

THE LOG-BALANCEDNESS OF COMBINATORIAL SEQUENCES

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ABSTRACT. In this paper, we discuss the log-balancedness of combinatorial sequences. We consider operators on sequences that preserve the log-balancedness property. We also give a sufficient condition for the log-balancedness of the product of two sequences. As applications, we prove that some combinatorial sequences are log-balanced. In addition, we discuss the reverse ultra log-concavity of some sequences involving the log-balanced sequence.

1. INTRODUCTION

For convenience, we first recall some definitions involved in this paper.

Definition 1.1. For a given sequence of positive real numbers $\{z_n\}_{n \geq 0}$, it is said to be *log-concave* (or *log-convex*) if $z_n^2 \geq z_{n-1}z_{n+1}$ (or $z_n^2 \leq z_{n-1}z_{n+1}$) for all $n \geq 1$ and it is said to be *log-balanced* if it is log-convex and $\{z_n/n!\}_{n \geq 0}$ is log-concave.

Definition 1.2. For a sequence of positive real numbers $\{z_k\}_{0 \leq k \leq n}$ ($n \geq 3$), $\{z_k\}_{0 \leq k \leq n}$ ($n \geq 3$) is said to be reverse ultra log-concave if $\{z_k/\binom{n}{k}\}_{0 \leq k \leq n}$ is log-convex.

Note that the condition of the reverse ultra log-concavity can be restated as follows:

$$k(n-k)z_k^2 - (n-k+1)(k+1)z_{k-1}z_{k+1} \leq 0, \quad 1 \leq k \leq n-1.$$

There are some examples of reverse ultra log-concave polynomials in [2, 9].

Log-concavity and log-convexity are instrumental in obtaining the growth rate of a sequence and they play important roles in many subjects such as

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combinatorics, algebra, geometry, computer science, quantum physics, white noise theory, probability, economics and mathematical biology. It is clear that a sequence $\{z_n\}_{n \geq 0}$ is log-convex (log-concave) if and only if its quotient sequence $\{z_{n+1}/z_n\}_{n \geq 0}$ is nondecreasing (nonincreasing). A log-balanced sequence is a special case of log-convex sequences, whose quotient sequence does not grow too fast. For a log-convex sequence $\{z_n\}_{n \geq 0}$, it is log-balanced if and only if $(n+1)z_n^2 - nz_{n-1}z_{n+1} \geq 0$ for all $n \geq 1$. Došlić [7] gave some sufficient conditions for the log-balancedness of linear recurrence sequences and showed that a number of combinatorial sequences such as the Motzkin numbers, the fine numbers, the Apéry numbers, the large Schröder numbers, the central Delannoy numbers, and the Franel numbers of order 3 and 4 are log-balanced. Zhao [12] proved that the Catalan-Larcombe-French sequence is also log-balanced. In this paper, we still focus on the log-balancedness of combinatorial sequences. It is well known that the log-concavity (log-convexity) of sequences is preserved under some transformations. For example, the log-concavity of sequences is preserved under both ordinary and binomial convolutions. See for instance [11]. Liu & Wang [10] prove that log-convexity is preserved under componentwise sum, under binomial convolution, and the linear transformations given by the matrices of binomial coefficients, Stirling numbers of the second kind, and signles Stirling numbers of the first kind. The main purpose of this paper is to give operators preserving the log-balancedness of sequences and a sufficient condition for the log-balancedness of the product of two sequences. As applications, we prove that some combinatorial sequences are log-balanced. In addition, we discuss the reverse ultra log-concavity of some sequences involving the log-balanced sequence.

Some other notations are used in this paper. Throughout, let $S(n, k)$ denote Stirling numbers of the second kind. It is well known that $S(0, 0) = 1$ and $S(n, 0) = 0$ for $n \geq 1$. For a nonnegative integer n and a real number x , $(x)_n$ is defined by

$$(x)_n = \begin{cases} x(x-1) \cdots (x-n+1), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

For two sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$, their ordinary convolution is given by

$$z_n = \sum_{k=0}^n x_k y_{n-k}$$

and their binomial convolution is given by

$$z_n = \sum_{k=0}^n \binom{n}{k} x_k y_{n-k}.$$

2. OPERATORS PRESERVING THE LOG-BALANCEDNESS

In this section, we consider operators on sequences that preserve log-balancedness. The following lemmas will be useful.

