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# FIXED POINT THEOREMS FOR GENERALIZED $F-$CONTRACTION ON METRIC SPACE

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ABSTRACT. In this paper, we have generalized $F-$contraction to $MF-$contraction, a new of notation in metric space and we have proved some results with examples and corollaries.

## 1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is an important part of mathematics. In 1922, Banach investigated the fixed point theorem in metric spaces. It is used not only in Mathematics but also in other sciences such as Biology, Physics to find the existance of a unique solution to an equation. There are two types of generalization in literature. One is due to change to more generalized spaces such as $b-$metric space, $b_v(s)$metric space etc. and the other is due to change of various contractions such as Kannan, Chatterjea, Reich, Ćirić type contraction etc. on this spaces. Let us first recall the definition of a metric space. In this paper, we denote the metric space $(X,d)$ by $X$.

**Definition 1.1.** *Let $X$ be a non-empty set and $d : X \to [0,+\infty)$ be a function satisfying*

*i)* $d(x,y) \geq 0$ *and* $d(x,y) = 0$ *if and only if* $x = y$;
*ii)* $d(x,y) = d(y,x)$;
*iii)* $d(x,y) \leq d(x,z) + d(z,y)$

*for all $x,y,z \in X$. Then $d$ is called a metric on $X$ and $(X,d)$ is called a metric space.*

**Definition 1.2.** *Let $\{x_n\}$ be a sequence in $X$. Then the sequence is called convergent if there exists an $x \in X$ such that $d(x_n,x) \mapsto 0$ as $n \to +\infty$.*
*The sequence is called a Cauchy sequence if $\lim\limits_{m,n\to+\infty} d(x_m,x_n) = 0$ for all $m,n \in \mathbb{N}$.*
*The space $(X,d)$ is called complete if every Cauchy sequence converges in $X$.*

In 2012, D. Wardowski [10] introduced the notion of $F-$contraction as follows:

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**Definition 1.3.** [10] *Let* $F : (0,+\infty) \to (-\infty,+\infty)$ *be a mapping satisfying:*

**(F1)** $F$ *is strictly increasing, i.e., for all* $a,b \in (0,+\infty), a < b$ *implies* $F(a) < F(b)$;

**(F2)** *For each sequence* $\{\alpha_n\} \in (0,+\infty)$, $\lim_{n\to+\infty} \alpha_n = 0$ *if and only if* $\lim_{n\to+\infty} F(\alpha_n) = -\infty$;

**(F3)** *There exists* $k \in (0,1)$ *such that* $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

*A mapping* $T : X \to X$ *is a* $F-$*contraction on a metric space* $(X,d)$ *if there exists* $\tau > 0$ *such that for all* $x,y \in X$,

$$d(Tx,Ty) > 0 \text{ implies } \tau + F(d(Tx,Ty)) \leq F(d(x,y)).$$

It is noted that every $F-$contraction is continuous in the metric space. Many authors used this contraction partially or fully in various ways.

Turinci [7] showed that the condition **(F2)** can be relaxed to the form

**(F2′)** $\lim_{t\to 0+} F(t) = -\infty$.

After that Wardowski [11] also introduced $(\phi,F)-$ contraction as:

**Definition 1.4.** [11] *A self-mapping* $T$ *on metric space* $(X,d)$, *for some functions* $F : (0,+\infty) \to (-\infty,+\infty)$ *and* $\phi : (0,+\infty) \to (0,+\infty)$ *is said to be* $(\phi,F)-$*contraction if the following condition holds:*

**(H1)** $F$ *satisfies* ***(F1)*** *and* ***(F2')****;*

**(H2)** $\lim inf_{s\to t^+} \phi(s) > 0$ *for all* $t \geq 0$;

**(H3)** $\phi(d(x,y)) + F(d(Tx,Ty)) \leq F(d(x,y))$ *for all* $x,y \in X$ *such that* $Tx \neq Ty$.

Cosentino and Vetro [3] defined the notion of $F-$contraction of Hardy-Rogers type and generalized the main result of Wardowski [10].

