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# AN INTRODUCTION TO IMPLICATIVE SEMIGROUPS WITH APARTNESS

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*Dedicated to the memory of Professor Mahmut Bajraktarević*

Abstract. The setting of this research is Bishop's constructive mathematics. Following ideas of Chan and Shum, exposed in their famous paper "Homomorphisms of implicative semigroups", we discuss the structure of implicative semigroups on sets with tight apartness. Moreover, we use anti-orders instead of partial orders. We study concomitant issues induced by existence of apartness and anti-orders giving some specific characterizations of these semigroups. In addition, we introduce the notion of anti-filter in implicative semigroups and give some equivalent conditions that the inhabited real subset of an implicative semigroup is an ordered anti-filter.

## 1. Introduction

The notions of implicative semigroup and ordered filter were introduced by Chan and Shum [10]. For the first generalization of implicative semilattice see Nemitz [18] and Blyth [7]. Moreover, there exists a close relationship between implicative semigroups and other domains. For example, there are a lot of implications in mathematical logic and set theory (see Birkhoff [6]). For the general development of implicative semilattice theory the ordered filters play an important role. This is shown by Nemitz [18]. Motivated by this, Chan and Shum [10] established some elementary properties and constructed quotient structure of implicative semigroups via ordered filters. Jun [13], [14], Jun, Meng and Xin [15] and Jun and Kim [16] discussed ordered filters of implicative semigroups. Bang and So [1] analyzed some special substructures in implicative semigroups.

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In this paper, in the setting of Bishop's constructive mathematics, following the ideas of Chan and Shum and other authors mentioned above, we introduce the notion of implicative semigroups with (tight) apartness and give some fundamental characterization of these semigroups. Here, we use sets with apartness and anti-order relations introduced by the author (see, for example, [20] and [21]) (in this case, excise relation, researched by Barony [2], Greenleaf [12] and von Plato [23]), instead of partial order. So, in this research, we study side effects induced by existence of apartness and anti-orders. Additionally, we introduce the notion of anti-filter in an implicative semigroup and describe its connections with filter.

In the first part of Section 3, we introduce the notion of negatively anti-ordered semigroup and give some characteristics of these semigroups (Theorem 3.1). After that, we introduce the notion of implicative semigroup and give one example of such a semigroup. In the second part of Section 3, we describe fundamental characteristics of implicative semigroups. It is shown that any implicative semigroups has the greatest element. In the third part of Section 3, we introduce the notion of anti-filter in implicative semigroup and give some equivalent conditions that the inhabited real subset $G$ of $S$ is an ordered anti-filter (Theorem 3.7 and Theorem 3.8).

## 2. Preliminaries

This investigation is in Bishop's constructive algebra in the sense of papers [11, 20, 21, 22, 21] and books [3, 4, 5, 8, 9], [22](Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). The diversity relation "$\neq$" ([4]) is a binary relation on $S$, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z\ .$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z)),$$

then it is called apartness (A. Heyting). In this paper, we assume that the basic apartness is tight, i.e. it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

For a subset $X$ of $S$, we say that it is a strongly extensional subset of $S$ if and only if $(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S)$. Following Bridges and Vita's (see, for example, [9]) definition for subsets $X$ and $Y$ of $S$, we say that set $X$ is set-set apartness from $Y$, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$. We set $x \bowtie Y$, instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^C = \{x \in S : x \bowtie X\}$ we denote the apartness complement of $X$. For a function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$

we say that it is a strongly extensional if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b).$$

For relation $\alpha \subseteq S \times S$, we say that it is an anti-order relation on semigroup $S$, if it is consistent, cotransitive and linear

$$\alpha \subseteq \neq,\ \alpha \subseteq \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1},$$

where $\alpha$ has to be compatible with the semigroup operation in the following way

$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \Longrightarrow (x, y) \in \alpha).$$

Here, $*$ is the filed product between relations defined by the following way: If $\alpha$ and $\beta$ are relations on set $S$, then the filed product $\beta * \alpha$ of relation $\alpha$ and $\beta$ is the relation given by $\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in \alpha \vee (y, z) \in \beta)\}$.

For undefined notions and notations we refer to the following papers: [20, 21, 22, 23].

