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(w,c)-ALMOST PERIODIC GENERALIZED FUNCTIONS

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ABSTRACT. The aim of this work is to introduce and study a new space of (w,c) −almost periodic generalized functions containing the (w,c) − almost periodic functions, (w,c) −almost periodic Schwartz distributions, the algebra of periodic generalized functions, the space of Bloch-periodic generalized functions as well as the space of (w,c) −periodic generalized functions.

1. INTRODUCTION

The class of Bloch periodic functions, which extends the classes of periodic functions and anti-periodic functions, plays an important role in the quantum mechanics and solid state physics. The class of (w,c) −periodic functions, which extends the class of Bloch periodic functions, has been introduced and investigated by Alvarez, Gómez and Pinto [1]. This class of (w,c)−periodic functions is of major relevance in the qualitative analysis of solutions to the Mathieu linear differential equation

$$y''(t)+[a-2q\cos 2t]\,y(t)=0,$$

arising in modeling of seasonally forced population dynamics.

The concept of (w,c) −almost periodicity of functions (resp. distributions) and their Stepanov generalizations, is a generalization of the well known Bohr (resp. Schwartz) almost periodicity, these concepts have recently been introduced by M. T. Khalladi, M. Kostić, A. Rahmani and D. Velinov, see [9] and [10]. In [9], the authors briefly explain how the established theoretical results can be applied in the qualitative analysis of (w,c)−almost periodic solutions to the abstract integro-differential equations and inclusions in Banach spaces. Furthermore, as an application in [10], the authors have analyzed the existence of distributional (w,c)−almost periodic solutions of the linear ordinary differential systems.

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The Colombeau algebra $\mathcal{G}$ of generalized functions gives an answer to the problem of multiplication of distributions. For a detailed study of these generalized functions see the book [7]. An algebra of almost periodic generalized functions of Colombeau type containing classical Bohr almost periodic functions and almost periodic Schwartz distributions has been introduced and studied in [5].

The main purpose of this paper is to continue the above mentioned researches [9] and [10] by introducing and investigating a new space of $(w,c)-$almost periodic generalized functions containing $(w,c)-$almost periodic functions, $(w,c)-$almost periodic Schwartz distributions, the algebra of periodic generalized functions, the space of Bloch-periodic generalized functions as well as the space of $(w,c)-$periodic generalized functions. In section 2, we recall the basic definitions and results that we shall use in this paper. The main results are given in the third section. First, we construct the space $\mathcal{G}_{L^\infty_{w,c}}$ of $(w,c)-$bounded generalized functions in which we study the $(w,c)-$almost periodicity. The new space $\mathcal{G}_{AP_{w,c}}$ of $(w,c)-$almost periodic generalized functions of Colombeau type is given in Definition 8. A characterization of elements of $\mathcal{G}_{AP_{w,c}}$ is given by Proposition 14. Some algebraic properties of $(w,c)-$almost periodic generalized functions are given. Other important results related to the composition principle and convolution product in $\mathcal{G}_{AP_{w,c}}$, are established by Proposition 19 and Proposition 20. An extension of classical Bohl-Bohr theorem to the case of $(w,c)-$almost periodic generalized functions is given. Finally, a new space $\mathcal{G}_{P_{w,c}}$ of $(w,c)-$periodic generalized functions containing the space of Bloch-periodic generalized functions introduced by M. F. Hasler in [8] as well as the space of periodic generalized functions introduced by V. F. Valmorin in [12], is also given. We refer the reader to [5], [6] and [7] from which the results of this paper were inspired. In this paper we shall be concerned with functions and distributions defined on the whole line $\mathbb{R}$.

2. $(w,c)-$Almost periodic functions and distributions

The aims of this section are twofold. The first one is to briefly recall the definition of $(w,c)-$almost periodic functions and some basic properties of them. The second one is devoted to summarizing the definition and important results of $(w,c)-$almost periodicity in the setting of Sobolev-Schwartz distributions. For the proofs and more details see [9] and [10].

Recall $(\mathcal{C}_b, \|\ \ \|_{L^\infty})$ the Banach algebra of bounded and continuous complex valued functions on $\mathbb{R}$ endowed with the norm $\|\ \ \|_{L^\infty}$ of uniform convergence on $\mathbb{R}$. Denote by AP the well-known space of Bohr almost periodic functions on $\mathbb{R}$. In the sequel we will use the following notation:

$$\varphi_{w,c}(.) = c^{-\frac{(.)}{w}}\varphi(.), \forall\varphi \in \mathcal{C}^\infty \text{ or } L^p, 1 \le p \le +\infty \text{ and } T_{w,c} = c^{-\frac{(.)}{w}}T, \forall T \in \mathcal{D}',$$

where the equality is taken in the usual (resp. Lebesgue, distributional) sense.

We first recall the space $AP_{w,c}$ of $(w,c)-$almost periodic functions.

Definition 2.1. *Let $c \in \mathbb{C}\backslash\{0\}$ and $w > 0$. A complex-valued function f defined and continuous on $\mathbb{R}$ is called $(w,c)-$almost periodic, if and only if, $f_{\omega,c} \in AP$. Denote by $AP_{w,c}$ the set of all such functions.*

When $c = 1$ and $w > 0$ arbitrary, we obtain $AP_{w,c} := AP$. The space $AP_{w,c}$ is a vector space together with the usual operations of addition and pointwise multiplication with scalars. Some properties of $(w,c)-$almost periodic functions are summarized in the following result.

Proposition 2.1. i) The space $AP_{w,c}$ endowed with the $(w,c)-$norm

$$\|f\|_{w,c} = \sup_{t\in\mathbb{R}} |f_{w,c}(t)|,$$

is a Banach space.

ii) If $f \in AP_{w,c}$, then $\widetilde{f}(.) = f(-.) \in AP_{w,1/c}$.

iii) If $w > 0, c \in \mathbb{C}\backslash\{0\}$ such that $|c| = 1$ and if $f \in AP_{w,c}$ such that $\inf_{x\in\mathbb{R}} |f(x)| > 0$, then $1/f \in AP_{w,1/c}$.

iv) If $f \in AP_{w,c}$ and $g_{w,c} \in L^1$, then $f * g \in AP_{w,c}$.

To recall the concept of $(w,c)-$almost periodicity in the setting of Sobolev-Schwartz distributions, we first present the space $\mathcal{D}_{L^p_{w,c}}$, $1 \le p \le \infty$, and its topological dual, which were initially introduced in [10]. Let $p \in [1,+\infty]$ and f be a complex valued measurable function on $\mathbb{R}$. We denote by $L^p_{w,c}$ the set of $(w,c)-$Lebesgue functions with exponent p, i.e.

$$L^p_{w,c} = \{f : \mathbb{R} \longrightarrow \mathbb{C} \text{ measurable} : f_{w,c} \in L^p\}.$$

When $c = 1$, $L^p_{w,c} = L^p$ the classical Lebesgue space over $\mathbb{R}$.

