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ON THE SPACES OF FIBONACCI DIFFERENCE ABSOLUTELY p-SUMMABLE, NULL AND CONVERGENT SEQUENCES

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Abstract. Let $0 < p < 1$. In the present paper, as the domain of the band matrix $\widehat{F}$ defined by the Fibonacci sequence in the classical sequence spaces ℓ_p, c_0 and c, we introduce the sequence spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$, respectively. Also, we give some inclusion relations and construct the bases of the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$. Finally, we compute the alpha, beta, gamma duals of these spaces and characterize the classes $(\ell_p(\widehat{F}), \mu)$ of infinite matrices with $\mu \in \{\ell_\infty, c, c_0\}$.

1. Introduction

By $\mathbb{N}$ and $\mathbb{R}$, we denote the sets of all natural and real numbers, respectively. Let ω be the vector space of all real sequences. Any vector subspace of ω is called a *sequence space*. Let ℓ_∞, c, c_0 and ℓ_p denote the classes of all bounded, convergent, null and absolutely p-summable sequences, respectively; where $0 < p < \infty$. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. Also, we use the conventions that $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the n^{th} place for each $n \in \mathbb{N}$.

Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the A-transform $Ax = \{A_n(x)\}$ of x is in μ, where

$$A_n(x) = \sum_k a_{nk} x_k \quad \text{for each} \quad n \in \mathbb{N}. \tag{1.1}$$

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For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By (λ,μ), we denote the class of all matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda,\mu)$ if and only if the series on the right side of (1.1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax\in\mu$ for all $x\in\lambda$. Also, we write $A_n=(a_{nk})_{k\in\mathbb{N}}$ for the sequence in the n-th row of A.

The *domain* λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A:=\{x=(x_k)\in\omega : Ax\in\lambda\} \tag{1.2}$$

which is a sequence space. Depending on the choice of the matrix A, λ_A may include the original space λ or λ_A may be included by the original space λ and sometimes λ_A may be identical to the space λ. Indeed if we choose $A=\Delta$, the backward difference matrix, then $c_\Delta\supset c$ but in the case $A=\Delta^{-1}=S$, the summation matrix, $c_S=cs\supset c$, where both of two inclusions are proper. However, if we define $\lambda=c_0\oplus span\{z\}$ with $z=\{(-1)^k\}$, i.e., $x\in\lambda$ if and only if $x=s+\alpha z$ for some $s\in c_0$ and some $\alpha\in\mathbb{C}$, and consider the matrix A with the rows A_n defined by $A_n=(-1)^n e^{(n)}$ for all $n\in\mathbb{N}$, we have $Ae=z\in\lambda$ but $Az=e\notin\lambda$ which gives that $z\in\lambda\setminus\lambda_A$ and $e\in\lambda_A\setminus\lambda$. That is to say that the sequence spaces λ_A and λ overlap but neither contains the other.

By using the domain of a triangle infinite matrix, many sequence spaces have recently been defined by several authors, see for instance [1, 28, 2, 7]. In the literature, the matrix domain λ_Δ is called the *difference sequence space* whenever λ is a normed or paranormed sequence space, where $\Delta=(d_{nk})$ denotes the backward difference matrix defined by

$$d_{nk}=\begin{cases}(-1)^{n-k} & , \quad n-1\le k\le n,\\ 0 & , \quad 0\le k<n-1 \text{ or } k>n\end{cases}$$

for all $k,n\in\mathbb{N}$. The notion of difference sequence spaces was introduced by Kızmaz [23], who defined the sequence spaces

$$X(\Delta)=\{x=(x_k)\in\omega : (x_k-x_{k+1})\in X\}$$

for $X=\ell_\infty$, c and c_0. The difference space bv_p, consisting of all sequences (x_k) such that (x_k-x_{k-1}) is in the sequence space ℓ_p, was studied in the case $0<p<1$ by Altay and Başar [4] and in the case $1\le p\le\infty$ by Başar and Altay [8], and Çolak et al. [12]. Kirişçi and Başar [24] have introduced and studied the generalized difference sequence spaces