## 3. Implicative semigroups with apartness

3.1. **Definition and examples.** We recall some definitions and results. By a *negatively anti-ordered* semigroup (briefly, n.a-o. semigroup) we mean a set $S$ with an anti-order $\alpha$ and a binary operation '$\cdot$' (we will write $xy$ instead $x \cdot y$ ) such that for all $x, y, z \in S$, we have
(1) $(xy)z = x(yz)$,
(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and
(3) $(xy, x) \bowtie \alpha$ and $(xy, y) \bowtie \alpha$.
In that case for an anti-order $\alpha$ we will say that it is a *negative anti-order relation* on semigroup. The operation '$\cdot$' is extensional and strongly extensional function from $S \times S$ into $S$, i.e. it satisfies

$$(x, y) = (x', y') \Longrightarrow xy = x'y'$$
$$(xy \neq x'y \vee yx \neq yx') \Longrightarrow x \neq x'$$

for any elements $x, x', y$ of $S$.

Let us note that, in that case, we have
(3') $(x, xy) \bowtie \alpha^{-1}$ and $(y, xy) \bowtie \alpha^{-1}$ .
In fact, for $(v, u) \in \alpha^{-1}$ we have

$$(u, v) \in \alpha \Longrightarrow ((u, xy) \in \alpha \vee (xy, x) \in \alpha \vee (x, v) \in \alpha).$$

Thus, by (3), we have $u \neq xy$ or $x \neq v$. So, we have proved $(x, xy) \neq (v, u) \in \alpha^{-1}$. The second part of (3') we could prove analogously.

Let $\alpha$ be a relation on $S$. For an element $a$ of $S$ we put $a\alpha = \{x \in S : (a, x) \in \alpha\}$ and $\alpha a = \{x \in S : (x, a) \in \alpha\}$. In the following proposition we give some properties of negative anti-order relation on semigroups.

**Theorem 3.1.** *If $\alpha \subseteq S \times S$ is an anti-order relation on a semigroup $S$, then the following statements are equivalent:*
(i) *$\alpha$ is a negative anti-order relation;*
(ii) *$\alpha b$ is a consistent subset of $S$ for any $b$ in $S$;*
(iii) *$(\forall a, b \in S)(\alpha a \cup \alpha b \subseteq \alpha(ab))$;*
(iv) *$a\alpha$ is an ideal of $S$ for any $a$ in $S$;*
(v) *$(\forall a, b \in S)((ab)\alpha \subseteq a\alpha \cap b\alpha)$.*

*Proof.* (i) $\Longrightarrow$ (ii). Let $x, y, a, b$ be arbitrary elements of $S$ and let $\alpha$ be negative anti-order relation on semigroup $S$. If the product $xy$ lies in the set $\alpha b$, i.e. if $(xy, b) \in \alpha$ holds, then we have $(xy, x) \in \alpha \vee (x, b) \in \alpha$ and $(xy, y) \in \alpha \vee (y, b) \in \alpha$. Since $\alpha$ is n.a-o. relation on semigroup $S$, the first cases in these disjunctions are impossible. So, we have $x \in \alpha b$ and $y \in \alpha b$. Therefore, the set $\alpha b$ is a consistent subset of $S$.
(ii) $\Longrightarrow$ (iii). If $x \in \alpha a \cup \alpha b$, i.e. if $(x, a) \in \alpha \vee (x, b) \in \alpha$, then $(x, ab) \in \alpha \vee (ab, a) \in \alpha \vee (x, ab) \in \alpha \vee (ab, b) \in \alpha$ holds by cotransitivity of $\alpha$. Since $ab \in \alpha a$ and $ab \in \alpha b$ implies $a \in \alpha a$ and $b \in \alpha b$ respectively, and since this is forbidden, in both cases we have $x \in \alpha(ab)$.
(iii) $\Longrightarrow$ (i). Let $(u, v)$ be an arbitrary element of $\alpha$. Then we have $(u, xy) \in \alpha$ or $(xy, x) \in \alpha$ or $(x, v) \in \alpha$. Thus, $u \neq xy \vee xy \in \alpha x \subseteq \alpha x \cup \alpha y \subseteq \alpha(xy) \vee x \neq v$. Since the second case is imposible, we have $(xy, x) \neq (u, v) \in \alpha$. So, we have $(xy, x) \bowtie \alpha$. The proof of $(xy, y) \bowtie \alpha$ we obtain analogously. Therefore, the relation $\alpha$ is a negative anti-order relation on $S$.
(i) $\Longrightarrow$ (iv). Suppose that $x \in a\alpha$ or $y \in a\alpha$, i.e. assume that $(a, x) \in \alpha$ or $(a, y) \in \alpha$. Thus, we have $(a, xy) \in \alpha \vee (xy, x) \in \alpha$ and $(a, xy) \in \alpha \vee (xy, y) \in \alpha$. So, in both cases we have $xy \in a\alpha$. Therefore, the set $a\alpha$ is an ideal of semigroup $S$ for any element $a \in S$.
(iv) $\Longrightarrow$ (v). Let $x$ be an arbitrary element of $(ab)\alpha$, i.e. assume that $(ab, x) \in \alpha$. Thus follows $(ab, a) \in \alpha \vee (a, x) \in \alpha$ and $(ab, b) \in \alpha \vee (b, x) \in \alpha$. Since $\alpha$ is a negative anti-order relation on semigroup $S$, we have $x \in a\alpha$ and $x \in b\alpha$. So, finally we have $x \in a\alpha \cap b\alpha$.
(v) $\Longrightarrow$ (i). Suppose that the inclusion (v) holds for any two elements $a, b$ of semigroup $S$ and suppose that $(u, v)$ is an arbitrary element of $\alpha$. Because of cotransitivity of $\alpha$, we have $(u, xy) \in \alpha$ or $(xy, x) \in \alpha$ or $(x, v) \in \alpha$. In the second case we have $x \in (xy)\alpha \subseteq x\alpha \cap y\alpha \subseteq x\alpha$, which is impossible because $x \bowtie x\alpha$. So, it follows that $(xy, x) \neq (u, v) \in \alpha$ and we have $(xy, x) \bowtie \alpha$. The proof of $(xy, y) \bowtie \alpha$ is obtained analogously. Therefore, the relation $\alpha$ is a negative anti-order relation on semigroup $S$. $\square$