Proposition 2.2. The space $L^p_{w,c}$ endowed with the $(w,c)-$norm

$$\|f\|_{L^p_{w,c}} := \|f_{w,c}\|_{L^p}, \text{ for } 1 \le p < +\infty,$$

and

$$\|f\|_{L^\infty_{w,c}} := \|f\|_{w,c}, \text{ for } p = +\infty,$$

is a Banach space.

Proposition 2.3. $\mathcal{D}$ is dense in $L^p_{w,c}; 1 \le p < \infty$.

We recall that the following functional spaces

$$\mathcal{D}_{L^p} := \left\{\varphi \in \mathcal{C}^\infty : \varphi^{(j)} \in L^p, \forall j \in \mathbb{Z}_+\right\},$$

endowed with the topology defined by the countable family of norms

$$|\varphi|_{k,p} := \sum_{j\le k} \left\|\varphi^{(j)}\right\|_{L^p}, k \in \mathbb{Z}_+,$$

are differential Fréchet subalgebras of $\mathcal{C}^\infty$.

Define

$$\mathcal{D}_{L^p_{w,c}} := \{\varphi \in \mathcal{C}^\infty : \varphi_{w,c} \in \mathcal{D}_{L^p}\}.$$

When $c = 1$, we get $\mathcal{D}_{L^p_{w,c}} = \mathcal{D}_{L^p}$. Moreover, it is easy to show that the space $\mathcal{D}_{L^p_{w,c}}, 1 \leq p \leq \infty$, endowed with the topology defined by the following countable family of norms

$$|\varphi|_{k,p;w,c} := \sum_{j \leq k} \left\| (\varphi_{w,c})^{(j)} \right\|_{L^p},\ k \in \mathbb{Z}_+,$$

is a Fréchet subspace of $\mathcal{C}^\infty$.

Proposition 2.4. Let $1 \leq p \leq \infty$, if $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$, then $\varphi\psi \in \mathcal{D}_{L^p_{w,c}}$.

The following result shows that the family of norms $|.|_{k,p;w,c}$ is submultiplicative.

Proposition 2.5. Let $1 \leq p \leq \infty$, if $\varphi, \psi \in \mathcal{D}_{L^p_{2w,c}}$, then for all $k \in \mathbb{Z}_+$, there exists $C_k > 0$ such that

$$|\varphi\psi|_{k,p;w,c} \leq C_k\, |\varphi|_{k,p;2w,c} \cdot |\psi|_{k,p;2w,c}\,.$$

For $1 \leq p < \infty$, we have $\mathcal{D} \subset \mathcal{D}_{L^p_{w,c}} \subset \mathcal{D}_{L^\infty_{w,c}}$. Moreover, we have the following result.

Proposition 2.6. For $1 \leq p < \infty$, the space $\mathcal{D}$ is dense in $\mathcal{D}_{L^p_{w,c}}$.

Since he space $\mathcal{D}$ is not dense in $\mathcal{D}_{L^\infty_{w,c}}$, we then define $\dot{\mathcal{D}}_{L^\infty_{w,c}}$as the subspace of all functions in $\mathcal{D}_{L^\infty_{w,c}}$ which vanish at infinity with all their derivatives. This space is the closure in $\mathcal{D}$ of the space $\mathcal{D}_{L^\infty_{w,c}}$. It is clear that $\dot{\mathcal{D}}_{L^\infty_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$, hence it is a Fréchet space. Moreover, it is easy to check the following properties on the structure of $\mathcal{D}_{L^p_{w,c}}$.

Proposition 2.7. For $1 \leq p < \infty$, we have

$$\mathcal{D}_{L^p_{w,c}} \hookrightarrow \dot{\mathcal{D}}_{L^\infty_{w,c}} \hookrightarrow \mathcal{D}_{L^\infty_{w,c}},$$

with continuous embedding.

Thanks to the density of the space $\mathcal{D}$ in $\mathcal{D}_{L^p_{w,c}}, 1 \leq p < \infty$, (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$), we have that the space $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) is a normal space of distributions, i.e. the elements of the topological dual of $\mathcal{D}_{L^p_{w,c}}$ (resp. $\dot{\mathcal{D}}_{L^\infty_{w,c}}$) can be identified with continuous linear forms on $\mathcal{D}$.

Definition 2.2. *For* $1 < p \leq \infty$, *we denote by* $\mathcal{D}'_{L^p_{w,c}}$ *the topological dual of* $\mathcal{D}_{L^q_{w,c}}$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 2.3. i) *The topological dual of* $\mathcal{D}_{L^1_{w,c}}$, *denoted by* $\mathcal{B}'_{w,c}$, *is called the space of* $(w,c)-$*bounded distributions.*

ii) *The topological dual of* $\dot{\mathcal{D}}_{L^\infty_{w,c}}$, *denoted by* $\mathcal{D}'_{L^1_{w,c}}$, *is called the space of* (w,c)-*integrable distributions.*

We can easily show the following characterizations of $L^p_{w,c}-$distributions.

Theorem 2.1. *Let $T \in \mathcal{D}'$, the following statements are equivalent:*

i) $T \in \mathcal{D}'_{L^p_{w,c}}$.

ii) $c^{\frac{t}{w}}(T_{w,c} * \varphi) \in L^p_{w,c}$, $\forall \varphi \in \mathcal{D}$.

iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_{0\leq j\leq k} \subset L^p_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^{k} (f_{w,c})_j^{(j)}$, *where*

$$\left((f_{w,c})_j\right)_{0\leq j\leq k} = \left(c^{-\frac{t}{w}} f_j\right)_{0\leq j\leq k}.$$

Remark 2.1. As a consequence of Theorem 2.1, a distribution $T \in \mathcal{D}'_{L^p_{w,c}}$, if and only if, $T_{w,c} \in \mathcal{D}'_{L^p}$.

Proposition 2.8. Let $T \in \mathcal{D}'$. Then $T \in \mathcal{D}'_{L^p_{w,c}}, 1 \leq p \leq \infty$, if and only if, there exists $S \in \mathcal{D}'_{L^p}, 1 \leq p \leq \infty$, such that $T = c^{\frac{t}{w}} S$ in $\mathcal{D}'$.

Recall that the space $\mathcal{B}'_{ap}$ of almost periodic distributions, which was introduced and studied by L. Schwartz, is based on the topological definition of Bochner's almost periodic functions. Let $h \in \mathbb{R}$ and $T \in \mathcal{D}'$, the translated of T by h, denoted by $\tau_h T$, is defined as:

$$\langle \tau_h T, \varphi \rangle = \langle T, \tau_{-h}\varphi \rangle, \ \varphi \in \mathcal{D},$$

where $\tau_{-h}\varphi(x) = \varphi(x+h)$.