**Definition 1.5.** [3] *Let* $(X,d)$ *be a metric space. A mapping* $T : X \to X$ *is called a* $F-$*contraction of Hardy-Rogers-type if there exists* $\tau > 0$ *and* $F : (0,+\infty) \to (-\infty,+\infty)$ *satisfying* ***(F1), (F2)*** *and* ***(F3)*** *such that*

$$\tau + F(d(Tx,Ty)) \leq F(ad(x,y) + bd(x,Tx) + \gamma d(y,Ty) + \delta d(x,Ty) + Ld(y,Tx))$$

*holds for any* $x,y \in X$ *with* $d(Tx,Ty) > 0$, *where* $a,b,\gamma,\delta,L$ *are non-negative numbers,* $\gamma \neq 1$ *and* $a+b+\gamma+2\delta = 1$.

Also Wardowski-Dung [12] introduce the notion of $F-$weak contraction as follows:

**Definition 1.6.** [12] *Let* $(X,d)$ *be a metric space. A mapping* $T : X \to X$ *is said to be a* $F-$*weak contraction on* $(X,d)$ *if there exists* $F : (0,+\infty) \to (-\infty,+\infty)$ *satisfying* ***(F1)***, ***(F2)*** *and* ***(F3)*** *and* $\tau > 0$ *such that for all* $x,y \in X$ *with* $d(Tx,Ty) > 0$, *the following holds:*

$$\tau + F(d(Tx,Ty)) \leq F(d(M(x,y))),$$

*where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \tfrac{d(x,Ty)+d(y,Tx)}{2}\}.$$

They have proved the generalized Banach contraction principle using this $F$-weak contraction.

Recently in a similar way, Vujaković et. al. [8] have introduced the notion of $(\phi,F)-$weak contraction, $(\phi,F)-$weak contraction of Hardy Rogers type and $(\phi,F)-$weak contraction of Ćirić type contraction. They have proved some results considering the functions $F$ and $\phi$ satisfying only the conditions **(F1)** and **(H2)** respectively.

Hussain and Salimi [5] worked on $F-$contraction with another two functions $\alpha$ admissible mapping with respect to $\eta$. Also they have introduced the notion of $\alpha-\eta-GF-$contraction as bellow and proved some results on it.

Let $\Delta_G$ denote the set of all continuous functions $G:[0,+\infty)^4\to[0,+\infty)$ satisfying: for all $t_1,t_2,t_3,t_4\in[0,+\infty)$ with $t_1.t_2.t_3.t_4=0$ there exists a $\tau>0$ such that $G(t_1,t_2,t_3,t_4)=\tau$.

**Definition 1.7.** [5] *Let $(X,d)$ be a metric space and $T$ be a self-mapping on $X$. Also suppose that $\alpha,\eta:X\times X\to[0,+\infty)$ be two functions. The mapping $T$ is said to be an $\alpha-\eta-GF-$contraction if for $x,y\in X$ with $\eta(x,Tx)\le\alpha(x,y)$ and $d(Tx,Ty)>0$ holds*

$$G(d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx))+F(d(Tx,Ty))\le F(d(x,y))$$

*where $G\in\Delta_G$ and $F$ satisfying **(F1), (F2)** and **(F3)**.*

Many authors such as Ahamad et. al. [1], Ansari et al. [2], Dey et. al. [4], Piri and Kumam [6], Vujaković and Radenović [9] and others have worked on the $F-$conteaction in metric spaces and generalized many existing results.