Let us note that for any $a$, $b$ in $S$ the following implication

$$(a, b) \in \alpha \Longrightarrow a\alpha \cup \alpha b = S$$

is valid.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that for any elements $x, y, z$ of $S$, the following is true
(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$.

Let us point out, as in the classical case, that in the definition of implicative semigroup we can make the stronger demand
(4') $(z, x \otimes y) \bowtie \alpha \Longleftrightarrow (zx, y) \bowtie \alpha$
instead of demand (4).

**Theorem 3.2.** (4) *implies* (4').

*Proof.* Assume that $(zx, y) \bowtie \alpha$ holds for any $x, y, z \in S$. Let $(u, v)$ be an arbitrary element of $\alpha$ . Then

$$\begin{aligned}(u, v) \in \alpha &\Longrightarrow (u, z) \in \alpha \vee (z, x \otimes y) \in \alpha \vee (x \otimes y, v) \in \alpha \\ &\Longrightarrow u \neq z \vee (z, x \otimes y) \in \alpha \vee x \otimes y \neq v \\ &\Longrightarrow u \neq z \vee (zx, y) \in \alpha \vee x \otimes y \neq v \\ &\Longrightarrow (z, x \otimes y) \neq (u, v) \in \alpha\ .\end{aligned}$$

Now we assume that $(z, x \otimes y) \bowtie \alpha$. From $(u, v) \in \alpha$ it follows that

$$\begin{aligned}(u, v) \in \alpha &\Longrightarrow ((u, zx) \in \alpha \vee (zx, y) \in \alpha \vee (y, v) \in \alpha) \\ &\Longrightarrow u \neq zx \vee (z, x \otimes y) \in \alpha \vee y \neq v \\ &\Longrightarrow (zx, y) \neq (u, v) \in \alpha.\end{aligned}$$

□

In addition, let us recall that the internal binary operation must satisfy the following implications:

$$(a, b) = (u, v) \Longrightarrow a \otimes b = u \otimes v,$$
$$a \otimes b \neq u \otimes v \Longrightarrow (a, b) \neq (u, v).$$

The operation $\otimes$ is called *implication.* From now on, an implicative n.a-o. semigroup is simply called *implicative semigroup.*