Lemma 2.1. (Davenport-Pólya Theorem [5]) *If both $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are log-convex, then so is their binomial convolution*

$$z_n = \sum_{k=0}^n \binom{n}{k} x_k y_{n-k}.$$

Lemma 2.2. [11] *If the sequences $\{x_n\}$ and $\{y_n\}$ are both log-concave, then so is their ordinary convolution*

$$z_n = \sum_{k=0}^n x_k y_{n-k}, \quad n = 0, 1, 2, \dots$$

Lemma 2.3. [10] *The Stirling transformation of the second kind $z_n = \sum_{k=0}^n S(n, k) x_k$ preserves the log-convexity.*

Now we give the main results of this section.

Theorem 2.1. *If the sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are both log-balanced, then so is their binomial convolution*

$$z_n = \sum_{k=0}^n \binom{n}{k} x_k y_{n-k}, \quad n = 0, 1, 2, \dots$$

Proof. Since the sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are both log-balanced, $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are log-convex and $\{x_n/n!\}_{n \geq 0}$ and $\{y_n/n!\}_{n \geq 0}$ are log-concave. It follows from Lemma 2.1 that the sequence $\{z_n\}_{n \geq 0}$ is log-convex. We note that

$$\frac{z_n}{n!} = \sum_{k=0}^n \frac{x_k y_{n-k}}{k!(n-k)!}.$$

It follows from Lemma 2.2 that the sequence $\{z_n/n!\}_{n \geq 0}$ is log-concave. Hence $\{z_n\}_{n \geq 0}$ is log-balanced. $\square$

Corollary 2.1. *For $n \geq 0$, let $f_n(q) = \sum_{k=0}^n S(n, k) q^k$, where $q > 0$. If $q \geq 1$, the sequence $\{f_n(q)\}_{n \geq 0}$ is log-balanced.*

Proof. It follows from Lemma 2.3 that $\{f_n(q)\}_{n \geq 0}$ is log-convex. We prove by induction that $f_n(q) = \sum_{k=0}^n S(n, k)q^k$ is log-balanced. For $q \geq 1$, it is clear that

$$\begin{aligned} 2f_1^2(q) - f_0(q)f_2(q) &= q^2 - q \\ &\geq 0, \\ 3f_2^2(q) - 2f_1(q)f_3(q) &= -2q + q^2 + 5q^3 + 3q^4 \\ &\geq 0. \end{aligned}$$

Then $\{f_0(q), f_1(q), f_2(q), f_3(q)\}$ is log-balanced. For $n \geq 3$, assume that $\{f_0(q), f_1(q), \dots, f_{n-1}(q)\}$ is log-balanced. Since (see [3])

$$\begin{aligned} S(n, k) &= \sum_{j=k}^n \binom{n-1}{j-1} S(j-1, k-1), \\ f_n(q) &= \sum_{k=1}^n \sum_{j=k}^n \binom{n-1}{j-1} S(j-1, k-1) q^k \\ &= \sum_{j=1}^n \binom{n-1}{j-1} \sum_{k=1}^j S(j-1, k-1) q^k \\ &= \sum_{j=1}^n \binom{n-1}{j-1} \sum_{k=0}^{j-1} S(j-1, k) q^{k+1} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} q \sum_{k=0}^j S(j, k) q^k. \end{aligned}$$

For $0 \leq j \leq n-1$, let $y_j = q \sum_{k=0}^j S(j, k) q^k$. By the induction hypothesis, we deduce that $\{y_0, y_1, \dots, y_{n-1}\}$ is log-balanced. It follows from Theorem 2.1 that $\{f_0(q), f_1(q), \dots, f_{n-1}(q), f_n(q)\}$ is log-balanced. $\square$

Now we discuss the log-balancedness of some combinatorial sequences.

Example 2.1. Consider the Bell sequence $\{B_n\}_{n \geq 0}$. The Bell number B_n (also called exponential number) is the total number of partitions $\{1, 2, \dots, n\}$, i.e.,

$$B_n = \sum_{k=1}^n S(n, k), \quad (n \geq 1), \quad B_0 = 1.$$

Some initial values of $\{B_n\}$ are as follow:

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	1	2	5	15	52	203	877	4140	21147	115975

It follows from Corollary 2.1 that $\{B_n\}_{n \geq 0}$ is log-balanced.