## 2. Main Results

In view of this generalization and the works of Wardowski [10], Wardowski-Dung [12], Hussain and Salimi [5], Vujaković et. al. [8] and others we introduce a new notion as follows:

**Definition 2.1.** *Let, $(X,d)$ be a metric space. A self-map $T:X\to X$ is said to be a $MF-$ contraction if there exists $\tau>0$ such that for $x,y\in X$*

$$M(Tx,Ty)>0 \text{ implies } \tau+F(M(Tx,Ty))\le F(M(x,y)),$$

*where $M(x,y)$ is a function of $d(x,y)$ and $F:(0,+\infty)\to(-\infty,+\infty)$ satisfies **(F1)** and **(F2)**.*

*Remark* 2.1. Let $(X,d)$ be a complete metric space and self-map $T$ is $MF-$ contraction. If $M(x,y)=d(x,y)$, then $T$ has a unique fixed point in $X$ according to Wardowski's theorem.

In this case $T$ is continuous since it is a $F-$contraction.

**Theorem 2.1.** *Let $(X,d)$ be a complete metric space and $T$ continuous self-map that is $MF-$contraction where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\}$, then $T$ has a unique fixed point in $X$.*

*Proof.* Let, $\{x_n\}$ be a sequence in $X$ with an initial approximation $x_0 \in X$ such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly, if $x_{n+1} = x_n$, then $x_n$ is a fixed point of $T$ and is unique.

Now we show that $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$. We have

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Similarly we can show

$$M(Tx_n, Tx_{n+1}) = \max\{d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_{n+3})\}.$$

If $d(x_n, x_{n+1}) \le d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}$, then

$$M(x_n, x_{n+1}) = d(x_{n+1}, x_{n+2}) \text{ and } M(Tx_n, Tx_{n+1}) = d(x_{n+2}, x_{n+3}).$$

Then by the given condition

$$\tau + F(M(Tx_n, Tx_{n+1})) \le F(M(x_n, x_{n+1}))$$

implies,

$$\begin{aligned} F(d(x_{n+2}, x_{n+3})) &\le F(d(x_{n+1}, x_{n+2})) - \tau \le F(d(x_n, x_{n+1})) - 2\tau \le \dots \\ &\le F(d(x_0, x_1)) - (n+2)\tau. \end{aligned}$$

Applying **(F2)**, we get

$\lim_{n\to+\infty} F(d(x_{n+2}, x_{n+3})) = -\infty$ implies, $\lim_{n\to+\infty} d(x_{n+2}, x_{n+3}) = 0$.

Therefore $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

If $d(x_{n+1}, x_{n+2}) \le d(x_n, x_{n+1})$, then $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$ and $M(Tx_n, Tx_{n+1}) = d(x_{n+1}, x_{n+2})$.

Then as above we can show $\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence.

Let, $n, p \in \mathbb{N}$. Then

$$\begin{aligned} d(x_n, x_{n+p}) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p}) \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+p}) \\ &\quad \vdots \\ &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p}). \end{aligned}$$

Taking the limit as $n \to +\infty$ we get

$$\lim_{n\to+\infty} d(x_n, x_{n+p}) = 0 \quad \text{i.e.,} \quad \lim_{n\to+\infty} d(x_{n+p}, x_n) = 0.$$

Thus $\{x_n\}$ is a Cauchy sequence. Since the space is complete, there exists an $a$ in $X$ such that $\lim_{n\to+\infty} x_n = a$. Again $T$ is continuous. Therefore $\lim_{n\to+\infty} Tx_n = Ta$ i.e., $\lim_{n\to+\infty} x_{n+1} = a = Ta$.

Thus $a$ is a fixed point of $T$.

Again suppose $F$ is continuous and $Ta \ne a$. So $d(Ta, a) > 0$.