**Example 3.1.** Consider a set $S = \{1, a, b, c, d, 0\}$ with the operation tables

| $\cdot$ | *1* | *a* | *b* | *c* | *d* | *0* |
|---|---|---|---|---|---|---|
| *1* | *1* | *a* | *b* | *c* | *d* | *0* |
| *a* | *a* | *b* | *b* | *d* | *0* | *0* |
| *b* | *b* | *b* | *b* | *0* | *0* | *0* |
| *c* | *c* | *d* | *0* | *c* | *d* | *0* |
| *d* | *d* | *0* | *0* | *d* | *0* | *0* |
| *0* | *0* | *0* | *0* | *0* | *0* | *0* |

and

| $\otimes$ | 1 | a | b | c | d | 0 |
|---|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d | 0 |
| a | 1 | 1 | a | c | c | d |
| b | 1 | 1 | 1 | c | c | c |
| c | 1 | a | b | 1 | a | b |
| d | 1 | 1 | a | 1 | 1 | a |
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |

under the anti-order relation $\alpha = \{(a,0),(a,b),(a,c),(a,d),(b,0),(b,c),(b,d),(c,0),(c,a),(c,b),(c,d),(d,0),(d,b),(1,0),(1,a),(1,b),(1,c),(1,d)\}$. Then $S$ is an implicative semigroup.

3.2. **Important properties.** In any implicative semigroup $S$ there exist a special element of $S$, the biggest element in $(S, \alpha^C)$, which is an almost neutral element in $(S, \cdot)$.

**Theorem 3.3.** *In any inhabited implicative semigroup* $(S, =, \neq, \cdot, \alpha, \otimes)$ *we have*

$$x \otimes x = y \otimes y \text{ for every } x, y \in S$$

*and this element is the greatest element, written as* $1$, *of* $(S, \alpha^C)$.

*Proof.* By (3) we have $(tx, x) \bowtie \alpha$ for any $t, x$ in $S$. By definition (4), we have $(t, x \otimes x) \bowtie \alpha$. Thus, particularly for any elements $x$ and $y$ in $S$, we have $(y \otimes y, x \otimes x) \bowtie \alpha$ and $(x \otimes x, y \otimes y) \bowtie \alpha$. Therefore, we have $(x \otimes x, y \otimes y) \bowtie \alpha \cup \alpha^{-1} = \neq$ and $x \otimes x = y \otimes y$ since the apartness is tight. Finally, we conclude that the element $x \otimes x$, for any $x$ in $S$, is the greatest element in $(S, =, \neq, \alpha^C)$. The greatest element in $(S, =, \neq, \alpha^C)$ we denote by 1. So, for any element $t$ in $S$ we have $(t, 1) \bowtie \alpha$. $\square$

**Corollary 3.1.** *In semigroup* $(S, \cdot)$ *the following equation* $t = 1 \cdot t$ *holds.*

*Proof.* In the semigroup $(S, \cdot)$ holds $t = 1 \cdot t$. Indeed. For any element $t$ in $S$, by (3), we have $(1 \cdot t, t) \bowtie \alpha$. Additionally, let $(u, v)$ be an arbitrary element of $\alpha^{-1}$. Further on, we have $(v, u) \in \alpha \Longrightarrow v \neq 1 \cdot t \vee (1 \cdot t, t) \in \alpha \vee t \neq u$. Therefore, we have $(1 \cdot t, t) \neq (v, u) \in \alpha^{-1}$ because $(1 \cdot t, t) \bowtie \alpha$ holds. Finally, we have $(1 \cdot t, t) \bowtie \alpha \cup \alpha^{-1} = \neq$. $\square$

**Remark 3.1.** From (3) we immediately conclude $(t \cdot 1, t) \bowtie \alpha$ but we cannot conclude $t = t \cdot 1$. For that equation we also need $(t, t \cdot 1) \bowtie \alpha$. In fact, Chan and Shum in [10] demonstrate in Example 1.2 that the top element does not have to be a multiplicative unit.

In the following theorem we describe the role of this special element.