Example 2.2. Let S_n denote the number of ways of placing n labeled balls into n unlabeled (but 2-colored) boxes ($S_0 = 1$). It is well known that

$$S_n = \sum_{k=1}^n 2^k S(n, k), \quad n \geq 1.$$

Some initial values of $\{S_n\}$ are as follow:

n	1	2	3	4	5	6	7	8
S_n	2	6	22	94	454	2430	14214	89918

It follows from Corollary 2.1 that $\{S_n\}_{n \geq 0}$ is log-balanced.

Corollary 2.2. *If the sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$ are both log-balanced, then so is their convolution*

$$z_n = \sum_{k=m}^n \binom{n}{k} \binom{k}{m} x_{k-m} y_{n-k}, \quad n = m, m+1, \dots,$$

where m is a fixed nonnegative integer.

Proof. For $n \geq m+2$, we have

$$\begin{aligned} z_n &= \binom{n}{m} \sum_{k=m}^n \binom{n-m}{k-m} x_{k-m} y_{n-k} \\ &= \binom{n}{m} \sum_{k=0}^{n-m} \binom{n-m}{k} x_k y_{n-m-k} \\ &= \frac{(n)_m}{m!} \sum_{k=0}^{n-m} \binom{n-m}{k} x_k y_{n-m-k}, \\ \frac{z_n}{n!} &= \frac{1}{m!(n-m)!} \sum_{k=0}^{n-m} \binom{n-m}{k} x_k y_{n-m-k} \\ &= \frac{1}{m!} \sum_{k=0}^{n-m} \frac{x_k y_{n-m-k}}{k!(n-m-k)!}. \end{aligned}$$

It follows from Lemma 2.1 that $\sum_{k=0}^{n-m} \frac{x_k y_{n-m-k}}{k!(n-m-k)!}$ is log-convex for $n = m, m+1, \dots$. On the other hand, we can prove that $\{(n)_m\}$ is log-convex

by the definition of the log-convexity. Hence $\{z_n\}$ is log-convex. It follows from Lemma 2.2 that $\{z_n/n!\}$ is log-concave. Therefore $\{z_n\}_{n \geq m}$ is log-balanced. $\square$

3. A SUFFICIENT CONDITION FOR THE LOG-BALANCEDNESS OF THE PRODUCT OF TWO SEQUENCES

In this section, we give a sufficient condition for the log-balancedness of the product of a log-balanced sequence and a log-concave sequence.

Theorem 3.1. *For two sequences $\{x_n\}_{n \geq 0}$ and $\{y_n\}_{n \geq 0}$, suppose that $\{x_n\}_{n \geq 0}$ is log-balanced and $\{y_n\}_{n \geq 0}$ is log-concave. For $n \geq 0$, let $s_n = x_{n+1}y_{n+1}/(x_n y_n)$. If the sequence $\{s_n\}_{n \geq 0}$ is increasing, $\{x_n y_n\}_{n \geq 0}$ is log-balanced.*

Proof. Since $\{s_n\}_{n \geq 0}$ is increasing, $\{x_n y_n\}_{n \geq 0}$ is log-convex. Noting that $\{x_n\}_{n \geq 0}$ is log-balanced and $\{y_n\}_{n \geq 0}$ is log-concave, we have $(n+1)x_n^2 - nx_{n-1}x_{n+1} \geq 0$ and $y_n^2 - y_{n-1}y_{n+1} \geq 0$ for all $n \geq 1$. Then, for $n \geq 1$,

$$(n+1)x_n^2 y_n^2 - nx_{n-1}y_{n-1}x_{n+1}y_{n+1} \geq 0.$$

Hence the sequence $\{x_n y_n\}_{n \geq 0}$ is log-balanced. $\square$

Now we give an application of Theorem 3.1.

Example 3.1. Let H_n ($n \geq 1$) denote the harmonic number. It is well known that

$$H_n = \sum_{k=1}^n \frac{1}{k}, \quad n \geq 1.$$

For positive integers n and r , the hyperharmonic number $H_n^{[r]}$ is defined by

$$H_n^{[r]} = \begin{cases} H_n, & r = 1, \\ \sum_{k=1}^n H_k^{[r-1]}, & r \geq 2. \end{cases}$$

For the combinatorial connections of hyperharmonic numbers, see [1, 4].