The definition and characterizations of Schwartz almost periodic distributions are given in the following result.

Theorem 2.2. *For any bounded distribution $T \in \mathcal{D}'_{L^\infty}$, the following statements are equivalent:*

(i) *The set $\{\tau_h T, h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$.*

(ii) $T * \varphi \in AP$, $\forall \varphi \in \mathcal{D}$.

(iii) $\exists k \in \mathbb{Z}_+, \exists (f_j)_{0\leq j\leq k} \subset AP : T = \sum_{j=0}^{k} f_j^{(j)}$.

The concept of $(w,c)-$almost periodicity of Schwartz distributions is given by the following

Definition 2.4. *A distribution $T \in \mathcal{B}'_{w,c}$ is said to be $(w,c)-$almost periodic, if and only if, $T_{w,c} \in \mathcal{B}'_{ap}$, i.e. the set $\{\tau_h T_{w,c}, h \in \mathbb{R}\}$ is relatively compact in $\mathcal{D}'_{L^\infty}$. The set of $(w,c)-$almost periodic distributions is denoted by $\mathcal{B}'_{AP_{w,c}}$.*

Example 2.1. (i) *The associated distribution of an $(w,c)-$almost periodic function (resp. Stepanov $(p,w,c)-$almost periodic function) is an $(w,c)-$almost periodic distribution, i.e.*

$$AP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}} \text{ (resp. } S^pAP_{w,c} \hookrightarrow \mathcal{B}'_{AP_{w,c}}).$$

(ii) *When $c = 1$ it follows that $\mathcal{B}'_{AP_{w,c}} := \mathcal{B}'_{ap}$.*

Characterizations of $(w,c)-$almost periodic distributions are given in the following result.

Theorem 2.3. *Let $T \in \mathcal{B}'_{w,c}$, the following statements are equivalent:*

(i) $T \in \mathcal{B}'_{AP_{w,c}}$.

(ii) $c^{\frac{t}{w}}\left(T_{w,c} * \varphi\right) \in AP_{w,c}, \forall \varphi \in \mathcal{D}$.

(iii) $\exists k \in \mathbb{Z}_+, \exists \left(f_j\right)_{0 \leq j \leq k} \subset AP_{w,c} : T = c^{\frac{t}{w}} \sum_{j=0}^{k} \left(f_{w,c}\right)_j^{(j)}$, *where*

$$\left(\left(f_{w,c}\right)_j\right)_{0 \leq j \leq k} = \left(c^{-\frac{t}{w}} f_j\right)_{0 \leq j \leq k}.$$

We recall also the following space of smooth $(w,c)-$almost periodic functions

$$\mathcal{B}_{AP_{w,c}} := \left\{\varphi \in \mathcal{D}_{L^\infty_{w,c}} : \varphi_{w,c} \in \mathcal{B}_{ap}\right\},$$

where

$$\mathcal{B}_{ap} := \left\{\varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in AP, \forall j \in \mathbb{Z}_+\right\},$$

is the space of smooth almost periodic functions introduced by L. Schwartz. We endow $\mathcal{B}_{AP_{w,c}}$ with the topology induced by $\mathcal{D}_{L^\infty_{w,c}}$. Some properties of $\mathcal{B}_{AP_{w,c}}$ are given in the following

Proposition 2.9. i) $\mathcal{B}_{AP_{w,c}} = AP_{w,c} \cap \mathcal{D}_{L^\infty_{w,c}}$.

ii) $\mathcal{B}_{AP_{w,c}}$ is a closed subspace of $\mathcal{D}_{L^\infty_{w,c}}$.

iii) If $f \in L^1_{w,c}$ and $\varphi \in \mathcal{B}_{AP_{w,c}}$, then $c^{\frac{t}{w}}\left(f_{w,c} * \varphi_{w,c}\right) \in \mathcal{B}_{AP_{w,c}}$.

Corollary 2.1. *If $f \in \mathcal{D}_{L^\infty_{w,c}}$ and $c^{\frac{t}{w}}\left(f_{w,c} * \varphi_{w,c}\right) \in AP_{w,c}, \forall \varphi \in \mathcal{D}$, then $f \in \mathcal{B}_{AP_{w,c}}$.*

Remark 2.2. It is clear that $\mathcal{B}_{AP_{w,c}} \subset AP_{w,c} \cap \mathcal{C}^\infty$, whereas the converse inclusion is not true. Indeed, the function

$$f(t) = 2^{-t}\sqrt{2 + \cos t + \cos\sqrt{2}t}$$

is an element of $AP_{w,c} \cap \mathcal{C}^\infty$ with $c = 2$ and $w = 1$. However

$$f'(t) = 2^{-t}\left(\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2 + \cos t + \cos\sqrt{2}t}} - \ln 2\sqrt{2 + \cos t + \cos\sqrt{2}t}\right)$$

is not bounded, because $\inf_{t \in \mathbb{R}}\left(2 + \cos t + \cos\sqrt{2}t\right) = 0$ and therefore

$$\frac{-\sin t - \sqrt{2}\sin\sqrt{2}t}{2\sqrt{2 + \cos t + \cos\sqrt{2}t}} \notin AP,$$

hence $f \notin \mathcal{B}_{AP_{w,c}}$.

The main properties of $\mathcal{B}'_{AP_{w,c}}$ are given in the following proposition.

Proposition 2.10. i) If $T \in \mathcal{B}'_{AP_{w,c}}$, then $c^{\frac{t}{w}}\left(T_{w,c}\right)^{(j)} \in \mathcal{B}'_{AP_{w,c}}, \forall j \in \mathbb{Z}_+$.

ii) If $\varphi \in \mathcal{B}_{AP_{w,c}}$ and $T \in \mathcal{B}'_{AP_{w,c}}$, then $\varphi_{w,c} T \in \mathcal{B}'_{AP_{w,c}}$.

iii) If $T \in \mathcal{B}'_{AP_{w,c}}$ and $S \in \mathcal{D}'_{L^1_{w,c}}$, then $c^{\frac{t}{w}}\left(T_{w,c} * S_{\ w,c}\right) \in \mathcal{B}'_{AP_{w,c}}$.

iv) $\mathcal{B}_{AP_{w,c}}$ is dense in $\mathcal{B}'_{AP_{w,c}}$.

