SARAJEVO JOURNAL OF MATHEMATICS
Vol.11 (24), No.2, (2015), 155–164

DOI: 10.5644/SJM.11.2.02

ON SEMIGROUP IDEALS AND GENERALIZED n-DERIVATIONS IN NEAR-RINGS

MOHAMMAD ASHRAF AND MOHAMMAD ASLAM SIDDEEQUE

Abstract. In the present paper, we investigate the commutativity of addition and ring behavior of 3-prime near-rings satisfying certain conditions involving generalized n-derivations on semigroup ideals. Moreover, examples justifying the necessity of the 3-primeness condition in all the results are provided.

1. Introduction

Throughout the paper, N will denote a zero-symmetric left near-ring. N is called zero- symmetric if $0x = 0$ holds for all $x \in N$ (recall that in a left near-ring $x0 = 0$ for all $x \in N$). N is called a 3-prime near-ring if $xNy = \{0\}$ implies $x = 0$ or $y = 0$. It is called semiprime if $xNx = \{0\}$ implies $x = 0$. A nonempty subset U of N is called semigroup left ideal (resp. semigroup right ideal) if $NU \subseteq U$ (resp. $UN \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it will be called a semigroup ideal. The symbol Z will denote the multiplicative center of N, that is, $Z = \{x \in N \mid xy = yx$ for all $y \in N\}$. For any $x, y \in N$ the symbols $[x, y] = xy - yx$ and $(x, y) = x + y - x - y$ stand for multiplicative commutator and additive commutator of x and y respectively. For terminologies concerning near-rings, we refer to G. Pilz [8].

An additive map $d : N \longrightarrow N$ is called a derivation if $d(xy) = d(x)y + xd(y)$ (or equivalently $d(xy) = xd(y) + d(x)y$ as noted in [9, Proposition 1]) holds for all $x, y \in N$. In the year 2006, Gölbasi [5] introduced the notion of generalized derivation in near-rings as follows: An additive mapping $f : N \longrightarrow N$ is called a right generalized derivation with associated derivation d if $f(xy) = f(x)y + xd(y)$, for all $x, y \in N$ and f is called a left generalized derivation with associated derivation d if $f(xy) = d(x)y + xf(y)$, for all $x, y \in N$. f is called a generalized derivation with associated derivation d

---

2010 *Mathematics Subject Classification.* 16W25,16Y30.

Key words and phrases. 3-prime near-ring, semigroup ideals, derivation, n-derivation, generalized n-derivation and commutativity.

if it is both a left as well as a right generalized derivation with associated derivation d. Very recently Ashraf *et al.* [2, 3] generalized the notion of derivation by introducing the notions of n-derivation and generalized n-derivation, where n is a positive integer, which are given below:

A map $D : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \longrightarrow N$ is said to be permuting if the equation $D(x_1, x_2, \dots, x_n) = D(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(n)})$ holds for all $x_1, x_2, \dots, x_n \in N$ and for every permutation $\pi \in S_n$ where S_n is the permutation group on $\{1, 2, \dots, n\}$. A map $d : N \to N$ defined by $d(x) = D(x, x, \dots, x)$ for all $x \in N$ where $D : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \to N$ is a map, is called the trace of D.

An n-additive (i.e. additive in each argument) mapping $D : N \times N \times \cdots \times N \longrightarrow N$ is called an *n-derivation* if the relations

$$\begin{aligned}D(x_1, x_2, \dots, x_{i-1}, x_i x_i', x_{i+1}, \dots, x_n) = {} & D(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) x_i' \\ & + x_i D(x_1, x_2, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)\end{aligned}$$

hold for all $x_1, x_2, \dots, x_{i-1}, x_i, x_i', x_{i+1}, \dots, x_n \in N$, $i = 1, 2, 3, \dots, n$. If in addition, D is a permuting map then all the above conditions are equivalent and in this case D is called a permuting n-derivation of N (see [2] for further reference).