Corollary 3.1. *The sequences $\{n!H_n\}_{n \geq 1}$ and $\{n!H_n^{[2]}\}_{n \geq 1}$ are log-balanced.*

Proof. It is evident that $\{n!\}_{n \geq 1}$ is log-balanced. It follows from the definition of the log-concavity that $\{H_n\}_{n \geq 1}$ is log-concave. For $n \geq 1$, let

$s_n = (n+1)H_{n+1}/H_n$. For $n \geq 1$, we note that

$$\begin{aligned} s_{n+1} - s_n &= 1 - \frac{H_{n+1} - H_n}{H_n H_{n+1}} \\ &= 1 - \frac{1}{(n+1)H_n H_{n+1}} \\ &> 0. \end{aligned}$$

Then $\{s_n\}_{n \geq 1}$ is increasing. Hence, the sequence $\{n!H_n\}_{n \geq 1}$ is log-balanced by Theorem 3.1.

For $n \geq 1$, let $t_n = (n+1)H_{n+1}^{[2]}/H_n^{[2]}$. From the definition of harmonic numbers and hyperharmonic numbers, we get

$$H_n^{[2]} = (n+1)H_n - n.$$

Then we have

$$\begin{aligned} t_{n+1} - t_n &= \frac{(n+2)H_{n+2}^{[2]}H_n^{[2]} - (n+1)(H_{n+1}^{[2]})^2}{H_{n+1}^{[2]}H_n^{[2]}} \\ &= \frac{(n+2)(H_{n+1}^{[2]} + H_{n+2})H_n^{[2]} - (n+1)(H_{n+1}^{[2]})^2}{H_{n+1}^{[2]}H_n^{[2]}} \\ &= \frac{[(n+2)H_n^{[2]} - (n+1)H_{n+1}^{[2]}]H_{n+1}^{[2]} + (n+2)H_{n+2}H_n^{[2]}}{H_{n+1}^{[2]}H_n^{[2]}} \\ &= \frac{(n+2)H_{n+2}H_n^{[2]} - (n+1)H_{n+1}^{[2]}}{H_{n+1}^{[2]}H_n^{[2]}}. \end{aligned}$$

For $n \geq 1$, let $w_n = (n+2)H_{n+2}H_n^{[2]} - (n+1)H_{n+1}^{[2]}$. It is obvious that $w_n > 0$ ($n = 1, 2, 3$). Now we prove that $w_n > 0$ for $n \geq 4$. Through computation, we obtain

$$\begin{aligned} w_n &= (n+1)(H_{n+2}H_n^{[2]} - H_{n+1}^{[2]}) + H_{n+2}H_n^{[2]} \\ &= (n+1)(H_{n+2} - 1)H_n^{[2]} - (n+1)H_{n+1} + H_{n+2}H_n^{[2]} \\ &> H_{n+2}H_n^{[2]} - (n+1)H_{n+1} \\ &= [(n+1)H_n - n]H_{n+2} - (n+1)H_{n+1} \\ &\geq (n+2)H_{n+2} - (n+1)H_{n+1} \\ &> 0. \end{aligned}$$

Then $\{t_n\}_{n \geq 1}$ is increasing. On the other hand, it follows from Lemma 2.2 that $\{H_n^{[2]}\}_{n \geq 1}$ is log-concave. Therefore, the sequence $\{n!H_n^{[2]}\}_{n \geq 1}$ is log-balanced by Theorem 3.1. $\square$

Example 3.2. Dil and Mezö [6] introduced the definition of hyperfibonacci numbers $F_n^{[r]}$ and hyperlucas numbers $L_n^{[r]}$ as follows:

$$F_n^{[r]} = \sum_{j=0}^n F_j^{[r-1]}, \quad L_n^{[r]} = \sum_{j=0}^n L_j^{[r-1]}, \quad r \geq 1,$$

$$F_n^{[0]} = F_n, \quad L_n^{[0]} = L_n,$$

where r is a positive integer, and $\{F_n\}_{n \geq 0}$ and $\{L_n\}_{n \geq 0}$ denotes the Fibonacci and Lucas sequence, respectively. Some initial values of $\{F_n^{[1]}\}$ and $\{L_n^{[1]}\}$ are as follows:

n	0	1	2	3	4	5	6	7	8	9	10
$F_n^{[1]}$	0	1	2	4	7	12	20	33	54	88	143
$L_n^{[1]}$	2	3	6	10	17	28	46	75	122	198	321

Corollary 3.2. The sequences $\{n!F_n^{[1]}\}_{n \geq 1}$ and $\{n!L_n^{[1]}\}_{n \geq 3}$ are log-balanced.