**Theorem 3.4.** *If* $S$ *is an implicative semigroup, then for every* $x, y \in S$ *holds*

$$(x, y) \bowtie \alpha \Longleftrightarrow 1 = x \otimes y$$
$$(x, y) \in \alpha \Longleftrightarrow 1 \neq x \otimes y.$$

*Proof.* (1) It is clear that the implication $1 = x \otimes y \Longrightarrow (x, y) \bowtie \alpha$ is valid. Indeed,

$$\begin{aligned} 1 = x \otimes y &\Longrightarrow (1, x \otimes y) \bowtie \alpha \cup \alpha^{-1} \\ &\Longrightarrow (1, x \otimes y) \bowtie \alpha \\ &\Longleftrightarrow (1 \cdot x, y) \bowtie \alpha \\ &\Longleftrightarrow (x, y) \bowtie \alpha. \end{aligned}$$

In the opposite, suppose $(x, y) \bowtie \alpha$, i.e. $(1, x \otimes y) \bowtie \alpha$. Since $(xy, 1) \bowtie \alpha$ holds for any $x, y$ in $S$, we have $(1, x \otimes y) \bowtie$ and $(1, x \otimes y) \bowtie \alpha^{-1}$. Thus, we have $(1, x \otimes y) \bowtie \alpha \cup \alpha^{-1} =\neq$. So, $1 = x \otimes y$.
(2) $(x, y) \in \alpha$, i.e. $(1 \cdot x, y) \in \alpha$ is equivalent to $(1, x \otimes y) \in \alpha$. Thus, we have $1 \neq x \otimes y$. Now, suppose that $1 \neq x \otimes y$. Hence, $(1, x \otimes y) \in \alpha$ or $(x \otimes y, 1) \in \alpha$. Thus, we have $(x, y) \in \alpha$ because the case $(x \otimes y, 1) \in \alpha$ is impossible by Theorem 3.3. $\square$

As a consequence of above theorem we immediately get the following corollary.

**Corollary 3.2.** *For any implicative semigroup $(S, =, \neq, \cdot, \alpha, \otimes)$, it holds that $1 = x \otimes 1$ for every element $x \in S$.*

As we saw, the greatest element 1 is a right annihilator for the semigroup operation $\otimes$. Further fundamental properties of this operation are given in the following statements.

**Theorem 3.5.** *Let $(S, =, \neq, \cdot, \alpha, \otimes)$ be an implicative semigroup. Then, for every $x, y, z \in S$, the following hold:*
(i) $(x, y \otimes (x \cdot y)) \bowtie \alpha$;
(ii) $(x, x \otimes x^2) \bowtie \alpha$;
(iii) $(x, y \otimes x) \bowtie \alpha$;
(iv) *If* $(z \otimes x, z \otimes y) \in \alpha$ *or* $(y \otimes z, x \otimes z) \in \alpha$ *then* $(x, y) \in \alpha$.

*Proof.* Since the proofs of parts (i), (ii) and (iii) are trivial, we will prove only part (iv).
1. From $(z \otimes x, z \otimes y) \in \alpha$ follows $((z \otimes x) \cdot z, y) \in \alpha$ and thus $((z \otimes x) \cdot z, x) \in \alpha$ or $(x, y) \in \alpha$. Since out of the first case we have $(z \otimes x, z \otimes x) \in \alpha$, which is impossible, we have to have $(x, y) \in \alpha$.
2.
$$\begin{aligned} (y \otimes z, x \otimes z) \in \alpha &\Longleftrightarrow ((y \otimes z) \cdot x, z) \in \alpha \\ &\Longrightarrow ((y \otimes z) \cdot x, (y \otimes z) \cdot y) \in \alpha \vee ((y \otimes z) \cdot y, z) \in \alpha \\ &\Longrightarrow (x, y) \in \alpha \vee (y \otimes z, y \otimes z) \in \alpha \\ &\Longrightarrow (x, y) \in \alpha. \end{aligned}$$
$\square$

As we saw from (iv) of the above theorem, the relation $\alpha$ is not compatible with the operation $\otimes$ on both sides. There exists only left compatibility and so-called right 'anti-compatibility' between the anti-order $\alpha$ and the operation $\otimes$.