Since $F$ is strictly increasing, we take the limit as $n \to +\infty$

$$\tau + F\big(\lim_{n\to+\infty} M(Tx_n, Ta)\big) \leq F\big(\lim_{n\to+\infty} M(x_n, a)\big). \tag{2.1}$$

Now

$$\begin{aligned}\lim_{n\to+\infty} M(x_n,a) &= \lim_{n\to+\infty} \max\left\{d(x_n,a), d(x_n,Tx_n), d(a,Ta), \frac{d(x_n,Ta)+d(a,Tx_n)}{2}\right\}\\ &= \max\{0,0,0,0,0\} = 0,\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to+\infty} M(Tx_n, Ta) =\\ &= \lim_{n\to+\infty} \max\left\{d(Tx_n,Ta), d(Tx_n,T^2x_n), d(Ta,T^2a), \frac{d(Tx_n,T^2a)+d(Ta,T^2x_n)}{2}\right\}\\ &= \max\{0,0,0,0,0\} = 0.\end{aligned}$$

From (2.1) we get,

$$\tau + F(0) \leq F(0),$$

which is a contradiction.

Thus $Ta = a$.

To show the uniqueness, let $b$ be another fixed point of $T$.

Then by given condition

$$\begin{aligned}&\tau + F(M(Ta,Tb)) \leq F(M(a,b))\\ i.e.,\ &\tau + F(M(a,b)) \leq F(M(a,b))\\ i.e.,\ &\tau \leq 0,\end{aligned}$$

which is a contradiction.

Therefore $T$ has a unique fixed point in $X$. □

*Remark* 2.2. The theorem can also be proved without considering the condition **(F2)**.

**Corollary 2.1.** *Let $(X,d)$ be a complete metric space and $T$ be a continuous self-map satisfying $MF-$ contraction where*

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

*Then $T$ has a unique fixed point in $X$.*

*Proof.* The proof follows from **Theorem 2.1**. □

**Theorem 2.2.** *Let $(X,d)$ be a complete metric space and $T$ be a continuous self-map satisfying $MF-$ contraction of Hardy Rogers type where $M(x,y) = \alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty) + \alpha_4 d(x,Ty) + \alpha_5 d(y,Tx)$ and each of $\alpha_i \geq 0$, $(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5) < 1$. Then $T$ has a unique fixed point in $X$.*

*Proof.* Construct a sequence $\{x_n\}$ as **Theorem 2.1** and assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}\cup\{0\}$.

By the given condition,

$$\tau + F(M(Tx_n, Tx_{n+1})) \leq F(M(x_n, x_{n+1})), \tag{2.2}$$

where,

$$\begin{aligned} M(x_n,x_{n+1}) &= \alpha_1 d(x_n,x_{n+1}) + \alpha_2 d(x_n,Tx_n) + \alpha_3 d(x_{n+1},Tx_{n+1}) \\ &\quad + \alpha_4 d(x_n,Tx_{n+1}) + \alpha_5 d(x_{n+1},Tx_n) \\ &= \alpha_1 d(x_n,x_{n+1}) + \alpha_2 d(x_n,x_{n+1}) + \alpha_3 d(x_{n+1},x_{n+2}) \\ &\quad + \alpha_4 d(x_n,x_{n+2}) + \alpha_5 d(x_{n+1},x_{n+1}) \\ &\leq (\alpha_1+\alpha_2+\alpha_4) d(x_n,x_{n+1}) + (\alpha_3+\alpha_4) d(x_{n+1},x_{n+2}), \end{aligned}$$

and

$$\begin{aligned} M(Tx_n,Tx_{n+1}) &= \alpha_1 d(Tx_n,Tx_{n+1}) + \alpha_2 d(Tx_n,T^2x_n) + \alpha_3 d(Tx_{n+1},T^2x_{n+1}) \\ &\quad + \alpha_4 d(Tx_n,T^2x_{n+1}) + \alpha_5 d(Tx_{n+1},T^2x_n) \\ &= \alpha_1 d(x_{n+1},x_{n+2}) + \alpha_2 d(x_{n+1},x_{n+2}) + \alpha_3 d(x_{n+2},x_{n+3}) \\ &\quad + \alpha_4 d(x_{n+1},x_{n+3}) + \alpha_5 d(x_{n+2},x_{n+3}) \\ &\leq (\alpha_1+\alpha_2+\alpha_4) d(x_{n+1},x_{n+2}) + (\alpha_3+\alpha_4) d(x_{n+2},x_{n+3}). \end{aligned}$$