Proof. The log-balancedness of $\{n!F_n^{[1]}\}_{n \geq 1}$ and $\{n!L_n^{[1]}\}_{n \geq 3}$ have been proved in [13]. Here we prove their log-balancedness by using Theorem 3.1.

For $n \geq 1$, let

$$u_n = \frac{(n+1)F_{n+1}^{[1]}}{F_n^{[1]}}, \quad v_n = \frac{(n+1)L_{n+1}^{[1]}}{L_n^{[1]}}.$$

It is evident that

$$F_n^{[1]} = F_{n+2} - 1, \quad L_n^{[1]} = L_{n+2} - 1. \quad (1)$$

By applying (1), we have

$$(F_{n+1}^{[1]})^2 - F_n^{[1]}F_{n+2}^{[1]} = (-1)^n + F_n,$$

$$(L_{n+1}^{[1]})^2 - L_n^{[1]}L_{n+2}^{[1]} = 5(-1)^{n+1} + L_n.$$

Then we derive

$$\begin{aligned} u_{n+1} - u_n &= \frac{(n+2)F_{n+2}^{[1]}F_n^{[1]} - (n+1)(F_{n+1}^{[1]})^2}{F_{n+1}^{[1]}F_n^{[1]}} \\ &= \frac{F_{n+2}^{[1]}F_n^{[1]} - (n+1)(-1)^n - (n+1)F_n}{F_{n+1}^{[1]}F_n^{[1]}} \\ &= \frac{(F_{n+4} - 1)(F_{n+2} - 1) - (n+1)(-1)^n - (n+1)F_n}{F_{n+1}^{[1]}F_n^{[1]}}, \end{aligned}$$

$$\begin{aligned}
v_{n+1} - v_n &= \frac{L_{n+2}^{[1]}L_n^{[1]} - 5(n+1)(-1)^n - (n+1)L_n}{L_{n+1}^{[1]}L_n^{[1]}} \\
&= \frac{(L_{n+4} - 1)(L_{n+2} - 1) - 5(n+1)(-1)^n - (n+1)L_n}{L_{n+1}^{[1]}L_n^{[1]}}.
\end{aligned}$$

It is clear that

$$u_2 - u_1 > 0, \quad u_3 - u_2 > 0, \quad v_2 - v_1 > 0, \quad \text{and} \quad v_3 - v_2 > 0.$$

We can prove by induction that

$$F_n \geq n \quad (n \geq 5), \quad L_n > n \quad (n \geq 2). \quad (2)$$

When $n \geq 3$, it follows from (2) that

$$\begin{aligned}
u_{n+1} - u_n &\geq \frac{(F_{n+4} - 1)(n+1) - (n+1) - (n+1)F_n}{F_{n+1}^{[1]}F_n^{[1]}} \\
&= \frac{(n+1)(F_{n+3} + F_{n+1} - 2)}{F_{n+1}^{[1]}F_n^{[1]}} \\
&> 0, \\
v_{n+1} - v_n &\geq \frac{(n+1)(L_{n+3} + L_{n+1} - 5)}{L_{n+1}^{[1]}L_n^{[1]}} \\
&> 0.
\end{aligned}$$

Then the sequences $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$ are increasing. On the other hand, the sequences $\{F_n^{[1]}\}_{n \geq 1}$ and $\{L_n^{[1]}\}_{n \geq 3}$ are log-concave (see [13]). It follows from Theorem 3.1 that $\{n!F_n^{[1]}\}_{n \geq 1}$ and $\{n!L_n^{[1]}\}_{n \geq 3}$ are log-balanced. $\square$

Corollary 3.3. *For the Fibonacci and Lucas sequence $\{F_n\}_{n \geq 0}$ and $\{L_n\}_{n \geq 0}$, the sequences $\{n!F_{2n}\}_{n \geq 1}$ and $\{n!L_{2n+1}\}_{n \geq 1}$ are log-balanced.*