**Theorem 3.6.** *Let $S$ be an implicative semigroup. Then for every $x, y, z \in S$, the following holds:*
(1) *If $(x, y) \bowtie \alpha$ then $(y \otimes z, x \otimes z) \bowtie \alpha$ and $(z \otimes x, z \otimes y) \bowtie \alpha$,*
(2) $x \otimes (y \otimes z) = (x \cdot y) \otimes z$,

*Proof.* The first assertion directly follows from (iv) of Theorem 3.5.
Let $u = x \otimes (y \otimes z)$ and $v = (x \cdot y) \otimes z$. Out of (1) we have $(u \cdot x, y \otimes z) \bowtie \alpha$ and $(u, x \otimes (y \otimes z)) \bowtie \alpha$. Thus, we have $(u \cdot (x \cdot y), z) = ((u \cdot x) \cdot y, z) \bowtie \alpha$. Hence $(u, (x \cdot y) \otimes z) \bowtie \alpha$. So, we have $(x \otimes (y \otimes z), (x \cdot y) \otimes z) \bowtie \alpha$. On the other hand, $(v \cdot x) \cdot y = v \cdot (x \cdot y)$ and $v = (x \cdot y) \otimes z$ implies $((v \cdot x) \cdot y, z) \bowtie \alpha$ and $(v \cdot x, y \otimes z) \bowtie \alpha$. Hence $(v, x \otimes (y \otimes z)) \bowtie \alpha$. Therefore, we have shown that $u = v$. □

As a corollary of above theorem we conclude the following:

**Corollary 3.3.** *For any implicative semigroup $(S, =, \neq, \cdot, \alpha, \otimes)$, it holds that $x = 1 \otimes x$ for every element $x \in S$.*

*Proof.* Firstly, we have $(x, 1 \otimes x) \bowtie \alpha$ for any $x$ in $S$ because $(x \cdot 1, x) = (x, x) \bowtie \alpha$ is valid for any $x$ in $S$. Secondly, from $x \otimes x = 1$ it follows that $(1 \cdot x) \otimes x = 1$. Thus, $(1 \otimes x) \otimes x = 1$ by assertion (2) in the above theorem. Thus, by Theorem 3.4, we have $((1 \otimes x), x) \bowtie \alpha$. Finally, we have $1 \otimes x = x$. □

As we saw the biggest element of the ordered semigroup $S$ is a left unity in $(S, \otimes)$.

3.3. **Important substructures.** In this subsection we will begin with the standard definition (Chan and Shum [[10], Definition 2.1]) of an ordered filter. Let $S$ be an implicative semigroup and let $F$ be a nonempty subset of $S$. Then $F$ is called an *ordered filter* of $S$ if
(F1) $xy \in F$ for every $x, y \in F$, that is, $F$ is a subsemigroup of $S$, and
(F2) If $x \in F$ and $(x, y) \bowtie \alpha$, then $y \in F$.
As we saw, condition (F1) supplies subset F with subsemigroup structure, and the condition (F2) implies that F is an upper set. As usual in the Constructive mathematics, we can introduce a special (inhabited) proper subset of implicative semigroup $S$ claiming that subset $G$ of $S$ satisfies the following conditions:
(G1) $xy \in G \Longrightarrow x \in G \vee y \in G$, that is, $G$ is a cosubsemigroup of $S$ and
(G2) $y \in G \Longrightarrow (x, y) \in \alpha \vee x \in G$.
This subset of $S$ we called *anti-filter*. It is easy to check that an anti-filter is a strongly extensional subset of $S$. Moreover, the strong compliment $G^C$ of

an anti-filter $G$ is a filter in $S$. In fact, strong compliment $G^C$ is obviously a subsemigroup of $S$. Assume that $x \bowtie G$ and $(x, y) \bowtie \alpha$ and let $u$ be an arbitrary element of $G$. Then, by strongly extensionality of $G$, we have $u \neq y$ or $y \in G$. In the second case we have $(x, y) \in \alpha \vee x \in G$. As both cases are impossible by hypothesis, we have $y \neq u \in G$. So, we have $y \in G^C$.

The following two theorems give equivalent conditions for $G$ to be an ordered anti-filter.

**Theorem 3.7.** *An inhabited proper subset $G$ of an implicative semigroup $S$ is an ordered anti-filter of $S$ if and only if it satisfies the following conditions:*
(i) $1 \bowtie G$;
(ii) $y \in G \Longrightarrow x \otimes y \in G \vee x \in G$.