$$\widehat{X}=\{x=(x_k)\in\omega : B(r,s)x\in X\},$$

where X denotes any of the spaces ℓ_∞, c, c_0 and ℓ_p, and $B(r,s)x=(sx_{k-1}+rx_k)$ with $r,s\in\mathbb{R}\setminus\{0\}$, and $1\le p<\infty$. Following Kirişçi and Başar [24], Sönmez [33] have examined the sequence space $X(B)$ as the set of all sequences whose $B(r,s,t)$-transforms $B(r,s,t)x = (rx_k + sx_{k+1} + tx_{k+2})$ are in the space $X \in \{\ell_\infty, \ell_p, c, c_0\}$ where $r, s, t \in \mathbb{R} \setminus \{0\}$. Also in [14, 32, 29, 18, 9, 34, 17], authors studied certain difference sequence spaces. Furthermore, quite recently, Kara [21] has defined the Fibonacci difference matrix $\widehat{F}$ by means of the Fibonacci sequence $(f_n)_{n\in\mathbb{N}}$ and introduced the new sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ which are derived by the matrix domain of $\widehat{F}$ in the sequence spaces ℓ_p and ℓ_∞, respectively; where $1 \leq p < \infty$.

In this paper, we introduce the sequence spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ by using the Fibonacci difference matrix $\widehat{F}$, where $0 < p < 1$. The rest of this paper is organized, as follows:

In Section 2, we give some notations and basic concepts. In Section 3, we introduce the sequence spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$, and establish some inclusion relations. Also, we construct the bases of these spaces. In Section 4, the alpha, beta and gamma duals of the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ are determined, and the classes $(\ell_p(\widehat{F}), \ell_\infty)$, $(\ell_p(\widehat{F}), c)$ and $(\ell_p(\widehat{F}), c_0)$ of matrix transformations are characterized.

2. Preliminaries

A sequence space λ is called an FK-space if it is a complete linear metric space with continuous coordinates $p_n : \lambda \to \mathbb{C}$ with $p_n(x) = x_n$ for all $x = (x_n) \in \lambda$ and every $n \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A normed FK-space is called a BK-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space ℓ_p is a BK-space with $\|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}$ and c_0, c and ℓ_∞ are BK-spaces with $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$, where $1 \leq p < \infty$. The sequence space λ is said to be *solid* (cf. [20, p. 48]) if and only if

$$\widetilde{\lambda} := \{(u_k) \in \omega : \exists (x_k) \in \lambda \text{ such that } |u_k| \leq |x_k| \ \text{ for all } \ k \in \mathbb{N}\} \subset \lambda.$$

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\|x - \sum_{n=0}^m \alpha_n b_n\| \to 0$, as $m \to \infty$.

The alpha, beta and gamma duals λ^α, λ^β and λ^γ of a sequence space λ are respectively defined by

$$\lambda^\alpha := \{a = (a_k) \in \omega : ax = (a_k x_k) \in \ell_1 \text{ for all } x = (x_k) \in \lambda\},$$
$$\lambda^\beta := \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in \lambda\},$$
$$\lambda^\gamma := \{a = (a_k) \in \omega : ax = (a_k x_k) \in bs \text{ for all } x = (x_k) \in \lambda\}.$$

The sequence (f_n) of Fibonacci numbers is defined by the linear recurrence equalities

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ with } n \geq 2.$$

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converge to the golden ratio which is important in sciences and arts. Some basic properties of sequences of Fibonacci numbers are also given, below (cf. Koshy [25]):

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \varphi \text{ (Golden Ratio)}, \tag{2.1}$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \text{ for each } n \in \mathbb{N},$$

$$\sum_k \frac{1}{f_k} \text{ converges},$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \text{ for all } n \geq 1 \text{ (Cassini Formula)}.$$

One can easily derive by substituting f_{n+1} in Cassini's formula that $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

Now, let $A = (a_{nk})$ be an infinite matrix and $\mathcal{F}$ be the collection of all finite subsets of $\mathbb{N}$. We list the following conditions:

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty \tag{2.2}$$

$$\lim_{n\to\infty} a_{nk} = 0 \text{ for each } k \in \mathbb{N} \tag{2.3}$$

$$\exists \alpha_k \in \mathbb{R} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \text{ for each } k \in \mathbb{N} \tag{2.4}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{2.5}$$

$$\exists \alpha \in \mathbb{R} \ni \lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{2.6}$$

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} \right| < \infty \tag{2.7}$$

$$\sup_{k,n\in\mathbb{N}} |a_{nk}| < \infty \tag{2.8}$$

Now, we can give the following lemma on the characterization of the classes of the matrix transformations between some classical sequence spaces.