Proof. For $n \geq 1$, we have

$$\begin{aligned}
\frac{(n+2)F_{2n+4}}{F_{2n+2}} - \frac{(n+1)F_{2n+2}}{F_{2n}} &= \frac{(n+1)(F_{2n+4}F_{2n} - F_{2n+2}^2) + F_{2n+4}F_{2n}}{F_{2n+2}F_{2n}}, \\
\frac{(n+2)L_{2n+3}}{L_{2n+1}} - \frac{(n+1)L_{2n+1}}{L_{2n-1}} &= \frac{(n+1)(L_{2n+3}L_{2n-1} - L_{2n+1}^2) + L_{2n+3}L_{2n-1}}{L_{2n+1}L_{2n-1}}.
\end{aligned}$$

Using $F_{2n+4}F_{2n} - F_{2n+2}^2 = -1$ and $L_{2n+3}L_{2n-1} - L_{2n+1}^2 = -5$, we get

$$\begin{aligned}
\frac{(n+2)F_{2n+4}}{F_{2n+2}} - \frac{(n+1)F_{2n+2}}{F_{2n}} &= \frac{-(n+1) + F_{2n+4}F_{2n}}{F_{2n+2}F_{2n}}, \\
\frac{(n+2)L_{2n+3}}{L_{2n+1}} - \frac{(n+1)L_{2n+1}}{L_{2n-1}} &= \frac{-5(n+1) + L_{2n+3}L_{2n-1}}{L_{2n+1}L_{2n-1}}.
\end{aligned}$$

For $n = 1$, it is evident that

$$\frac{(n+2)F_{2n+4}}{F_{2n+2}} - \frac{(n+1)F_{2n+2}}{F_{2n}} > 0, \quad \frac{(n+2)L_{2n+3}}{L_{2n+1}} - \frac{(n+1)L_{2n+1}}{L_{2n-1}} > 0.$$

For $n \geq 2$, it follows from (2) that

$$\begin{aligned} \frac{(n+2)F_{2n+4}}{F_{2n+2}} - \frac{(n+1)F_{2n+2}}{F_{2n}} &\geq \frac{-(n+1) + 2n + 4}{F_{2n+2}F_{2n}} \\ &> 0, \\ \frac{(n+2)L_{2n+3}}{L_{2n+1}} - \frac{(n+1)L_{2n+1}}{L_{2n-1}} &\geq \frac{-5(n+1) + 4(2n+3)}{L_{2n+1}L_{2n-1}} \\ &> 0. \end{aligned}$$

Then the sequences $\{\frac{(n+1)F_{2n+2}}{F_{2n}}\}_{n \geq 1}$ and $\{\frac{(n+1)L_{2n+1}}{L_{2n-1}}\}_{n \geq 1}$ are increasing. Naturally, the sequences $\{n!F_{2n}\}_{n \geq 1}$ and $\{n!L_{2n+1}\}_{n \geq 1}$ are both log-convex. On the other hand, we note that $\{F_{2n}\}_{n \geq 1}$ and $\{L_{2n-1}\}_{n \geq 1}$ are log-concave. It follows from Theorem 3.1 that the sequences $\{n!F_{2n}\}_{n \geq 1}$ and $\{n!L_{2n+1}\}_{n \geq 1}$ are log-balanced. $\square$

In fact, by using (2), we obtain

$$\begin{aligned} \frac{F_n^2}{n^2} - \frac{F_{n-1}F_{n+1}}{n^2-1} &= \frac{(n^2-1)F_n^2 - n^2F_{n-1}F_{n+1}}{n^2(n^2-1)} \\ &= \frac{(-1)^{n+1}(n^2-1) - F_{n-1}F_{n+1}}{n^2(n^2-1)} \\ &\leq 0, \quad (n \geq 6), \\ (n+1)\frac{F_n^2}{n^2} - n\frac{F_{n-1}F_{n+1}}{n^2-1} &= \frac{(n-1)(n+1)^2F_n^2 - n^3F_{n-1}F_{n+1}}{n^2(n^2-1)} \\ &= \frac{(-1)^{n+1}n^3 + (n^2-n-1)F_n^2}{n^2(n^2-1)} \\ &\geq \frac{n^2-2n-1}{n^2-1}, \quad (n \geq 6) \\ &> 0. \end{aligned}$$

Hence the sequence $\{\frac{F_n}{n}\}_{n \geq 5}$ is log-balanced.