Now if $d(x_n,x_{n+1}) \leq d(x_{n+1},x_{n+2})$, then

$$M(x_n,x_{n+1}) = (\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+1},x_{n+2})$$

and

$$M(x_{n+1},x_{n+2}) = (\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+2},x_{n+3}).$$

From (2.2), we get

$$\tau + F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+2},x_{n+3})) \leq F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+1},x_{n+2}))$$

implying,

$$\begin{aligned} F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+2},x_{n+3})) &\leq F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+1},x_{n+2})) - \tau \\ &\leq F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4)^2 d(x_n,x_{n+1})) - 2\tau \\ &\quad \vdots \\ &\leq F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4)^{n+2} d(x_0,x_1)) - (n+1)\tau. \end{aligned}$$

Applying **(F2)**, we get

$$\lim_{n\to+\infty} F((\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+2},x_{n+3})) = -\infty$$

implies,

$\lim_{n\to+\infty}(\alpha_1+\alpha_2+\alpha_3+2\alpha_4) d(x_{n+2},x_{n+3}) = 0$ i.e., $\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0$ [since $\alpha_1+\alpha_2+\alpha_3+2\alpha_4 \neq 0$].

We can easily show as above in **Theorem 2.1** that $\{x_n\}$ is a Cauchy sequence. Since the space $X$ is complete, there exists an $a \in X$ such that $\lim_{n\to+\infty} x_n = a$.

Also $T$ is continuous. So we have

$$\lim_{n\to+\infty} Tx_n = Ta \text{ i.e., } \lim_{n\to+\infty} x_{n+1} = a = Ta.$$

Thus $a$ is a fixed point of $T$.

Let, $b$ be an another fixed point of $T$ i.e., $d(a,b) > 0$. Then

$$\tau + F(M(Ta,Tb)) \leq F(M(a,b)).$$

Or,

$$\tau + F(M(a,b)) \leq F(M(a,b)),$$

which is a contradiction.

Therefore $a = b$.

Thus $T$ has a unique fixed point in $X$. □

*Remark* 2.3. This theorem can also be prove without using condition **(F2)**. Also if $F$ is continuous, then $T$ has a unique fixed point in $X$.

**Corollary 2.2.** *Let $(X,d)$ be a complete metric space and $T$ be a continuous self-map satisfying $MF-$ contraction of Banach type where $M(x,y) = \alpha d(x,y)$ and $0 < \alpha < 1$. Then $T$ has a unique fixed point in $X$.*

**Corollary 2.3.** *Let $(X,d)$ be a complete metric space and $T$ be a continuous self-map satisfying $MF-$ contraction of Kannan type where $M(x,y) = \alpha d(x,Tx) + \beta d(y,Ty)$ and $0 \leq (\alpha+\beta) < 1$. Then $T$ has a unique fixed point in $X$.*

**Corollary 2.4.** *Let $(X,d)$ be a complete metric space and $T$ be a continuous self-map satisfying $MF-$ contraction of Reich type where $M(x,y) = \alpha_1 d(x,y) + \alpha_2 d(x,Tx) + \alpha_3 d(y,Ty)$ and $\alpha_i \geq 0, (\alpha_1+\alpha_2+\alpha_3) < 1$, then $T$ has a unique fixed point in $X$.*

**Corollary 2.5.** *Let $(X,d)$ be a complete metric space and $T$ be a continuous self-map satisfying $MF-$ contraction of Chaterjea type where $M(x,y) = \alpha_4 d(x,Ty) + \alpha_5 d(y,Tx)$ and each of $\alpha_4, \alpha_5 \geq 0, (\alpha_4+\alpha_5) < 1$. Then $T$ has a unique fixed point in $X$.*