An n-additive mapping $F : N \times N \times \cdots \times N \longrightarrow N$ is called a *right generalized n-derivation* of N with associated n-derivation D if the relations

$$\begin{aligned}F(x_1, x_2, \dots, x_{i-1}, x_i x_i', x_{i+1}, \dots, x_n) = {} & F(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) x_i' \\ & + x_i D(x_1, x_2, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)\end{aligned}$$

hold for all $x_1, x_2, \dots, x_{i-1}, x_i, x_i', x_{i+1}, \dots, x_n \in N$, $i = 1, 2, 3, \dots, n$. If in addition, both F and D are permuting maps then all the above conditions are equivalent and in this case F is called a permuting right generalized n-derivation of N with associated permuting n-derivation D. An n-additive mapping $F : N \times N \times \cdots \times N \longrightarrow N$ is called a *left generalized n-derivation* of N with associated n-derivation D if the relations

$$\begin{aligned}F(x_1, x_2, \dots, x_{i-1}, x_i x_i', x_{i+1}, \dots, x_n) = {} & D(x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) x_i' \\ & + x_i F(x_1, x_2, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)\end{aligned}$$

hold for all $x_1, x_2, \dots, x_{i-1}, x_i, x_i', x_{i+1}, \dots, x_n \in N$, $i = 1, 2, 3, \dots, n$. If in addition, both F and D are permuting maps then all the above conditions are equivalent and in this case F is called a permuting left generalized n-derivation of N with associated permuting n-derivation D. An n-additive mapping $F : N \times N \times \cdots \times N \longrightarrow N$ is called a *generalized n-derivation* of N with associated n-derivation D if it is both a right generalized n-derivation as well as a left generalized n-derivation of N with associated n-derivation D. If in addition, both F and D are permuting maps then F is called a permuting generalized n-derivation of N with associated permuting

n-derivation D (see [3] for further reference). If N is a commutative ring, then it is trivial to see that the set of all left generalized n-derivations of N is equal to the set of all right generalized n-derivations of N.

Recently many authors have studied commutativity of addition and ring behavior of 3-prime near-rings satisfying certain properties and identities involving derivations and generalized derivations on semigroup ideals (see [1, 4, 6, 7] where further references can be found). The purpose of the present paper is to study the commutativity of addition and ring behavior of 3-prime near-rings satisfying certain properties and identities involving generalized n-derivations on semigroup ideals. In fact, our results generalize, extend, compliment and improve several results obtained earlier on derivations, generalized derivations, permuting n-derivations and generalized n-derivations for 3-prime near-rings; for example Theorem 12 of [1], Theorems 3.2 – 3.4 & 3.7 of [2], Theorems 3.1, 3.11, 3.15, 3.16 of [3], Theorem 3.2 – 3.3 of [4] etc.- to mention a few only.

2. Preliminary results

We begin with the following known results which will be used extensively. The proofs of Lemmas 2.1 – 2.3 can be found in [4], whereas Lemma 2.4 is proved in [3].

Lemma 2.1. *Let N be a 3-prime near-ring. If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.*

Lemma 2.2. *Let N be a 3-prime near-ring and U a nonzero semigroup ideal of N. If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*

Lemma 2.3. *Let N be a 3-prime near-ring and Z contains a nonzero semigroup left ideal or nonzero semigroup right ideal, then N is a commutative ring.*

Lemma 2.4. *Let N be a near-ring admitting a generalized n-derivation F with associated n-derivation D of N. Then,*

$$\begin{aligned}&\{D(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)x_i^{'}\\ &\qquad\qquad +x_iF(x_1, x_2, \ldots, x_{i-1}, x_i^{'}, x_{i+1}, \ldots, x_n)\}y\\ &= D(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)x_i^{'}y\\ &\qquad\qquad +x_iF(x_1, x_2, \ldots, x_{i-1}, x_i^{'}, x_{i+1}, \ldots, x_n)y\end{aligned}$$

for all $x_1, x_2, \ldots, x_{i-1}, x_i, x_i^{'}, x_{i+1}, \ldots, x_n, y \in N$, $i = 1, 2, 3, \ldots, n$.