By mathematical induction, we prove that $L_n \geq 3n$ for $n \geq 6$. Then we get

$$\begin{aligned} \frac{L_n^2}{n^2} - \frac{L_{n-1}L_{n+1}}{n^2-1} &= \frac{5(-1)^n(n^2-1) - L_{n-1}L_{n+1}}{n^2(n^2-1)} \\ &\leq \frac{5(n^2-1) - 9(n^2-1)}{n^2(n^2-1)}, \quad (n \geq 7), \\ &\leq 0, \\ (n+1)\frac{L_n^2}{n^2} - n\frac{L_{n-1}L_{n+1}}{n^2-1} &= \frac{(n-1)(n+1)^2L_n^2 - n^3L_{n-1}L_{n+1}}{n^2(n^2-1)} \\ &= \frac{5(-1)^nn^3 + (n^2-n-1)L_n^2}{n^2(n^2-1)} \\ &\geq \frac{9n^2(n^2-n-1) - 5n^3}{n^2(n^2-1)}, \quad (n \geq 7), \\ &> 0. \end{aligned}$$

Hence the sequence $\{\frac{L_n}{n}\}_{n \geq 6}$ is also log-balanced.

Corollary 3.4. *For the Catalan sequence $\{C_n\}_{n \geq 1}$, the sequence $\{nC_n\}_{n \geq 2}$ is log-balanced.*

Proof. It is well known that

$$C_n = \frac{1}{n} \binom{2n-2}{n-1}, \quad n \geq 1.$$

Some values of C_n are as follow:

n	1	2	3	4	5	6	7
C_n	1	1	2	5	14	42	132

It follows from the definition of the log-balancedness that $\{c_n\}_{n \geq 2}$ is log-balanced. It is apparent that $\{\frac{(n+1)C_{n+1}}{nC_n}\}$ is increasing. Clearly, $\{n\}_{n \geq 1}$ is log-concave. It follows from Theorem 3.1 that the sequence $\{nC_n\}_{n \geq 2}$ is log-balanced. $\square$

Corollary 3.5. *For the sequence of Motzkin numbers $\{M_n\}_{n \geq 0}$, the sequences $\{nM_n\}_{n \geq 3}$ and $\{\frac{M_n}{n}\}_{n \geq 4}$ are log-balanced.*

Proof. The Motzkin numbers satisfy the following recurrence relation

$$M_n = \frac{2n+1}{n+2}M_{n-1} + \frac{3(n-1)}{n+2}M_{n-2}, \quad n \geq 2, \quad (3)$$

where $M_0 = M_1 = 1$. For $n \geq 0$, let $q_n = M_{n+1}/M_n$. Now we show that the sequence $\{\frac{(n+1)q_n}{n}\}$ is increasing and $\{\frac{nq_n}{(n+1)^2}\}$ is decreasing. Since

$$\begin{aligned}\frac{(n+2)q_{n+1}}{n+1} &\geq \frac{(n+1)q_n}{n} \iff n(n+2)q_{n+1} - (n+1)^2q_n \geq 0, \\ \frac{(n+1)q_{n+1}}{(n+2)^2} &\leq \frac{nq_n}{(n+1)^2} \iff (n+1)^3q_{n+1} - n(n+2)^2q_n \leq 0,\end{aligned}$$

we only need to prove

$$\begin{aligned}n(n+2)q_{n+1} - (n+1)^2q_n &\geq 0, \quad n \geq 3, \\ (n+1)^3q_{n+1} - n(n+2)^2q_n &\leq 0, \quad n \geq 4.\end{aligned}$$

From (3), we derive

$$q_n = \frac{2n+3}{n+3} + \frac{3n}{(n+3)q_{n-1}}, \quad n \geq 1.$$

Then we obtain

$$\begin{aligned}&n(n+2)q_{n+1} - (n+1)^2q_n \\ &= -\frac{(n+1)^2(n+4)q_n^2 - n(n+2)(2n+5)q_n - 3n(n+1)(n+2)}{(n+4)q_n},\end{aligned}$$

$$\begin{aligned}&(n+1)^3q_{n+1} - n(n+2)^2q_n \\ &= -\frac{n(n+2)^2(n+4)q_n^2 - (n+1)^3(2n+5)q_n - 3(n+1)^4}{(n+4)q_n}.\end{aligned}$$