3. $(w,c)-$ALMOST PERIODIC GENERALIZED FUNCTIONS

In this section, we will introduce the space of $(w,c)-$almost periodic generalized functions of Colombeau type and give their elementary properties. First, let us recall the simplified Colombeau algebra of generalized functions on $\mathbb{R}$. For a detailed study see [7]. Set $I := (0,1]$. We denote by $\left(\mathcal{C}^\infty(\mathbb{R})\right)^I$ the set of all families of maps of class $\mathcal{C}^\infty$ on $\mathbb{R}$ indexed by I and endowed with the following operations:

i) $(u_\varepsilon)_\varepsilon + (v_\varepsilon)_\varepsilon = (u_\varepsilon + v_\varepsilon)_\varepsilon$, for all $\varepsilon \in I$.

ii) $(u_\varepsilon)_\varepsilon . (v_\varepsilon)_\varepsilon = (u_\varepsilon v_\varepsilon)_\varepsilon$, for all $\varepsilon \in I$.

iii) $\lambda (u_\varepsilon)_\varepsilon = (\lambda u_\varepsilon)_\varepsilon$, for all $\varepsilon \in I$ and $\lambda \in \mathbb{C}$.

iv) $(u_\varepsilon)_\varepsilon^{(j)} = \left(u_\varepsilon^{(j)}\right)_\varepsilon$, for all $\varepsilon \in I$ and $j \in \mathbb{Z}_+$.

Definition 3.1. *The algebra of generalized functions on* $\mathbb{R}$, *denoted by* $\mathcal{G}$, *is defined as elements of the quotient space*

$$\mathcal{G} := \frac{\mathcal{E}_M[\mathbb{R}]}{\mathcal{N}(\mathbb{R})},$$

where

$$\mathcal{E}_M[\mathbb{R}] = \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in \left(\mathcal{C}^\infty(\mathbb{R})\right)^I; \forall K \Subset \mathbb{R}, \forall \alpha \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \\ \sup\limits_{x \in K}\left|u_\varepsilon^{(\alpha)}(x)\right| = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \end{array} \right\},$$

is the space of moderate elements and

$$\mathcal{N}(\mathbb{R}) = \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in \left(\mathcal{C}^\infty(\mathbb{R})\right)^I; \forall K \Subset \mathbb{R}, \forall \alpha \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \\ \sup\limits_{x \in K}\left|u_\varepsilon^{(\alpha)}(x)\right| = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \end{array} \right\},$$

is the space of negligible elements.

Recall the algebra of bounded generalized functions on $\mathbb{R}$

$$\mathcal{G}_{L^\infty} := \frac{\mathcal{M}_{L^\infty}}{\mathcal{N}_{L^\infty}},$$

where

$$\mathcal{M}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in \left(\mathcal{D}_{L^\infty}\right)^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0 \right\},$$

and

$$\mathcal{N}_{L^\infty} = \left\{ (u_\varepsilon)_\varepsilon \in \left(\mathcal{D}_{L^\infty}\right)^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+ |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right), \varepsilon \longrightarrow 0 \right\}.$$

The algebra $\mathcal{G}_{ap}$ of almost periodic generalized functions, introduced and studied in [5], contains the classical Bohr almost periodic functions as well as almost periodic Schwartz distributions. We refer to [5] and [6] for details on the construction of this algebra and their main properties. Recall the space of smooth almost periodic functions introduced above,

$$\mathcal{B}_{ap} := \left\{ \varphi \in \mathcal{D}_{L^\infty} : \varphi^{(j)} \in AP, \forall j \in \mathbb{Z}_+ \right\}.$$

The algebra of almost periodic generalized functions on $\mathbb{R}$, is defined by the quotient algebra

$$\mathcal{G}_{ap} := \frac{\mathcal{M}_{ap}}{\mathcal{N}_{ap}},$$

where

$$\mathcal{M}_{ap} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0 \right\},$$

is called the space of almost periodic moderate elements and

$$\mathcal{N}_{ap} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{ap})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, |u_\varepsilon|_{k,\infty} = O\left(\varepsilon^{m}\right), \varepsilon \longrightarrow 0 \right\},$$

is called the space of almost periodic negligible elements.

Remark 3.1. The space $\mathcal{M}_{ap}$ is a subalgebra of $(\mathcal{B}_{ap})^I$ and $\mathcal{N}_{ap}$ is an ideal of $\mathcal{M}_{ap}$.

Let us consider the following notation:

$$u_{w,c} := c^{-\frac{t}{w}} u, \forall t \in \mathbb{R}, \ \forall u \in \mathcal{G}.$$

This means that $\left((u_{w,c})_\varepsilon - c^{-\frac{t}{w}} u_\varepsilon\right)_\varepsilon \in \mathcal{N}(\mathbb{R})$, for all representatives $(u_{w,c})_\varepsilon, u_\varepsilon$ of $u_{w,c}, u$ respectively.

Definition 3.2. *The space of $(w,c)-$bounded generalized functions is denoted and defined by*

$$\mathcal{G}_{L^\infty_{w,c}} := \left\{ u \in \mathcal{G} : u_{w,c} \in \mathcal{G}_{L^\infty} \right\}.$$

Remark 3.2. It is easy to show that if $\lambda \in \mathbb{C}$ and $u, v \in \mathcal{G}_{L^\infty_{w,c}}$ then $\lambda u, u+v \in \mathcal{G}_{L^\infty_{w,c}}$ and $uv \in \mathcal{G}$. While $uv \notin \mathcal{G}_{L^\infty_{w,c}}$, in fact we have $uv \in \mathcal{G}_{L^\infty_{w/2,c}}$, because

$$(uv)_{w/2,c} = c^{-\frac{2t}{w}} (uv) = c^{-\frac{t}{w}} u c^{-\frac{t}{w}} v = u_{w,c} v_{w,c} \in \mathcal{G}_{L^\infty}.$$

It is clear that if $c^{-\frac{t}{w}} u = v$ almost everywhere on $\mathbb{R}$ in L^p, then $u = c^{\frac{t}{w}} v$ almost everywhere on $\mathbb{R}$ in $L^p_{w,c}$. Moreover, we have the following proposition.