Lemma 2.5. *Let N be a semiprime near-ring admitting a right generalized n-derivation (resp. a left generalized n-derivation) F with associated n-derivation D of N. If $F = 0$, then $D = 0$.*

Proof. Let F be a right generalized n-derivation of N with associated n-derivation D. Then $F(x_1x_1^{'}, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)x_1^{'} + x_1D(x_1^{'}, x_2, \ldots, x_n)$ holds for all $x_1, x_1^{'}, x_2, \ldots, x_n \in N$. Using the hypothesis, the latter relation takes the form $x_1D(x_1^{'}, x_2, \ldots, x_n) = 0$ for all $x_1, x_1^{'}, x_2, \ldots, x_n \in N$. This implies that $D(x_1^{'}, x_2, \ldots, x_n)x_1D(x_1^{'}, x_2, \ldots, x_n) = 0$ i.e. $D(x_1^{'}, x_2, \ldots, x_n)ND(x_1^{'}, x_2, \ldots, x_n) = \{0\}$. Now semiprimeness of N gives us $D(x_1^{'}, x_2, \ldots, x_n) = 0$ for all $x_1^{'}, x_2, \ldots, x_n \in N$. Hence $D = 0$. □

Lemma 2.6. *Let N be a 3-prime near-ring admitting a nonzero generalized n-derivation F with associated n-derivation D of N. Then $F(U_1, U_2, \ldots, U_n) \neq \{0\}$, where $U_1, U_2, \ldots, U_n$ are nonzero semigroup left ideals or nonzero semigroup right ideals of N.*

Proof. Let us suppose that $U_1, U_2, \ldots, U_n$ are nonzero semigroup left ideals of N. We have to prove that $F(U_1, U_2, \ldots, U_n) \neq \{0\}$. Suppose on contrary that $F(U_1, U_2, \ldots, U_n) = \{0\}$. Now we divide the proof in two cases:
Case I: Let $D \neq 0$. For all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, we have $F(u_1, u_2, \ldots, u_n) = 0$. Replacing u_1 by $u_1u_1^{'}$, where $u_1^{'} \in U_1$, the latter relation provides us $F(u_1u_1^{'}, u_2, \ldots, u_n) = 0$ i.e. $D(u_1, u_2, \ldots, u_n)u_1^{'} + u_1F(u_1^{'}, u_2, \ldots, u_n) = 0$. Using hypothesis, we arrive at $D(u_1, u_2, \ldots, u_n)u_1^{'} = 0$. Replacing $u_1^{'}$ by $s_1u_1^{'}$, where $s_1 \in N$, we obtain $D(u_1, u_2, \ldots, u_n)s_1u_1^{'} = 0$ i.e. $D(u_1, u_2, \ldots, u_n)Nu_1^{'} = \{0\}$. Since $U_1 \neq \{0\}$, the 3-primeness of N gives us $D(u_1, u_2, \ldots, u_n) = 0$ for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Putting r_1u_1, where $r_1 \in N$ for u_1 in the latter relation and using it we get $D(r_1, u_2, \ldots, u_n)u_1 = 0$. Now replacing u_1 by t_1u_1, where $t_1 \in N$, we arrive at $D(r_1, u_2, \ldots, u_n)t_1u_1 = 0$ i.e. $D(r_1, u_2, \ldots, u_n)Nu_1 = \{0\}$. Since $U_1 \neq \{0\}$, the 3-primeness of N provides us $D(r_1, u_2, \ldots, u_n) = 0$ for all $r_1 \in N, u_2 \in U_2, \ldots, u_n \in U_n$. Now putting r_2u_2 for u_2, where $r_2 \in N$ in the latter relation and using the similar arguments as above, it can be easily shown that $D(r_1, r_2, u_3, \ldots, u_n) = 0$ for all $r_1, r_2 \in N, u_3 \in U_3, \ldots, u_n \in U_n$. Now proceeding inductively we conclude that $D = 0$, leading to a contradiction.
Case II: Suppose that $D = 0$. As we are given that for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, $F(u_1, u_2, \ldots, u_n) = 0$. Replacing u_1 by p_1u_1, where $p_1 \in N$, the latter relation provides us $F(p_1u_1, u_2, \ldots, u_n) = 0$ i.e. $F(p_1, u_2, \ldots, u_n)u_1 + p_1D(u_1, u_2, \ldots, u_n) = 0$. Under this case, we get $F(p_1, u_2, \ldots, u_n)u_1 = 0$. Replacing u_1 by m_1u_1, where $m_1 \in N$, we obtain $F(p_1, u_2, \ldots, u_n)m_1u_1 = 0$ i.e. $F(p_1, u_2, \ldots, u_n)Nu_1 = \{0\}$. Since $U_1 \neq \{0\}$, the 3-primeness of N gives us $F(p_1, u_2, \ldots, u_n) = 0$ for all $p_1 \in N, u_2 \in U_2, \ldots, u_n \in U_n$. Putting p_2u_2, where $p_2 \in N$ for u_2 in the latter relation and using the similar arguments as above, it is easy to prove that $F(p_1, p_2, u_3, \ldots, u_n) = 0$

for all $p_1, p_2 \in N, u_3 \in U_3, \dots, u_n \in U_n$. Now proceeding inductively, we arrive at $F = 0$. This leads to a contradiction.