*Proof.* $(i) \wedge (ii) \Longrightarrow (G2)$ Let conditions (i) and (ii) be valid and $y$ be an arbitrary element of subset $G$. Thus, by (ii), we have $x \otimes y \in G$ or $x \in G$. Since $1 \bowtie G$, we have $x \otimes y \neq 1$ and $(x, y) \in \alpha$ by Theorem 3.4. Therefore, the condition (G2) is true.
$(i) \wedge (ii) \Longrightarrow (G1)$ Let us note that for any elements $u$ and $v$ of $S$ the following holds $((u \otimes v) \cdot u, v) \bowtie \alpha$. Indeed, it is equivalent with $(u \otimes v, u \otimes v) \bowtie \alpha$ which is true for any elements $u, v$ of semigroup $S$. Now, we have
$xy \in G \Longrightarrow y \otimes (x \cdot y) \in G \vee y \in G$
$\Longrightarrow (x, y \otimes (x \cdot y)) \in \alpha \vee x \in G \vee y \in G$
$\Longrightarrow x \in G \vee y \in G.$
$(G1) \wedge (G2) \Longrightarrow (ii)$ Let $y$ be an element of $G$, Thus, we have
$y \in G \Longrightarrow ((x \otimes y) \cdot x, y) \in \alpha \vee x \in G$ (by (G2))
$\Longrightarrow (x \otimes y, x \otimes y) \in \alpha \vee x \otimes y \in G \vee x \in G$ (by (G1)
$\Longrightarrow x \otimes y \in G \vee x \in G.$
$(G1) \wedge (G2) \Longrightarrow (i)$ If we suppose that $1 \in G$ by (G2) we have $(x, 1) \in \alpha \vee x \in G$ for any element $x \in S$. Since the first case is impossible because it is equivalent to $1 = x \otimes 1 \neq 1$, we have to have $x \in G$. This means $S = G$. So, $\neg(1 \in G)$. Thus, by strongly extensionality of $G$ follows $1 \bowtie G$. □

**Theorem 3.8.** *An inhabited proper subset $G$ of an implicative semigroup $S$ is an ordered anti-filter of $S$ if and only if it satisfies the following condition:*
(G3) $z \in G \Longrightarrow (x, y \otimes z) \in \alpha \vee x \in G \vee y \in G$.

*Proof.* (1) Assume that $G$ is an ordered anti-filter of implicative semigroup $S$ and let $x, y, z$ be elements of $S$. We have:
$z \in G \Longrightarrow y \otimes z \in G \vee y \in G$ (by (ii))
$\Longrightarrow (x, y \otimes z) \in \alpha \vee x \in G \vee y \in G$ (by (G2)).
(2) Conversely, suppose that $G$ satisfies the condition (G3). First, for the proof of (G2) we have:
$xy \in G \Longrightarrow (x, y \otimes xy) \in \alpha \vee x \in G \vee y \in G$

$$\Longrightarrow (x \otimes y, xy) \in \alpha \vee x \in G \vee y \in G$$
$$\Longrightarrow ((x \otimes y) \cdot x, y) \in \alpha \vee x \in G \vee y \in G$$
$$\Longrightarrow x \in G \vee y \in G.$$

In order to prove (G1) we have:

$$y \in G \Longrightarrow (x, x \otimes y) \in \alpha \vee x \in G$$
$$\Longrightarrow (x, y) \in \alpha \vee (y, x \otimes y) \in \alpha \vee x \in G$$
$$\Longrightarrow (x, y) \in \alpha \vee (yx, y) \in \alpha \vee x \in G$$
$$\Longrightarrow (x, y) \in \alpha \vee x \in G$$

by (3) in the definition of negatively anti-ordered semigroup. □

**Example 3.2.** Consider a set $S = \{1, a, b, c, d, 0\}$ with the operation tables as in the above example. It is not so hard to check that set $G = \{c, d, 0\}$ is an ordered anti-filter of implicative semigroup $S$.

Let $S$ be an implicative semigroup. For any $a \in S$, we define $G(a) = \{x \in S : (a, x) \in \alpha\}$, (i.e. $G(a)$ is the left class of relation $\alpha$ generated by the element $a$). Clearly that $1 \bowtie G(a)$ and $a \bowtie G(a)$. Generally speaking, $G(a)$ is not an ordered anti-filter in $S$. Let $S$ be an implicative semigroup in Example 1. Then $G(c) = \{0, a, b, d\}$ is an ordered anti-filter in $S$, but the set $G(a) = \{0, b, c, d\}$ is not an ordered anti-filter in $S$ because, for example, holds $b \in G(a)$, $a \bowtie G(a)$ and $a \otimes b = a \bowtie G(a)$. Using Theorem 3.7 we can give one condition for the set $G(t)$ $(t \in S)$ to be an order anti-filter in $S$: The set $G(t)$ is an ordered anti-filter in an implicative semigroup $S$ if and only if the following condition

$$(t, y) \in \alpha \Longrightarrow (t, x \otimes y) \in \alpha \vee (t, x) \in \alpha$$

is true for all $x, y \in S$.