Lemma 2.1. *The following statements hold:*

(a) $A = (a_{nk}) \in (c_0, c_0)$ *if and only if* (2.2) *and* (2.3) *hold.*
(b) $A = (a_{nk}) \in (c_0, c)$ *if and only if* (2.2) *and* (2.4) *hold.*
(c) $A = (a_{nk}) \in (c, c_0)$ *if and only if* (2.2), (2.3) *and* (2.5) *hold.*
(d) $A = (a_{nk}) \in (c, c)$ *if and only if* (2.2), (2.4) *and* (2.6) *hold.*
(e) $A = (a_{nk}) \in (c_0, \ell_\infty) = (c, \ell_\infty)$ *if and only if the condition* (2.2) *holds.*
(f) $A = (a_{nk}) \in (c_0, \ell_1) = (c, \ell_1)$ *if and only if the condition* (2.7) *holds.*
(g) $A = (a_{nk}) \in (\ell_p, c)$ *if and only if* (2.4) *and* (2.8) *hold, where* $0 < p < 1$.
(h) $A = (a_{nk}) \in (\ell_p, \ell_\infty)$ *if and only if the condition* (2.8) *holds, where* $0 < p < 1$.

3. The Fibonacci difference spaces of absolutely p-summable, null and convergent sequences

In this section, we define the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ of Fibonacci absolutely p-summable, Fibonacci null and Fibonacci convergent sequences, where $0 < p < 1$. Also, we present some inclusion theorems and construct the Schauder bases of the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$.

Recently, Kara [21] has defined the sequence space $\ell_p(\widehat{F})$ as follows:

$$\ell_p(\widehat{F}) = \left\{ x \in \omega : \widehat{F}x \in \ell_p \right\}, \ (1 \leq p \leq \infty),$$

where $\widehat{F} = (\widehat{f}_{nk})$ is the double band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$\widehat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ \frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Also, in [22], Kara et al. have characterized some classes of compact operators on the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, where $1 \leq p < \infty$.

One can derive by a straightforward calculation that the inverse $\widehat{F}^{-1} = (g_{nk})$ of the Fibonacci matrix $\widehat{F}$ is given by

$$g_{nk} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Now, we introduce the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ as the set of all sequences whose $\widehat{F}$-transforms are in the spaces

ℓ_p, c_0 and c, respectively, i.e.,

$$\ell_p(\widehat{F}) := \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\}, \ (0 < p < 1),$$

$$c_0(\widehat{F}) := \left\{ x = (x_n) \in \omega : \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\},$$

$$c(\widehat{F}) := \left\{ x = (x_n) \in \omega : \exists l \in \mathbb{R} \ni \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = l \right\}.$$

With the notation of (1.2), the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ can be redefined as follows:

$$\ell_p(\widehat{F}) = (\ell_p)_{\widehat{F}}, \quad c_0(\widehat{F}) = (c_0)_{\widehat{F}} \quad \text{and} \quad c(\widehat{F}) = c_{\widehat{F}}.$$

Here and after, we assume unless stated otherwise that $0 < p < 1$ and all the terms with negative subscript are equal to zero.

Define the sequence $y = (y_k)$ by the $\widehat{F}$-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = \widehat{F}_k(x) = \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \tag{3.1}$$

for all $k \in \mathbb{N}$. Therefore, one can derive by a straightforward calculation that

$$x_k = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \quad \text{for all} \quad k \in \mathbb{N}. \tag{3.2}$$

Throughout the text, we suppose that the sequences $x = (x_k)$ and $y = (y_k)$ are connected with the relation (3.1).

Theorem 3.1. *The following statements hold:*

(i) *The sets $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ are the linear spaces with the coordinatewise addition and scalar multiplication.*

(ii) *$\ell_p(\widehat{F})$ is a complete p-normed space with the p-norm $\|x\|_p = \sum_n \left|\widehat{F}_n(x)\right|^p$.*

(iii) *$c_0(\widehat{F})$ and $c(\widehat{F})$ are the BK-spaces with the norm $\|x\|_{c_0(\widehat{F})} = \|x\|_{c(\widehat{F})} = \|\widehat{F}x\|_\infty$.*

Proof. This is a routine verification and so we omit the detail. □

Remark 3.2. One can easily check that the absolute property does not hold on the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$, that is $\|x\|_p \neq \||x|\|_p$, $\|x\|_{c_0(\widehat{F})} \neq \||x|\|_{c_0(\widehat{F})}$ and $\|x\|_{c(\widehat{F})} \neq \||x|\|_{c(\widehat{F})}$ for at least one sequence in the spaces

$\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$, and this tells us that $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$.