Proposition 3.1. Let $u \in \mathcal{G}$. Then $u \in \mathcal{G}_{L^\infty_{w,c}}$, if and only if, there exists $v \in \mathcal{G}_{L^\infty}$ such that

$$u = c^{\frac{t}{w}} v \text{ in } \mathcal{G}. \tag{3.1}$$

Proof. $(\Longrightarrow)$: Let $u \in \mathcal{G}_{L^\infty_{w,c}}$, then $u_{w,c} = c^{-\frac{t}{w}} u \in \mathcal{G}_{L^\infty}$, $\forall t \in \mathbb{R}$. So for all representatives $(u_{w,c})_\varepsilon \in \mathcal{M}_{L^\infty}$ of $u_{w,c}$ and for all representatives $u_\varepsilon \in \mathcal{M}_{L^\infty}$ of u, we have $\left((u_{w,c})_\varepsilon - c^{-\frac{t}{w}} u_\varepsilon\right)_\varepsilon \in \mathcal{N}_{L^\infty}$. Therefore $\forall \varepsilon \in I$, $(u_{w,c})_\varepsilon = c^{-\frac{t}{w}} u_\varepsilon$ in $\mathcal{D}_{L^\infty}$, i.e. $\forall \varepsilon \in I$, $c^{\frac{t}{w}} (u_{w,c})_\varepsilon = u_\varepsilon$ in $\mathcal{D}_{L^\infty}$, hence $\left(c^{\frac{t}{w}} (u_{w,c})_\varepsilon - u_\varepsilon\right)_\varepsilon \in \mathcal{N}_{L^\infty} \subset \mathcal{N}(\mathbb{R})$. By taking $v = \left[(u_{w,c})_\varepsilon\right] \in \mathcal{G}_{L^\infty}$ we obtain that $u = c^{\frac{t}{w}} v$ in $\mathcal{G}$. On the other hand, if v_ε and $\widetilde{v}_\varepsilon$ are two representatives of v such that

$$\left(u_\varepsilon - c^{\frac{t}{w}} v_\varepsilon\right)_\varepsilon \in \mathcal{N}(\mathbb{R}) \text{ and } \left(\widetilde{u}_\varepsilon - c^{\frac{t}{w}} \widetilde{v}_\varepsilon\right)_\varepsilon \in \mathcal{N}(\mathbb{R}),$$

for all representatives u_ε and $\widetilde{u}_\varepsilon$ of u, then

$$\begin{aligned} &\left(u_\varepsilon - c^{\frac{t}{w}} v_\varepsilon\right)_\varepsilon - \left(\widetilde{u}_\varepsilon - c^{\frac{t}{w}} \widetilde{v}_\varepsilon\right)_\varepsilon \in \mathcal{N}(\mathbb{R}) \\ \Longrightarrow\ &(u_\varepsilon - \widetilde{u}_\varepsilon)_\varepsilon - c^{\frac{t}{w}} (v_\varepsilon - \widetilde{v}_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}) \\ &\Longrightarrow (u_\varepsilon - \widetilde{u}_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R}), \end{aligned}$$

because $(v_\varepsilon - \widetilde{v}_\varepsilon)_\varepsilon \in \mathcal{N}_{L^\infty}$ and $c^{\frac{t}{w}} (v_\varepsilon - \widetilde{v}_\varepsilon)_\varepsilon \in \mathcal{N}(\mathbb{R})$. This shows that (3.1) does not depend on the choice of representatives.

$(\Longleftarrow)$: Suppose that there exist $v \in \mathcal{G}_{L^\infty}$ such that $u = c^{\frac{t}{w}} v$ in $\mathcal{G}$, then $c^{-\frac{t}{w}} u = c^{-\frac{t}{w}} \left(c^{\frac{t}{w}} v\right) = v$ in $\mathcal{G}$, this shows that $u_{w,c} \in \mathcal{G}_{L^\infty}$, hence $u \in \mathcal{G}_{L^\infty_{w,c}}$. □

The following definition introduces the space of $(w,c)-$almost periodic generalized functions.

Definition 3.3. *The space of $(w,c)-$almost periodic generalized functions is denoted and defined by*

$$\mathcal{G}_{AP_{w,c}} := \left\{ u \in \mathcal{G}_{L^\infty_{w,c}} : u_{w,c} \in \mathcal{G}_{ap} \right\}.$$

By definition, it follows that

$$\mathcal{G}_{AP_{w,c}} \subset \mathcal{G}_{L^\infty_{w,c}} \subset \mathcal{G}.$$

As a consequence of Proposition 3.1, we have the following proposition.

Proposition 3.2. Let $u \in \mathcal{G}$. Then $u \in \mathcal{G}_{AP_{w,c}}$, if and only if, there exists $v \in \mathcal{G}_{ap}$ such that $u = c^{\frac{t}{w}} v$ in $\mathcal{G}$.

Corollary 3.1. *In the case where $c \in \mathbb{C}\backslash\{0\}$ is such that $|c| = 1$, we obtain that*

$$\mathcal{G}_{AP_{w,c}} = \mathcal{G}_{ap}.$$

Example 3.1. i) *We have*

$$AP_{w,c} \subset \mathcal{B}'_{AP_{w,c}} \subset \mathcal{G}_{AP_{w,c}}.$$

ii) *The generalized function $u = (u_\varepsilon)_\varepsilon + \mathcal{N}(\mathbb{R})$, where*

$$u_\varepsilon(t) = e^{\left(i\varepsilon - \frac{\ln c}{w}\right)t},\ t \in \mathbb{R},\ \varepsilon \in I,$$

is an element of $\mathcal{G}_{AP_{w,c}}$.

Let us recall the space of smooth (w,c) −almost periodic functions on $\mathbb{R}$

$$\mathcal{B}_{AP_{w,c}} := \left\{ \varphi \in \mathcal{D}_{L^\infty_{w,c}} : \varphi_{w,c} \in \mathcal{B}_{ap} \right\}.$$

A characterization of elements of $\mathcal{G}_{AP_{w,c}}$ similar to the result obtained for (w,c)-almost periodic distributions is given by the following.

Proposition 3.3. Let $u \in \mathcal{G}_{L^\infty_{w,c}}$, the following assertions are equivalent:

(i) u is (w,c)-almost periodic.
(ii) $c^{\frac{t}{w}} \left((u_{w,c})_\varepsilon * \varphi \right) \in \mathcal{B}_{AP_{w,c}}$, for all $\varphi \in \mathcal{D}$ and for all representatives $(u_{w,c})_\varepsilon$ of $u_{w,c}$.

Proof. $(i) \Longrightarrow (ii)$: Let $u \in \mathcal{G}_{AP_{w,c}}$. Then $u_{w,c} \in \mathcal{G}_{ap}$, therefore for every $\varepsilon \in I$, we have $(u_{w,c})_\varepsilon \in \mathcal{B}_{ap}$. Since $\mathcal{B}_{ap} * \mathcal{D} \subset \mathcal{B}_{ap}$, then $(u_{w,c})_\varepsilon * \varphi \in \mathcal{B}_{ap}$, $\forall \varepsilon \in I$, $\forall \varphi \in \mathcal{D}$, hence $c^{\frac{t}{w}} \left((u_{w,c})_\varepsilon * \varphi \right) \in \mathcal{B}_{AP_{w,c}}$, $\forall \varepsilon \in I$, $\forall \varphi \in \mathcal{D}$.