For $x \geq 0$, define two functions as follows:

$$\begin{aligned}f(x) &= (n+1)^2(n+4)x^2 - n(n+2)(2n+5)x - 3n(n+1)(n+2), \\ g(x) &= n(n+2)^2(n+4)x^2 - (n+1)^3(2n+5)x - 3(n+1)^4.\end{aligned}$$

It is evident that $f(x)$ is increasing on $[\frac{n(n+2)(2n+5)}{2(n+1)^2(n+4)}, +\infty)$ and $g(x)$ is increasing on $[\frac{(n+1)^3(2n+5)}{2n(n+2)^2(n+4)}, +\infty)$. On the other hand, there is an inequality (see [8])

$$b_n \leq q_n \leq b_{n+1}, \quad n \geq 3,$$

where $b_n = \frac{6(n+1)}{2n+5}$. Then $f(x)$ and $g(x)$ are both increasing on $[b_n, b_{n+1}]$. After straightforward computation, we have

$$\begin{aligned} f(b_{n+1}) &= -\frac{3(n+2)(9n^2+75n+96)}{(2n+7)^2} \\ &< 0, \\ g(b_n) &= \frac{9(n+1)^2(4n^3+11n^2-6n-25)}{(2n+5)^2} \\ &> 0. \end{aligned}$$

This implies that

$$n(n+2)q_{n+1} - (n+1)^2q_n > 0 \quad \text{and} \quad (n+1)^3q_{n+1} - n(n+2)^2q_n < 0$$

for $n \geq 3$. Therefore the sequence $\{\frac{(n+1)q_n}{n}\}$ is increasing and the sequence $\{\frac{nq_n}{(n+1)^2}\}$ is decreasing. We note that the log-balancedness of $\{M_n\}_{n \geq 3}$ has been proved in [7]. It follows from Theorem 3.1 that the sequence $\{nM_n\}_{n \geq 3}$ is log-balanced. It follows from the definition of the log-balancedness that the sequence $\{\frac{M_n}{n}\}_{n \geq 4}$ is log-balanced. $\square$

4. THE REVERSE ULTRA LOG-CONCAVITY OF SOME SEQUENCES INVOLVING THE LOG-BALANCED SEQUENCE

In this section, we investigate the reverse ultra log-concavity of some sequences involving the log-balanced sequence.

Theorem 4.1. *Let m be a nonnegative integer. If the sequence $\{z_n\}_{n \geq 0}$ is log-balanced, the sequence $\{z_k/(k+m)!\}_{0 \leq k \leq n}$ is reverse ultra log-concave.*

Proof. For $1 \leq k \leq n-1$, we have

$$\begin{aligned} &\frac{k(n-k)z_k^2}{[(k+m)!]^2} - \frac{(n-k+1)(k+1)z_{k-1}z_{k+1}}{(k+m-1)!(k+m+1)!} \\ &= \frac{k(n-k)(k+m+1)z_k^2 - (n-k+1)(k+1)(k+m)z_{k-1}z_{k+1}}{(k+m)!(k+m+1)!}. \end{aligned}$$

Since $\{z_n\}_{n \geq 0}$ is log-balanced, $\{z_n\}_{n \geq 0}$ is log-convex. Then we have

$$\begin{aligned} &\frac{k(n-k)z_k^2}{[(k+m)!]^2} - \frac{(n-k+1)(k+1)z_{k-1}z_{k+1}}{(k+m-1)!(k+m+1)!} \\ &\leq \frac{[k(n-k)(k+m+1) - (n-k+1)(k+1)(k+m)]z_{k-1}z_{k+1}}{(k+m)!(k+m+1)!} \leq 0. \end{aligned}$$

Hence $\{z_k/(k+m)!\}_{0 \leq k \leq n}$ is reverse ultra log-concave. $\square$

Clearly, for the Bell sequence $\{B_n\}$, the sequence $\{B_k/k!\}_{0 \leq k \leq n}$ is reverse ultra log-concave.

5. CONCLUSIONS

We have discussed some properties of the log-balanced combinatorial sequences. We give some operators on preserving the log-balancedness property. We also give a sufficient condition for the log-balancedness of the product of a log-balanced sequence and a log-concave sequence. As applications, we investigated the log-balancedness of some combinatorial sequences. For example, we prove that the Bell number sequence is log-balanced. Our future work is to study the log-behavior of some nonlinear recurrence sequences.

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