Let λ denotes any of the spaces ℓ_p, c_0 and c. With the notation of (3.1), since the transformation $T : \lambda(\widehat{F}) \to \lambda$ defined by $x \mapsto y = Tx = \widehat{F}x$ is a norm preserving linear bijection, we have the following:

Corollary 3.3. *The Fibonacci difference sequence spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ of non-absolute type are linearly p-norm/norm isomorphic to the spaces ℓ_p, c_0 and c, respectively, i.e., $\ell_p(\widehat{F}) \cong \ell_p$, $c_0(\widehat{F}) \cong c_0$ and $c(\widehat{F}) \cong c$.*

Now, we give some inclusion relations concerned with the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$.

Theorem 3.4. *The inclusion $c_0(\widehat{F}) \subset c(\widehat{F})$ strictly holds.*

Proof. It is clear that the inclusion $c_0(\widehat{F}) \subset c(\widehat{F})$ holds. Further, to show the strictness of the inclusion, consider the sequence $x = (x_k) = \left(\sum_{j=0}^{k} f_{k+1}^2 / f_j^2\right)$. Then, we obtain by (3.1) for all $k \in \mathbb{N}$ that

$$\widehat{F}_k(x) = \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j^2} - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_k^2}{f_j^2} = \frac{f_{k+1}}{f_k}$$

which tends to φ, as $k \to \infty$ by (2.1). That is to say that $\widehat{F}x \in c \setminus c_0$. Thus, the sequence x is in $c(\widehat{F})$ but is not in $c_0(\widehat{F})$. Hence, the inclusion $c_0(\widehat{F}) \subset c(\widehat{F})$ is strict. □

Theorem 3.5. *The space ℓ_∞ does not include the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$.*

Proof. Let us consider the sequence $x = (x_k) = (f_{k+1}^2)$. Since $f_{k+1}^2 \to \infty$, as $k \to \infty$ and $\widehat{F}x = e^{(0)} = (1, 0, 0, \ldots)$; the sequence x is in the space $c_0(\widehat{F})$ but is not in the space ℓ_∞. This shows that the space ℓ_∞ does not include both the space $c_0(\widehat{F})$ and the space $c(\widehat{F})$, as desired. □

Theorem 3.6. *The inclusions $c_0 \subset c_0(\widehat{F})$ and $c \subset c(\widehat{F})$ strictly hold.*

Proof. Let $\lambda = c_0$ or c. Since the matrix $\widehat{F} = (\widehat{f}_{nk})$ satisfies the conditions

$$\sup_{n\in\mathbb{N}} \sum_k \left|\widehat{f}_{nk}\right| = \sup_{n\in\mathbb{N}} \left(\frac{f_n}{f_{n+1}} + \frac{f_{n+1}}{f_n}\right) = 2 + \frac{1}{2} = \frac{5}{2},$$

$$\lim_{n\to\infty} \widehat{f}_{nk} = 0,$$

$$\lim_{n\to\infty} \sum_k \widehat{f}_{nk} = \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} - \frac{f_{n+1}}{f_n}\right) = \frac{1}{\varphi} - \varphi$$

we conclude by Parts (a) and (d) of Lemma 2.1 that $\widehat{F} \in (\lambda, \lambda)$. This leads to the fact that $\widehat{F}x \in \lambda$ for any $x \in \lambda$. Thus, $x \in \lambda(\widehat{F})$. This shows that $\lambda \subset \lambda(\widehat{F})$.

Now, let $x = (x_k) = (f_{k+1}^2)$. Then, it is clear that $x \in \lambda(\widehat{F}) \setminus \lambda$. This says that the inclusion $\lambda \subset \lambda(\widehat{F})$ is strict. □

Theorem 3.7. *The spaces $c_0(\widehat{F})$ and $c(\widehat{F})$ are not solid.*

Proof. Consider the sequences $u = (u_k)$ and $v = (v_k)$ defined by $u_k = f_{k+1}^2$ and $v_k = (-1)^{k+1}$ for all $k \in \mathbb{N}$. Then, it is clear that $u \in c_0(\widehat{F})$ and $v \in \ell_\infty$. Nevertheless $uv = \{(-1)^{k+1} f_{k+1}^2\}$ is not in the space $c_0(\widehat{F})$, since

$$\widehat{F}_k(uv) = \frac{f_k}{f_{k+1}}(-1)^{k+1} f_{k+1}^2 - \frac{f_{k+1}}{f_k}(-1)^k f_k^2 = 2(-1)^{k+1} f_k f_{k+1}$$

for all $k \in \mathbb{N}$. This shows that the multiplication $\ell_\infty c_0(\widehat{F})$ of the spaces ℓ_∞ and $c_0(\widehat{F})$ is not a subset of $c_0(\widehat{F})$. Hence, the space $c_0(\widehat{F})$ is not solid.