$(ii) \Longrightarrow (i)$: If $c^{\frac{t}{w}} \left((u_{w,c})_\varepsilon * \varphi \right) \in \mathcal{B}_{AP_{w,c}}$, for all $\varphi \in \mathcal{D}$ and for all representatives $(u_{w,c})_\varepsilon$ of $u_{w,c}$, then $(u_{w,c})_\varepsilon * \varphi \in \mathcal{B}_{ap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, so $(u_{w,c})_\varepsilon \in \mathcal{B}_{ap}$, $\forall \varepsilon \in I$, it suffices to show that

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \left| (u_{w,c})_\varepsilon \right|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0,$$

which follows from the fact that $(u_{w,c})_\varepsilon \in \mathcal{M}_{L^\infty}$. If $(u_{w,c})_\varepsilon \in \mathcal{N}_{L^\infty}$ and $(u_{w,c})_\varepsilon * \varphi \in \mathcal{B}_{ap}$, $\forall \varepsilon \in I, \forall \varphi \in \mathcal{D}$, we obtain the same result, because $\mathcal{N}_{L^\infty} \subset \mathcal{M}_{L^\infty}$. □

Remark 3.3. The characterization (ii) does not depend on the choice of representatives.

We have the following algebraic properties of (w,c) −almost periodic generalized functions.

Proposition 3.4. Let $c \in \mathbb{C} \backslash \{0\}$, $w_1, w_2 > 0$, $u \in \mathcal{G}_{AP_{w_1,c}}$ and $v \in \mathcal{G}_{AP_{w_2,c}}$. If there exists $w > 0$ such that $\frac{1}{w} = \frac{1}{w_1} + \frac{1}{w_2}$, then $uv \in \mathcal{G}_{AP_{w,c}}$.

Proof. If $u \in \mathcal{G}_{AP_{w_1,c}}$ and $v \in \mathcal{G}_{AP_{w_2,c}}$, then $u_{w_1,c} \in \mathcal{G}_{ap}$ and $v_{w_2,c} \in \mathcal{G}_{ap}$, so

$$(uv)_{w,c} = c^{-\frac{t}{w}} uv = c^{\left(\frac{1}{w_1} + \frac{1}{w_2} - \frac{1}{w}\right)t} \left(c^{-\frac{t}{w_1}} u c^{-\frac{t}{w_2}} v \right) = u_{w_1,c} v_{w_2,c} \text{ in } \mathcal{G}_{L^\infty}.$$

As $u_{w_1,c} v_{w_2,c} \in \mathcal{G}_{ap}$, then $(uv)_{w,c} \in \mathcal{G}_{ap}$, hence $uv \in \mathcal{G}_{AP_{w,c}}$. □

Proposition 3.5. Let $c \in \mathbb{C} \backslash \{0\}$ such that $|c| = 1$, $w > 0$ and $u, v \in \mathcal{G}_{AP_{w,c}}$. Then $uv \in \mathcal{G}_{AP_{w,c}}$.

Proof. According to Corollary 3.1, it follows that $u, v \in \mathcal{G}_{ap} = \mathcal{G}_{AP_{w,c}}$ and since $\mathcal{G}_{ap}$ is a subalgebra of $\mathcal{G}_{L^\infty}$, we deduce that $uv \in \mathcal{G}_{ap} = \mathcal{G}_{AP_{w,c}}$. □

It is clear that for any real numbers $c_1, c_2, w_1, w_2 > 0$, we have the equality

$$c_1^{-\frac{t}{w_1}} c_2^{-\frac{t}{w_2}} = \left(c_1^{\frac{w}{w_1}} c_2^{\frac{w}{w_2}} \right)^{-\frac{t}{w}}, \ t \in \mathbb{R}.$$

Furthermore, we have:

Proposition 3.6. Let $c_1, c_2 \in \mathbb{R}_+ \backslash \{0\}$, $w, w_1, w_2 > 0$, $u \in \mathcal{G}_{AP_{w_1,c_1}}$ and $v \in \mathcal{G}_{AP_{w_2,c_2}}$. Then $uv \in \mathcal{G}_{AP_{w,c}}$ where $c = c_1^{\frac{w}{w_1}} c_2^{\frac{w}{w_2}}$.

Proof. It follows from the fact that

$$(uv)_{w,c} = c^{-\frac{t}{w}} uv = \left(c_1^{\frac{w}{w_1}} c_2^{\frac{w}{w_2}} \right)^{-\frac{t}{w}} uv \ = c_1^{-\frac{t}{w_1}} u c_2^{-\frac{t}{w_2}} v = u_{w_1,c_1} v_{w_2,c_2} \text{ in } \mathcal{G}_{L^\infty}. \qquad \square$$

Motivated by the need of a good theory of a Fourier transform of generalized functions, J. F. Colombeau introduced the algebra $\mathcal{G}_{\mathcal{T}}(\mathbb{C})$ of tempered generalized functions on $\mathbb{C}$. For more details on this algebra see [7].

Proposition 3.7. If $u \in \mathcal{G}_{AP_{w,c}}$, then $c^{\frac{t}{w}} P(u_{w,c}) \in \mathcal{G}_{AP_{w,c}}$, where $P(x) = \sum_{k \leq m} a_k x^k$, $m \in \mathbb{Z}_+$ and $a_k \in \mathbb{C}$.

Proof. Let $u \in \mathcal{G}_{AP_{w,c}}$. Then $u_{w,c} \in \mathcal{G}_{ap}$ and from the fact that $\mathcal{G}_{ap}$ is an algebra, it follows that $P(u_{w,c}) = \left(P\left((u_{w,c})_\varepsilon \right) \right)_\varepsilon + \mathcal{N}_{ap} \in \mathcal{G}_{ap}$, hence $c^{\frac{t}{w}} P(u_{w,c}) \in \mathcal{G}_{AP_{w,c}}$. $\square$

The following proposition generalizes this result.

Proposition 3.8. Let $u \in \mathcal{G}_{AP_{w,c}}$ and $F \in \mathcal{G}_{\mathcal{T}}(\mathbb{C})$. Then the composition $c^{\frac{t}{w}} (F \circ u_{w,c})$ is a well defined element of $\mathcal{G}_{AP_{w,c}}$.