Finally including both cases we have shown that $F(U_1, U_2, \dots, U_n) \neq \{0\}$, where $U_1, U_2, \dots, U_n$ are nonzero semigroup left ideals. Similar arguments can be given for semigroup right ideals also. □

Lemma 2.7. *Let N be 3-prime near-ring admitting a generalized n-derivation F with associated nonzero n-derivation D of N and $U_1, U_2, \dots, U_n$ be nonzero semigroup ideals of N.*

(i) If $x \in N$ and $F(U_1, U_2, \dots, U_n)x = \{0\}$, then $x = 0$.

(ii) If $x \in N$ and $xF(U_1, U_2, \dots, U_n) = \{0\}$, then $x = 0$.

Proof. (i) Given that $F(u_1u_1^{'}, u_2, \dots, u_n)x = 0$ for all $u_1, u_1^{'} \in U_1, u_2 \in U_2, \dots, u_n \in U_n$. This yields that $\{D(u_1, u_2, \dots, u_n)u_1^{'} + u_1F(u_1^{'}, u_2, \dots, u_n)\}x = 0$. Using hypothesis and Lemma 2.4, we have $D(u_1, u_2, \dots, u_n)U_1x = \{0\}$. As N is a 3-prime near-ring, using Lemma 2.2 and Lemma 2.6, we conclude that $x = 0$.

(ii) It can be proved in a similar way. □

3. Main results

We start our investigation with the following results:

Theorem 3.1. *Let N be a 3-prime near-ring and $U_1, U_2, \dots, U_n$ be nonzero semigroup ideals of N. If N admits a nonzero generalized n-derivation F with associated n-derivation D of N such that $F(U_1, U_2, \dots, U_n) \subseteq Z$, then N is a commutative ring.*

Proof. Here we divide the proof in two cases:
Case I: Suppose that $D \neq 0$. For all $u_1, u_1^{'} \in U_1, u_2 \in U_2, \dots, u_n \in U_n$, we have

$$F(u_1u_1^{'}, u_2, \dots, u_n) = D(u_1, u_2, \dots, u_n)u_1^{'} + u_1F(u_1^{'}, u_2, \dots, u_n) \in Z. \quad (3.1)$$

Now commuting the equation (3.1) with the element u_1 we get $\{D(u_1, u_2, \dots, u_n)u_1^{'} + u_1F(u_1^{'}, u_2, \dots, u_n)\}u_1 = u_1\{D(u_1, u_2, \dots, u_n)u_1^{'} + u_1F(u_1^{'}, u_2, \dots, u_n)\}$. Using the hypothesis and Lemma 2.4, we get $D(u_1, u_2, \dots, u_n)u_1^{'}u_1 = u_1D(u_1, u_2, \dots, u_n)u_1^{'}$ for all $u_1, u_1^{'} \in U_1, u_2 \in U_2, \dots, u_n \in U_n$. Replacing $u_1^{'}$ by $u_1^{'}r$, where $r \in N$ in the latter relation and using it, we arrive at $D(u_1, u_2, \dots, u_n)\ u_1^{'}[r, u_1] = 0$ i.e. $D(u_1, u_2, \dots, u_n)U_1[r, u_1] = \{0\}$ for all $r \in N, u_1 \in U_1, u_2 \in U_2, \dots, u_n \in U_n$. For given $u_1 \in U_1$, Lemma 2.2 yields either $D(u_1, u_2, \dots, u_n) = 0$ or $[r, u_1] = 0$. If first case holds i.e. $D(u_1, u_2, \dots, u_n) = 0$ for all $u_2 \in U_2, \dots, u_n \in U_n$, then relation (3.1) provides us $F(u_1u_1^{'}, u_2, \dots, u_n) = u_1F(u_1^{'}, u_2, \dots, u_n) \in Z$ for all $u_1^{'} \in U_1, u_2 \in U_2, \dots, u_n \in U_n$. Using Lemmas 2.1 and 2.6, we get $u_1 \in Z$.