**Example 2.1.** *Let, $X = [0,2]$ and the metric $d$ be defined by $d(x,y) = |x-y|$. $T : X \to X$ be given by $Tx = \frac{2}{9}x^2$ and $F$ is given by $Fx = \ln x$.*
*Then clearly $(X,d)$ is a complete metric space.*
*Now, $M(x,y) = d(x,y) = |x-y|$ and*
$M(Tx,Ty) = d(Tx-Ty) = |Tx-Ty| = |\frac{2}{9}x^2 - \frac{2}{9}y^2| = \frac{2}{9}|x-y||x+y| \leq \frac{8}{9}|x-y|$.
*Therefore, $F(M(x,y)) = F(|x-y|) = \ln|x-y|$ and*
$F(M(Tx,Ty)) = F(\frac{8}{9}|x-y|) \leq \ln(\frac{8}{9}|x-y|) = (\ln(8)-\ln(9)) + \ln(|x-y|)$.
*Thus $(\ln(9)-\ln(8)) + F(M(Tx,Ty)) \leq F(M(x,y))$.*
*Therefore taking $\tau < \ln(9)-\ln(8))$, we get*
$\tau + F(M(Tx,Ty)) \leq F(M(x,y))$.
*Then by **Remark 2.1**, $T$ has a unique fixed point 0.*

**Example 2.2.** *Let, $X = [0,1]$ and the metric $d$ be defined by $d(x,y) = |x-y|$. $T : X \to X$ be given by $Tx = \frac{1}{2}x$ and $F$ is given by $Fx = \ln x$.*
*Then $(X,d)$ is a complete metric space. Suppose $y < x$.*

*Then*

$$\begin{aligned} M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty)\} \\ &= \max\{|x-y|, |x-Tx|, |y-Ty|\} \\ &= \max\{|x-y|, |x-\frac{x}{2}|, |y-\frac{y}{2}|\} \\ &= \max\{|x-y|, |\frac{x}{2}|, |\frac{y}{2}|\} \\ &= \max\{|x-y|, |\frac{x}{2}|\}. \end{aligned}$$

*If* $|x-y| \leq |\frac{x}{2}|$, *then* $\frac{1}{2}|x-y| \leq |\frac{x}{4}|$.
*Thus* $M(x,y) = |\frac{x}{2}|$ *and*

$$\begin{aligned} M(Tx,Ty) &= \max\{d(Tx,Ty), d(Tx,T^2x), d(Ty,T^2y)\} \\ &= \max\{|Tx-Ty|, |Tx-T^2x|, |Ty-T^2y|\} \\ &= \max\{\left|\frac{1}{2}x-\frac{1}{2}y\right|, \left|\frac{1}{2}x-\frac{1}{4}x\right|, \left|\frac{1}{2}y-\frac{1}{4}y\right|\} \\ &= \frac{1}{2}\left|\frac{x}{2}\right|. \end{aligned}$$

*Therefore,*

$$F(M(x,y)) = F\left(\left|\frac{x}{2}\right|\right) = \ln\left(\left|\frac{x}{2}\right|\right) = \ln(|x|) - \ln(2)$$

*and*

$$F(M(Tx,Ty)) = F\left(\left|\frac{x}{4}\right|\right) = \ln(|x|) - 2\ln(2).$$

*So,*

$$\ln(2) + F(M(Tx,Ty)) \leq F(M(x,y)).$$

*Taking* $\tau \leq \ln(2)$, *we get*

$$\tau + F(M(Tx,Ty)) \leq F(M(x,y)).$$

*Therefore by* ***Corollary 2.1***, $T$ *has a unique fixed point* $0$.

*Again, if* $|\frac{x}{2}| \leq |x-y|$, *then* $M(x,y) = |x-y|$ *and* $M(Tx,Ty) = \frac{1}{2}|x-y|$.