It is clear here that if the space $c_0(\widehat{F})$ is replaced by the space $c(\widehat{F})$, then we obtain the fact that $c(\widehat{F})$ is not solid. This completes the proof. □

It is known from Theorem 2.3 of Jarrah and Malkowsky [19] that if T is a triangle then the domain λ_T of T in a normed sequence space λ has a basis if and only if λ has a basis. As a direct consequence of this fact, we have:

Corollary 3.8. *Define the sequences $c^{(-1)} = \{c_k^{(-1)}\}_{k \in \mathbb{N}}$ and $c^{(n)} = \{c_k^{(n)}\}_{k \in \mathbb{N}}$ for every fixed $n \in \mathbb{N}$ by*

$$c_k^{(-1)} = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} \quad \text{and} \quad c_k^{(n)} := \begin{cases} 0 & , \quad 0 \leq k \leq n-1, \\ \frac{f_{k+1}^2}{f_n f_{n+1}} & , \quad k \geq n. \end{cases}$$

Then, the following statements hold:

(a) *The sequence $\{c^{(n)}\}_{n=0}^{\infty}$ is a basis for the spaces $\ell_p(\widehat{F})$ and $c_0(\widehat{F})$, and every sequence $x \in c_0(\widehat{F})$ or in the space $\ell_p(\widehat{F})$, has a unique representation of the form $x = \sum_n \widehat{F}_n(x) c^{(n)}$.*

(b) *The sequence $\{c^{(n)}\}_{n=-1}^{\infty}$ is a basis for the space $c(\widehat{F})$ and every sequence $z = (z_n) \in c(\widehat{F})$ has a unique representation of the form $z = l c^{(-1)} + \sum_n \left[\widehat{F}_n(z) - l\right] c^{(n)}$, where $l = \lim_{n \to \infty} \widehat{F}_n(z)$.*

4. The alpha, beta and gamma duals of the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$, and some matrix transformations

In this section, we determine the alpha, beta and gamma duals of the spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$, and characterize the classes of infinite matrices from the spaces $\ell_p(\widehat{F})$ to the spaces ℓ_∞, c and c_0.

Theorem 4.1. *The alpha dual of the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$ is the set*

$$d_1 := \left\{ a = (a_k) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} b_{nk} \right| < \infty \right\},$$

where the matrix $B = (b_{nk})$ is defined by

$$b_{nk} := \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$.

Proof. Let $a = (a_n) \in \omega$. Consider the equality

$$a_n x_n = \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = \sum_{k=0}^{n} b_{nk} y_k = B_n(y). \tag{4.1}$$

By (4.1), we obtain that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_k) \in c_0(\widehat{F})$ or $c(\widehat{F})$ if and only if $By \in \ell_1$ whenever $y = (y_k) \in c_0$ or c. That is, the sequence $a = (a_n)$ is in the alpha dual of the space $c_0(\widehat{F})$ or $c(\widehat{F})$ if and only if $B \in (c_0, \ell_1) = (c, \ell_1)$. By taking B instead of A in part (f) of Lemma 2.1, we obtain that $a \in \left[c_0(\widehat{F})\right]^\alpha = \left[c(\widehat{F})\right]^\alpha$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} b_{nk} \right| < \infty$$

which means that $\left[c_0(\widehat{F})\right]^\alpha = \left[c(\widehat{F})\right]^\alpha = d_1$. □

Theorem 4.2. *Define the sets d_2, d_3, d_4 and d_5, as follows:*

$$d_2 := \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| < \infty \right\},$$

$$d_3 := \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$d_4 := \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \text{ exists} \right\},$$

$$d_5 := \left\{ a = (a_k) \in \omega : \sup_{k,n \in \mathbb{N}} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| < \infty \right\}.$$

Then, the following statements hold:

(a) $\left[c_0(\widehat{F})\right]^\alpha = \left[c(\widehat{F})\right]^\alpha = d_1$.

(b) $\left[c_0(\widehat{F})\right]^\beta = d_2 \cap d_3$.

(c) $\left[c(\widehat{F})\right]^\beta = d_2 \cap d_3 \cap d_4$.