## References


[1] K. Bang and K. S. So, *Extended special sets in implicative semigroups*, Commun. Korean Math. Soc., 22 (1) (2007), 9–14.

[2] M. A. Baroni, *Constructive Order Completeness*, Report UCDMS 2004/12, University of Canterbury, Christchurch, New Zealand.

[3] M. J. Beeson, *Foundations of Constructive Mathematics*, Springer, Berlin, 1985.

[4] E. Bishop, *Foundations of Constructive Analysis*, McGraw-Hill, New York, 1967.

[5] E. Bishop and D. S. Bridges, *Constructive Analysis*, Grundlehren der mathematischen Wissenschaften 279, Springer, Berlin, 1985.

[6] G. Birkhoff, *Lattice Theory*, 3rd ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Rhode Island, 1967.

[7] T. S. Blyth, *Pseudo-residuals in semigroups*, J. London Math. Soc., 40 (1965), 441–454.

[8] D. S. Bridges and F. Richman, *Varieties of Constructive Mathematics*, London Mathematical Society Lecture Notes, No. 97, Cambridge University Press, Cambridge, 1987.

[9] D. S. Bridges and L. S. Vita, *Techniques of Constructive Analysis*, Springer, New York 2006.

[10] M. W. Chan and K. P. Shum, *Homomorphisms of implicative semigroups*, Semigroup Forum, 46 (1) (1993), 7–15.

[11] S. Crvenković, M. Mitrović and D. A. Romano, *Semigroups with Apartness*, Math. Logic Quart., 59 (6) (2013), 407–414.

[12] N. Greenleaf, *Linear Order in Lattices: A Constructive Study*, In: Rota G-C. (Ed.), Advances in Mathematics Supplementary Studies, (pp. 11-30), Academic Press, New York, 1978.

[13] Y. B. Jun, *A note on ordered filters of implicative semigroups*, Bull. Korean Math. Soc., 34 (2) (1997), 185–191.

[14] Y. B. Jun, *Some results on ordered filters of implicative semigroups*, Int. J. Math. Math. Sci., 26 (12) (2001), 731–735.

[15] Y. B. Jun, J. Meng and X. L. Xin, *On ordered filters of implicative semigroups*, Semigroup Forum, 54 (1) (1997), 75–82.

[16] Y. B. Jun and K. H. Kim, *On ideals of implicative semigroups*, Int. J. Math. Math. Sci., 27 (2) (2001), 77–82.

[17] M. Mandelkern, *Constructive complete finite sets*, Math. Logic Quart., (Formely: Zeitschrift fur Mathematische Logik und Grundlagen der Mathematik), 34 (1988), 97–103.

[18] W. C. Nemitz, *Implicative semi-lattices*, Trans. Amer. Math. Soc., 117 (1965), 128–142.

[19] S. Negri. *Sequent calculus proof theory of Intuitionistic apartness and order relations*, Arch. Math. Logic, 38 (1999), 521–547.

[20] D. A. Romano, *A Note on quasi-antiorder in semigroup*, Novi Sad J. Math., 37 (1) (2007), 3–8.

[21] D. A. Romano, *An isomorphism theorem for anti-ordered sets*, Filomat, 22 (1) (2008), 145–160.

[22] D. A. Romano, *On semilattice-ordered semigroups. A constructive point of view*, Sci. Stud. Res., Ser. Math. Inform., 21 (2) (2011), 117–134.

[23] D. A. Romano, *On Quasi-antiorder relation on semigroups*, Mat. Vesn., 64 (3) (2012), 190–199.

[24] A. S. Troelstra and D. van Dalen, *Constructivism in Mathematics: An Introduction*, North-Holland, Amsterdam, 1988.

[25] J. von Plato, *Positive lattices*, In: P.Schuster, U.Berger and H.Osswald (Eds.). *Re-uniting the Antipodes-Constructive and Nonstandard Views of the Continuum*, (pp. 185-197), Kluwer Academic Publishers, Dordrecht, 2001.


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