20 years with our professor we have gotten to know him, not only professionally, but as a person who was strict, solid, but extremely correct and just as a professor.

Academician Mahmut Bajraktarević was man whose total being is connected with the advance of the mathematical sciences and mathematics in general in Bosnia and Herzegovina. His name and his contributions leave a deep trace and hold a special place in the memories of all of us. Because of all of this we will remember with love and deep respect our very dear professor and teacher Academician Mahmut Bajraktarević.

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