(d) $\left[\ell_p(\widehat{F})\right]^\beta = d_3 \cap d_5$.

(e) $\left[c_0(\widehat{F})\right]^\gamma = \left[c(\widehat{F})\right]^\gamma = d_2$.

(f) $\left[\ell_p(\widehat{F})\right]^\gamma = d_5$.

Proof. Consider the equality

$$\sum_{k=0}^{n} a_k x_k = \sum_{k=0}^{n} a_k \left(\sum_{i=0}^{k} \frac{f_{k+1}^2}{f_i f_{i+1}} y_i \right) = \sum_{k=0}^{n} \left(\sum_{i=k}^{n} \frac{f_{i+1}^2}{f_k f_{k+1}} a_i \right) y_k = C_n(y), \tag{4.2}$$

where $C = (c_{nk})$ defined by

$$c_{nk} := \begin{cases} \sum_{i=k}^{n} \frac{f_{i+1}^2}{f_k f_{k+1}} a_i & , \quad 0 \le k \le n \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then, we observe by (4.2) that $ax = (a_n x_n) \in cs$ whenever $x \in c_0(\widehat{F})$ if and only if $Cy \in c$ whenever $y = (y_k) \in c_0$ which means that $a \in \left[c_0(\widehat{F})\right]^\beta$ if and only if $C \in (c_0, c)$. By using Part (b) of Lemma 2.1, we derive that

$$\sup_{n \in \mathbb{N}} \sum_k \left| \sum_{i=k}^{n} \frac{f_{i+1}^2}{f_k f_{k+1}} a_i \right| < \infty,$$

$$\lim_{n \to \infty} \sum_{i=k}^{n} \frac{f_{i+1}^2}{f_k f_{k+1}} a_i \text{ exists for each } k \in \mathbb{N}.$$

Hence, we conclude that $\left[c_0(\widehat{F})\right]^\beta = d_2 \cap d_3$.

In a similar way, one can easily show the facts about the beta duals of the spaces $c(\widehat{F})$, $\ell_p(\widehat{F})$ and gamma duals of the spaces $c_0(\widehat{F})$, $c(\widehat{F})$, $\ell_p(\widehat{F})$. □

Now, we give the theorems characterizing the classes $(\ell_p(\widehat{F}) : \ell_\infty)$, $(\ell_p(\widehat{F}) : c)$ and $(\ell_p(\widehat{F}) : c_0)$ of infinite matrices.

Theorem 4.3. *$A = (a_{nk}) \in (\ell_p(\widehat{F}) : \ell_\infty)$ if and only if*

$$\sup_{k,n \in \mathbb{N}} \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| < \infty. \tag{4.3}$$

Proof. $A = (a_{nk}) \in (\ell_p(\widehat{F}) : \ell_\infty)$ and $x = (x_k) \in \ell_p(\widehat{F})$. Then, Ax exists and belongs to the space ℓ_∞. Therefore, one can conclude for $x^{(k)} = \left\{x_j^{(k)}\right\} \in \ell_p(\widehat{F})$ defined by

$$x_j^{(k)} := \begin{cases} f_{j+1}^2/f_k f_{k+1} & , \quad j \geq k \\ 0 & , \quad 0 \leq j \leq k-1 \end{cases} \tag{4.4}$$

for all $j, k \in \mathbb{N}$ that $Ax^{(k)} = \left(\sum_{j=k}^{\infty} f_{j+1}^2 a_{nj}/f_k f_{k+1}\right) \in \ell_\infty$ for each $k \in \mathbb{N}$. Hence, the condition (4.3) is necessary.

Conversely, suppose that (4.3) holds and take any $x = (x_k) \in \ell_p(\widehat{F})$. Then, $A_n \in [\ell_p(\widehat{F})]^\beta$ for each $n \in \mathbb{N}$ which leads to the existence of Ax. Let $n \in \mathbb{N}$ be fixed. Consider the following relation derived from the m^{th} partial sum of the series $\sum_k a_{nk} x_k$ with (3.2):