Proof. If $u \in \mathcal{G}_{AP_{w,c}}$, then $u_{w,c} \in \mathcal{G}_{ap}$. Since $\mathcal{G}_{ap}$ is stable under convolution by any tempered generalized function, it follows from the composition theorem in $\mathcal{G}_{ap}$ that $F \circ u_{w,c} \in \mathcal{G}_{ap}$. Thus $c^{\frac{t}{w}} (F \circ u_{w,c}) \in \mathcal{G}_{AP_{w,c}}$. $\square$

Recall the algebra of integrable generalized functions on $\mathbb{R}$

$$\mathcal{G}_{L^1} := \frac{\mathcal{M}_{L^1}}{\mathcal{N}_{L^1}},$$

where

$$\mathcal{M}_{L^1} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^1})^I, \forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, |u_\varepsilon|_{k,1} = O\left(\varepsilon^{-m}\right), \ \varepsilon \longrightarrow 0 \right\},$$

and

$$\mathcal{N}_{L^1} = \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^1})^I, \forall k \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \ |u_\varepsilon|_{k,1} = O(\varepsilon^m), \ \varepsilon \longrightarrow 0 \right\}.$$

Definition 3.4. *A generalized function $u \in \mathcal{G}$ is said to be $(w,c)-$integrable on $\mathbb{R}$, if and only if, $u_{w,c} \in \mathcal{G}_{L^1}$. The space of $(w,c)-$integrable generalized functions on $\mathbb{R}$ is denoted by $\mathcal{G}_{L^1_{w,c}}$.*

Proposition 3.9. If $u \in \mathcal{G}_{AP_{w,c}}$ and $v \in \mathcal{G}_{L^1_{w,c}}$, then the convolution $u * v$ given by

$$u * v := c^{\frac{t}{w}} \left(u_{w,c} * v_{w,c}\right),$$

is a well defined element of $\mathcal{G}_{AP_{w,c}}$.

Proof. Let $(u_\varepsilon)_\varepsilon$ be a representative of u and $(v_\varepsilon)_\varepsilon$ be a representative of v, then for each $\varepsilon \in I$, we have

$$\begin{aligned}
(u_\varepsilon * v_\varepsilon)(t) &= \int_{\mathbb{R}} u_\varepsilon (t-s)\, v_\varepsilon (s)\, ds \\
&= \int_{\mathbb{R}} c^{\frac{t-s}{w}} \left(u_{w,c}\right)_\varepsilon (t-s)\, c^{\frac{s}{w}} \left(v_{w,c}\right)_\varepsilon (s)\, ds \\
&= c^{\frac{t}{w}} \int_{\mathbb{R}} \left(u_{w,c}\right)_\varepsilon (t-s) \left(v_{w,c}\right)_\varepsilon (s)\, ds \\
&= c^{\frac{t}{w}} \left(\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon\right)(t).
\end{aligned}$$

From Proposition $2.9-(iii)$, for each $\varepsilon \in I$, we have $c^{\frac{t}{w}} \left(f_{w,c} * \varphi_{w,c}\right) \in \mathcal{B}_{AP_{w,c}}$, therefore for each $\varepsilon \in I$, $\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon \in \mathcal{B}_{ap}$. On the other hand, since $\left(u_{w,c}\right)_\varepsilon \in \mathcal{M}_{ap}$ and $\left(v_{w,c}\right)_\varepsilon \in \mathcal{M}_{L^1}$, then

$$\begin{aligned}
&\forall k \in \mathbb{Z}_+, \exists m_1 \in \mathbb{Z}_+, \left|\left(u_{w,c}\right)_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{-m_1}\right),\ \varepsilon \longrightarrow 0, \\
&\forall k \in \mathbb{Z}_+, \exists m_2 \in \mathbb{Z}_+, \left|\left(v_{w,c}\right)_\varepsilon\right|_{k,1} = O\left(\varepsilon^{-m_2}\right),\ \varepsilon \longrightarrow 0.
\end{aligned}$$

By Young inequality there exists $c' > 0$ such that

$$\left\|\left(\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon\right)^{(j)}\right\|_{L^\infty} \leq c' \left\|\left(u_{w,c}\right)_\varepsilon^{(j)}\right\|_{L^\infty} \left\|\left(v_{w,c}\right)_\varepsilon\right\|_{L^1},$$

so there exists $c'' > 0$ such that $\forall k \in \mathbb{Z}_+$, we have

$$\left|\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon\right|_{k,\infty} \leq c'' \left|\left(u_{w,c}\right)_\varepsilon\right|_{k,\infty} \left|\left(v_{w,c}\right)_\varepsilon\right|_{k,1}. \tag{3.2}$$

Consequently there exists $m \in \mathbb{Z}_+$ such that

$$\left|\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0,$$

which shows that $\left(\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon\right)_\varepsilon \in \mathcal{M}_{ap}$. The inequality (3.2) shows that $\left(\left(u_{w,c}\right)_\varepsilon * \left(v_{w,c}\right)_\varepsilon\right)_\varepsilon$ where ε is independent of the representatives of $\left(u_{w,c}\right)_\varepsilon$ and $\left(v_{w,c}\right)_\varepsilon$. Hence $u_{w,c} * v_{w,c} \in \mathcal{G}_{ap}$ and $u * v = c^{\frac{t}{w}} \left(u_{w,c} * v_{w,c}\right) \in \mathcal{G}_{AP_{w,c}}$. □

Definition 3.5. *Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}$ and $x_0 \in \mathbb{R}$. A primitive of u is a generalized function U defined by*

$$U(x) := \left(\int_{x_0}^{x} u_\varepsilon (t)\, dt\right)_\varepsilon + \mathcal{N}[\mathbb{R}].$$

A generalized version of the classical Bohl-Bohr theorem is given and proved in [5]. We extend this result to the case of $(w,c)-$almost periodic generalized functions.

Proposition 3.10. A primitive of a $(w,c)-$almost periodic generalized function is $(w,c)-$almost periodic, if and only if, it is a $(w,c)-$bounded generalized function.

Proof. $(\Longrightarrow)$: It follows from the fact that $\mathcal{G}_{AP_{w,c}} \subset \mathcal{G}_{L^\infty_{w,c}}$. $(\Longleftarrow)$: Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{AP_{w,c}}$ and U its primitive. By the hypothesis, we have $U \in \mathcal{G}_{L^\infty_{w,c}}$, then $U_{w,c} = \left[(U_{w,c})_\varepsilon\right] \in \mathcal{G}_{L^\infty}$ and by definition

$$\forall \varepsilon \in I,\ \forall x \in \mathbb{R},\ (U_{w,c})_\varepsilon = \int_{x_0}^{x} (u_{w,c})_\varepsilon (t)\, dt \in \mathcal{D}_{L^\infty}.$$

Thus $(U_{w,c})_\varepsilon$ is a bounded primitive of $(u_{w,c})_\varepsilon \in \mathcal{C}_{ap}$. By the classical result of Bohl-Bohr we deduce that $(U_{w,c})_\varepsilon \in \mathcal{C}_{ap}$ and since $\mathcal{C}_{ap} \cap \mathcal{D}_{L^\infty} = \mathcal{B}_{ap}$, we obtain that for each $\varepsilon \in I$, $(U_{w,c})_\varepsilon \in \mathcal{B}_{ap}$. The moderateness condition of $\left((U_{w,c})_\varepsilon\right)_\varepsilon$ follows from the fact that $\left((U_{w,c})_\varepsilon\right)_\varepsilon \in \mathcal{M}_{L^\infty}$, i.e.

$$\forall k \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \left|(U_{w,c})_\varepsilon\right|_{k,\infty} = O\left(\varepsilon^{-m}\right), \varepsilon \longrightarrow 0.$$