*Therefore,*

$$F(M(x,y)) = \ln(|x-y|) \text{ and } F(M(Tx,Ty)) = F(\tfrac{1}{2}|x-y|) = \ln(|x-y|) - \ln(2).$$

*Thus,*

$$\ln(2) + F(M(Tx,Ty)) \leq F(M(x,y)).$$

*Taking* $\tau \leq \ln(2)$, *we get*

$$\tau + F(M(Tx,Ty)) \leq F(M(x,y)).$$

*Therefore, by* ***Corollary 2.1*** $T$ *has a unique fixed point* $0$.

**Example 2.3.** *Let, $X=[0,1]$ and the metric $d$ be defined by $d(x,y)=|x-y|$. $T:X\to X$ be given by $Tx=\frac{2}{3}x$ and $F$ is given by $Fx=\ln x+x, x>0$.*
*Then*

$$\begin{aligned} M(x,y) &= \alpha_1 d(x,y)+\alpha_2 d(x,Tx)+\alpha_3 d(y,Ty)+\alpha_4 d(x,Ty)+\alpha_5 d(y,Tx)\\ &= \alpha_1|x-y|+\alpha_2|x-Tx|+\alpha_3|y-Ty|+\alpha_4|x-Ty|+\alpha_5|y-Tx|\\ &= \alpha_1|x-y|+\alpha_2|x-\frac{2}{3}x|+\alpha_3|y-\frac{2}{3}y|+\alpha_4|x-\frac{2}{3}y|+\alpha_5|y-\frac{2}{3}x| \end{aligned}$$

*and*

$$\begin{aligned} & M(Tx,Ty)\\ &= \alpha_1 d(Tx,Ty)+\alpha_2 d(Tx,T^2x)+\alpha_3 d(Ty,T^2y)+\alpha_4 d(Tx,T^2y)+\alpha_5 d(Ty,T^2x)\\ &= \alpha_1|Tx-Ty|+\alpha_2|Tx-T^2x|+\alpha_3|Ty-T^2y|+\alpha_4|Tx-T^2y|+\alpha_5|Ty-T^2x|\\ &= \alpha_1\left|\frac{2}{3}x-\frac{2}{3}y\right|+\alpha_2\left|\frac{2}{3}x-\frac{4}{9}x\right|+\alpha_3\left|\frac{2}{3}y-\frac{4}{9}y\right|+\alpha_4\left|\frac{2}{3}x-\frac{4}{9}y\right|+\alpha_5\left|\frac{2}{3}y-\frac{4}{9}x\right|\\ &= \frac{2}{3}\left[\alpha_1|x-y|+\alpha_2\left|x-\frac{2}{3}x\right|+\alpha_3\left|y-\frac{2}{3}y\right|+\alpha_4\left|x-\frac{2}{3}y\right|+\alpha_5\left|y-\frac{2}{3}x\right|\right]. \end{aligned}$$

*Therefore,*

$$\begin{aligned} F(M(Tx,Ty)) &= \ln\left(\frac{2}{3}\left[\alpha_1\left|x-y\right|+\alpha_2\left|x-\frac{2}{3}x\right|+\alpha_3\left|y-\frac{2}{3}y\right|+\alpha_4\left|x-\frac{2}{3}y\right|+\alpha_5\left|y-\frac{2}{3}x\right|\right]\right)\\ &= \ln\left(\alpha_1\left|x-y\right|+\alpha_2\left|x-\frac{2}{3}x\right|+\alpha_3\left|y-\frac{2}{3}y\right|+\alpha_4\left|x-\frac{2}{3}y\right|+\alpha_5\left|y-\frac{2}{3}x\right|\right)+\ln\left(\frac{2}{3}\right)\\ &= \ln(M(x,y))+\ln\left(\frac{2}{3}\right). \end{aligned}$$

*Or,* $\ln\left(\frac{2}{3}\right)+F(M(Tx,Ty))\le F(M(x,y))$.

*Thus taking* $\tau\le\ln\left(\frac{2}{3}\right)$, *we get*

$$\tau+F(M(Tx,Ty))\le F(M(x,y)).$$

*Therefore by **Theorem 2.2**, $T$ has unique fixed point* $0$.

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