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \tag{4.5}$$

for all $m, n \in \mathbb{N}$. Then, by letting $m \to \infty$ in (4.5) we have

$$A_n(x) = \sum_k a_{nk} x_k = \sum_k e_{nk} y_k = E_n(y) \tag{4.6}$$

for all $n \in \mathbb{N}$, where the matrix $E = (e_{nk})$ is defined by

$$e_{nk} = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \tag{4.7}$$

for all $k, n \in \mathbb{N}$. Therefore, since

$$\begin{aligned} \left|\sum_k a_{nk} x_k\right|^p &= \left|\sum_k \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k\right|^p \\ &\leq \left(\sum_k \left|\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}\right| |y_k|\right)^p \\ &\leq \left(\sup_{k \in \mathbb{N}} \left|\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}\right|\right)^p \left(\sum_k |y_k|\right)^p \\ &\leq \left(\sup_{k \in \mathbb{N}} \left|\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}\right|\right)^p \sum_k |y_k|^p \end{aligned}$$

we obtain by taking supremum over $n \in \mathbb{N}$ that

$$\|Ax\|_\infty = \sup_{n\in\mathbb{N}} \left| \sum_k a_{nk}x_k \right| \leq \left(\|y\|_p \right)^{1/p} \sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| < \infty.$$

That is, $Ax \in \ell_\infty$, as desired.

This completes the proof. □

Theorem 4.4. *$A = (a_{nk}) \in (\ell_p(\widehat{F}) : c)$ if and only if (4.3) holds, and*

$$\exists \alpha_k \in \mathbb{R} \quad \text{such that} \quad \lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} = \alpha_k \quad \text{for each} \quad k \in \mathbb{N}. \tag{4.8}$$

Proof. Let $A = (a_{nk}) \in (\ell_p(\widehat{F}) : c)$. Then, Ax exists and is in the space c for all $x = (x_k) \in \ell_p(\widehat{F})$. Since the inclusion $c \subset \ell_\infty$ holds, the necessity of the condition (4.3) follows from Theorem 4.3. The necessity of the condition (4.8) is immediate by taking the sequence $x^{(k)} = \left\{ x_j^{(k)} \right\} \in \ell_p(\widehat{F})$ defined by (4.4).

Conversely, suppose that the conditions (4.3) and (4.8) hold, and take any $x = (x_k) \in \ell_p(\widehat{F})$. Then, since $A_n \in \{\ell_p(\widehat{F})\}^\beta$ for each $n \in \mathbb{N}$, Ax exists. Then, by taking into account the relation (4.6) one can see that the conditions (4.3) and (4.8) correspond to (2.8) and (2.4) with e_{nk} instead of a_{nk}, respectively; where e_{nk} is given by (4.7). Hence, $Ey \in c$ which gives by (4.6) that $A \in (\ell_p(\widehat{F}) : c)$. □

If we replace the space c_0 with the space c, then Theorem 4.4 yields the following:

Corollary 4.5. *$A = (a_{nk}) \in (\ell_p(\widehat{F}) : c_0)$ if and only if (4.3) holds, and (4.8) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Theorem 4.6. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $H = (h_{nk})$ are connected by the relation*

$$h_{nk} = -\frac{f_{n+1}}{f_n} a_{n-1,k} + \frac{f_n}{f_{n+1}} a_{nk} \tag{4.9}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then, $A \in (\mu, \lambda(\widehat{F}))$ if and only if $H \in (\mu, \lambda)$; where λ denotes any of the classical sequence spaces ℓ_p, c_0 or c.

Proof. Let $z = (z_k) \in \mu$. Then, by taking into account the relation (4.9) one can easily derive the following equality

$$\sum_{k=0}^{m} h_{nk} z_k = \sum_{k=0}^{m} \left(-\frac{f_{n+1}}{f_n} a_{n-1,k} + \frac{f_n}{f_{n+1}} a_{nk} \right) z_k \quad \text{for all} \quad m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $H_n(z) = (\widehat{F}A)_n(z)$. Therefore, we conclude that $Az \in \lambda(\widehat{F})$ whenever $z \in \mu$ if and only if $Hz \in \lambda$ whenever $z \in \mu$. This step completes the proof. □

It is trivial that combining Theorems 4.3, 4.4 and Corollary 4.5 with Theorem 4.6, one can derive the following results:

Corollary 4.7. *Let $A = (a_{nk})$ be an infinite matrix and $a(n,k) = \sum_{j=0}^{n} a_{jk}$ for all $k, n \in \mathbb{N}$. Then, the following statements hold:*