Hence $\left((U_{w,c})_\varepsilon\right)_\varepsilon \in \mathcal{M}_{ap}$ and $U_{w,c} \in \mathcal{G}_{ap}$. It is easy to show that the result does not depend on the choice of representatives and by taking $U = \left[\left(c^{\frac{t}{w}} (U_{w,c})_\varepsilon\right)_\varepsilon\right]$, we conclude that U is indeed a well-defined element of $\mathcal{G}_{AP_{w,c}}$. $\square$

We can also introduce the notion of $(w,c)-$periodicity in the framework of algebras of generalized functions. To this end, we recall that a continuous function f is said to be $(w,c)-$periodic, if and only if, $f(t+w) = cf(t)$, for all $t \in \mathbb{R}$. We denote by P_w the space of usual periodic functions with period w and by $P_{w,c}$ the space of $(w,c)-$periodic functions, then $P_w \subset P_{w,c}$; for more details on $P_{w,c}$, see [1]. We recall also the following notions and results of the algebra of periodic generalized functions which have been introduced by V. F. Valmorin in [12]. Let $S_1 := \{z \in \mathbb{C};\ |z| = 1\}$ be the compact subgroup of $\mathbb{C}$. In the same way as that of Colombeau algebra $\mathcal{G}$, we construct the algebra $\mathcal{G}(S_1)$ of generalized functions on S_1. We denote by $\mathcal{X}(S_1)$ the algebra $(\mathcal{E}(S_1))^I$ of families of applications of class $\mathcal{C}^\infty$ on S_1 indexed by I and we set

$$\mathcal{X}_M(S_1) = \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in \mathcal{X}(S_1),\ \forall \alpha \in \mathbb{Z}_+, \exists m \in \mathbb{Z}_+, \\ \left\|u_\varepsilon^{(\alpha)}\right\|_{L^\infty} = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \end{array} \right\},$$

$$\mathcal{N}(S_1) = \left\{ \begin{array}{c} (u_\varepsilon)_\varepsilon \in \mathcal{X}(S_1),\ \forall \alpha \in \mathbb{Z}_+, \forall m \in \mathbb{Z}_+, \\ \left\|u_\varepsilon^{(\alpha)}\right\|_{L^\infty} = O\left(\varepsilon^{m}\right),\ \varepsilon \longrightarrow 0 \end{array} \right\}.$$

The space $\mathcal{X}_M(S_1)$ is a subalgebra of $\mathcal{X}(S_1)$ and $\mathcal{N}(S_1)$ is an ideal of $\mathcal{X}_M(S_1)$. The algebra of generalized functions on S_1 is given by the quotient

$$\mathcal{G}(S_1) := \frac{\mathcal{X}_M(S_1)}{\mathcal{N}(S_1)}.$$

To recall the algebra of periodic generalized functions on $\mathbb{R}$ we denote firstly by $\mathcal{E}_w$ the space of functions of class $\mathcal{C}^\infty$ on $\mathbb{R}$ periodic with period w, and let $\mathcal{X}_w = (\mathcal{E}_w)^I$. Considering the following application

$$\begin{aligned} \theta_w : \mathbb{R} &\longrightarrow S_1 \\ t &\longmapsto e^{2\pi i \frac{t}{w}}, \end{aligned}$$

we define the following spaces

$$\begin{aligned} \mathcal{X}_{w,M} &= \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{X}_w : \exists (v_\varepsilon)_\varepsilon \in \mathcal{X}_M(S_1),\ u_\varepsilon = v_\varepsilon \circ \theta_w \right\}, \\ \mathcal{N}_w &= \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{X}_w : \exists (v_\varepsilon)_\varepsilon \in \mathcal{N}(S_1),\ u_\varepsilon = v_\varepsilon \circ \theta_w \right\}. \end{aligned}$$

The algebra of periodic generalized functions on $\mathbb{R}$ with period w, is defined by the quotient

$$\mathcal{G}_{P_w} := \frac{\mathcal{X}_{w,M}}{\mathcal{N}_w}.$$

The map

$$\begin{aligned} \mathcal{G}(S_1) &\longrightarrow \mathcal{G}_{P_w} \\ (u_\varepsilon)_\varepsilon + \mathcal{N}(S_1) &\longmapsto (u_\varepsilon \circ \theta_w)_\varepsilon + \mathcal{N}_w, \end{aligned}$$

is an isomorphism of differential algebras. Moreover, it is easy to see that

$$\mathcal{G}_{P_w} \subset \mathcal{G}_{ap}.$$

With the same approach that we adopted to define $(w,c)-$almost periodic generalize functions, we are able to introduce the following concept.

Definition 3.6. *The space of $(w,c)-$periodic generalized functions, for given $c \in \mathbb{C}\setminus\{0\}$ and $w > 0$, is denoted and defined by*

$$\mathcal{G}_{P_{w,c}} := \left\{ u \in \mathcal{G}_{L^\infty_{w,c}} : u_{w,c} \in \mathcal{G}_{P_w} \right\}.$$

When $c = 1$, we get

$$\mathcal{G}_{P_{w,c}} = \mathcal{G}_{P_w}.$$

By applying the standard results of classical setting on the level of representatives, we can prove the following characterizations of $(w,c)-$periodic generalized functions.

Theorem 3.1. *Let $u \in \mathcal{G}_{L^\infty_{w,c}}$. Then the following assertions are equivalent:*

(i) $u \in \mathcal{G}_{P_{w,c}}$.

(ii) *There exists $v \in \mathcal{G}_{P_w}$ such that $u = c^{\frac{t}{w}} v$ in $\mathcal{G}$.*

(iii) $u(\cdot + w) = cu(\cdot)$ *in $\mathcal{G}$.*

The space of Bloch-periodic generalized functions was introduced and studied by M. F. Hasler, see [8].

Definition 3.7. *For given $w,k \in \mathbb{R}$, the space of Bloch-periodic generalized functions is*

$$\mathcal{G}_{P_{w,k}} := \left\{ u \in \mathcal{G} : u(\cdot + w) = e^{ik.w} u(\cdot) \right\}.$$

When. $k.w = 2\pi$, we obtain that

$$\mathcal{G}_{P_{w,k}} = \mathcal{G}_{P_w},$$

and from Theorem $3.1-(iii)$, if $c = e^{ik.w}$, it follows that

$$\mathcal{G}_{P_{w,k}} = \mathcal{G}_{P_{w,c}}.$$

In conclusion, in the case where $|c| = 1$ and $w > 0$, we can simply show the following chain of inclusions

$$\mathcal{G}_{P_{w,c}} \subset \mathcal{G}_{AP_{w,c}} \subset \mathcal{G}_{L^{\infty}_{w,c}} \subset \mathcal{G}.$$

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