On the other hand if second case holds i.e. $[r, u_1] = 0$ for all $r \in N$, then $u_1 \in Z$. Including both cases, we conclude that $U_1 \subseteq Z$ and N is therefore a commutative ring by Lemma 2.3.
Case II: If $D = 0$, then the relation (3.1), takes the form $F(u_1u_1^{'}, u_2, \ldots, u_n)$ $= u_1F(u_1^{'}, u_2, \ldots, u_n) \in Z$ for all $u_1, u_1^{'} \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Now Lemmas 2.1 and 2.6 provide us $u_1 \in Z$ for all $u_1 \in U_1$ i.e. $U_1 \subseteq Z$. Using Lemma 2.3, we conclude that N is a commutative ring. □

Corollary 3.1. ([3], Theorem 3.1) *Let N be a 3-prime near-ring admitting a nonzero generalized n-derivation F with associated n-derivation D of N. If $F(N, N, \ldots, N) \subseteq Z$, then N is a commutative ring.*

The following example demonstrates that N to be 3-prime is essential in the hypothesis of the above theorem.

Example 3.1. Let $\mathbb{Z}$ be the usual ring of integers and $(\mathbb{C}, +, *)$ be the left near-ring of complex numbers. Here '$*$' is defined by $z_1 * z_2 = |z_1|.z_2$ for all $z_1, z_2 \in \mathbb{C}$, where '$+$' and '$.$' denote the usual addition and multiplication of complex numbers. Assume $N = \mathbb{Z} \times \mathbb{C}$ and $U_1 = m_1\mathbb{Z} \times \{0\}, U_2 = m_2\mathbb{Z} \times \{0\}, \ldots, U_n = m_n\mathbb{Z} \times \{0\}$, where $m_1, m_2, \ldots, m_n$ are different positive integers. Then it can be easily verified that N is a zero-symmetric left near-ring with regard to componentwise addition and multiplication, having $U_1, U_2, \ldots, U_n$ its nonzero semigroup ideals. Define $F : N \times N \times \cdots \times N \longrightarrow N$ such that $F((a_1, z_1), (a_2, z_2), \ldots, (a_n, z_n)) = (\lambda a_1 a_2 \ldots a_n, 0)$, where λ is any integer. It is easy to show that N is a semiprime near-ring but not a 3-prime near-ring and F is a nonzero generalized n-derivation of N with associated n-derivation $D = 0$, the zero map from $N \times N \times \cdots \times N$ to N such that $F(U_1, U_2, \ldots, U_n) \subseteq Z$. However, N is not a commutative ring.

Theorem 3.2. *Let N be a 3-prime near-ring and $U_1, U_2, \ldots, U_n$ nonzero semigroup ideals of N. If N admits generalized n-derivations F and G with associated nonzero n-derivations D and H of N respectively such that $F(x_1, x_2, \ldots, x_n)H(y_1, y_2, \ldots, y_n) = -G(x_1, x_2, \ldots, x_n)D(y_1, y_2, \ldots, y_n)$ for all $x_1, y_1 \in U_1; x_2, y_2 \in U_2; \ldots; x_n, y_n \in U_n$, then $(N, +)$ is abelian.*

Proof. If $u_1, u_1^{'} \in U_1$ are such that $u_1 + u_1^{'} \in U_1$, then replacing y_1 in the hypothesis by $u_1, u_1^{'}$ and $u_1 + u_1^{'}$ respectively and using these relations so obtained, we obtain
$F\left(x_1, x_2, \ldots, x_n\right) H\left(u_1, y_2, \ldots, y_n\right) + F\left(x_1, x_2, \ldots, x_n\right) H\left(u_1^{'}, y_2, \ldots, y_n\right) +$
$F(x_1, x_2, \ldots, x_n)H(-u_1, y_2, \ldots, y_n) + F(x_1, x_2, \ldots, x_n)H(-u_1^{'}, y_2, \ldots, y_n) = 0$
i.e. $F\left(x_1, x_2, \ldots, x_n\right) H\left((u_1, u_1^{'}), y_2, \ldots, y_n\right) = 0$.
This implies that $F(U_1, U_2, \ldots, U_n)H((u_1, u_1^{'}), y_2, \ldots, y_n) = \{0\}$. Now using Lemma 2.7$(i)$, we get