(a) *$A = (a_{nk}) \in (\ell_p(\widehat{F}), bs)$ if and only if* (4.3) *holds with $a(n,k)$ instead of a_{nk}.*
(b) *$A = (a_{nk}) \in (\ell_p(\widehat{F}), cs)$ if and only if* (4.3)*) and* (4.8) *hold with $a(n,k)$ instead of a_{nk}.*
(c) *$A = (a_{nk}) \in (\ell_p(\widehat{F}), cs_0)$ if and only if* (4.3) *and* (4.8) *hold with $a(n,k)$ instead of a_{nk} with $\alpha_k = 0$ for all $k \in \mathbb{N}$, where cs_0 denotes the space of series converging to zero.*
(d) *$A = (a_{nk}) \in (\ell_p(\widehat{F}), \ell_\infty(\widehat{F}))$ if and only if* (4.3) *holds with h_{nk} instead of a_{nk}.*
(e) *$A = (a_{nk}) \in (\ell_p(\widehat{F}), c(\widehat{F}))$ if and only if* (4.3) *and* (4.8) *hold with h_{nk} instead of a_{nk}.*
(f) *$A = (a_{nk}) \in (\ell_p(\widehat{F}), c_0(\widehat{F}))$ if and only if* (4.3) *and* (4.8) *hold with h_{nk} instead of a_{nk} with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

5. Conclusion

To review the relevant literature about the domain of the infinite matrix A in the sequence spaces ℓ_p, c_0 and c, the table on the next page may be useful.

The present paper is devoted to the sequence spaces $\ell_p(\widehat{F})$, $c_0(\widehat{F})$ and $c(\widehat{F})$ obtained as the domain of the double band matrix $\widehat{F}$ in the classical spaces ℓ_p, c_0 and c, respectively. Of course, the α-, β- and γ-duals of the spaces $c_0(\widehat{F})$, $c(\widehat{F})$ and $\ell_p(\widehat{F})$ with $0 < p < 1$ can be given, indirectly, in the light of Theorem 3.1 of Altay and Başar [3]. However, we prefer to do this by following the similar approach used in the proof of Theorems 4.1–4.3 of Başar and Altay [8].

We should state that although the double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ can be reduced to the matrix $\widehat{F}$ in the case $\widetilde{r} = (r_n)$ and $\widetilde{s} = (s_n)$ with $r_n = f_n/f_{n+1}$ and $s_{n-1} = -f_{n+1}/f_n$ for all $n \in \mathbb{N}$, the main results concerning the spaces $c_0(\widehat{F})$ and $c(\widehat{F})$ are obtained independently from Candan [13]. It is worth mentioning here that in spite of the domain of the matrix $\widehat{F}$ in the space ℓ_p of absolutely p-summable sequences has been recently studied by

Kara in [21] for $1 \leq p < \infty$, our results related to the space $\ell_p(\widehat{F})$ are new and are also complementary of [21].

A	λ	λ_A	refer to:
Δ	c_0 and c	$c_0(\Delta)$ and $c(\Delta)$	[23]
Δ	ℓ_p, $(0 < p \leq 1)$	bv_p	[4]
$B(r,s,t)$	c_0 and c	$B(c_0)$ and $B(c)$	[33]
C_1	c_0 and c	$\widetilde{c}_0$ and $\widetilde{c}$	[35]
A^r	c_0 and c	a_0^r and a_c^r	[6]
Δ^2	c_0 and c	$c_0(\Delta^2)$ and $c(\Delta^2)$	[15]
$u\Delta^2$	c_0 and c	$c_0(u;\Delta^2)$ and $c(u;\Delta^2)$	[30]
Δ^m	c_0 and c	$c_0(\Delta^m)$ and $c(\Delta^m)$	[16, 11]
R^q	c_0 and c	$(\overline{N},q)_0$ and $(\overline{N},q)$	[26]
$\Delta^{(m)}$	c_0 and c	$c_0(\Delta^{(m)})$ and $c(\Delta^{(m)})$	[27]
$G(u,v)$	c_0 and c	$c_0(u,v)$ and $c(u,v)$	[5]
Λ	c_0 and c	c_0^λ and c^λ	[31]
$B(r,s)$	c_0 and c	$\widehat{c}_0$ and $\widehat{c}$	[24]
E^r	c_0 and c	e_0^r and e_c^r	[1]
A_λ	c_0 and c	$A_\lambda(c_0)$ and $A_\lambda(c)$	[10]
$B(\widetilde{r},\widetilde{s})$	c_0 and c	$\widetilde{c}_0$ and $\widetilde{c}$	[13]
N^t	c_0 and c	$c_0(N^t)$ and $c(N^t)$	[36]

Table 1: The domains of some triangle matrices in the spaces ℓ_p, c_0 and c

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