$$H((u_1, u_1^{'}), y_2, \ldots, y_n) = 0 \tag{3.2}$$

for all $u_1, u_1^{'} \in U_1$ such that $u_1 + u_1^{'} \in U_1, y_2 \in U_2, \dots, y_n \in U_n$. Now take $u_1 = rx$ and $u_1^{'} = ry$ where $r \in U_1$ and $x, y \in N$, so that $u_1, u_1^{'}$ and $u_1 + u_1^{'}$ are all in U_1. It follows from relation (3.2) that $H(rx + ry - rx - ry, u_2, u_3, \dots, u_n) = 0$ for all $r \in U_1$ and for all $x, y \in N, u_2 \in U_2, \dots, u_n \in U_n$. Replacing r by wr, where $w \in U_1$, in the latter relation and using it, we arrive at $H(w, y_2, \dots, y_n)(rx + ry - rx - ry) = 0$ i.e. $H(U_1, U_2, \dots, U_n)U_1(x + y - x - y) = \{0\}$ for all $x, y \in N$. By Lemma 2.6, it is obvious that $H(U_1, U_2, \dots, U_n) \neq \{0\}$. Application of Lemma 2.2 yields that $(x + y - x - y) = 0$ for all $x, y \in N$. Hence $(N, +)$ is abelian. □

Corollary 3.2. ([3], Theorem 3.15) *Let F and G be generalized n-derivations of 3-prime near-ring N with associated nonzero n-derivations D and H of N respectively such that $F(x_1, x_2, \dots, x_n)H(y_1, y_2, \dots, y_n) = -G(x_1, x_2, \dots, x_n)\, D(y_1, y_2, \dots, y_n)$ for all $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in N$. Then $(N, +)$ is an abelian group.*

Let X and Y be nonempty subsets of N and $a \in N$. By the notations $[X, Y]$ and $[X, a]$ we mean the subsets of N defined by $[X, Y] = \{[x, y] \mid x \in X, y \in Y\}$ and $[X, a] = \{[x, a] \mid x \in X\}$ respectively.

Very recently A. Ali *et al.* [1, Theorem 12] proved that if N is a 3-prime near-ring, admitting a nonzero generalized derivation f with associated nonzero derivation d such that $[f(U), f(U)] = \{0\}$, where U is a nonzero semigroup ideal of N, then $(N, +)$ is abelian. We have improved and extended this result for generalized n-derivation in the setting of 3-prime near-rings. In fact we obtained the following.

Theorem 3.3. *Let N be a 3-prime near-ring and $U_1, U_2, \cdots, U_n$ nonzero semigroup ideals of N. If N admits generalized n-derivations F_1 and F_2 with associated nonzero n-derivations D_1 and D_2 of N respectively such that $[F_1(U_1, U_2, \dots, U_n), F_2(U_1, U_2, \dots, U_n)] = \{0\}$, then $(N, +)$ is abelian.*

Proof. It is straight forward to see that if $z \in N$ is such that $[z, F_2(u_1, u_2, \dots, u_n)] = [z + z, F_2(u_1, u_2, \dots, u_n)] = 0$ for all $u_1 \in U_1, u_2 \in U_2, \dots, u_n \in U_n$ and $x_1, x_1^{'} \in U_1$ are such that $x_1 + x_1^{'} \in U_1$, then $zF_2(c, u_2, \dots, u_n) = 0$, where c is the additive commutator $(x_1 + x_1^{'} - x_1 - x_1^{'}), u_2 \in U_2, \dots, u_n \in U_n$. If $r, s \in U_1$ we have $rs \in U_1$ and $rs + rs = r(s + s) \in U_1$. Since $[F_1(U_1, U_2, \dots, U_n), F_2(U_1, U_2, \dots, U_n)] = \{0\}$, taking $z = F_1(rs, u_2^{'}, \dots, u_n^{'})$ where $r, s \in U_1, u_2^{'} \in U_2, \dots, u_n^{'} \in U_n$ gives $F_1(U_1^2, U_2, \dots, U_n)F_2(c, u_2, \dots, u_n) = \{0\}$ because for all $r, s \in U_1$ implies that $rs \in U_1^2$. But $U_1^2 = \{pq \mid p, q \in U_1\}$ is a nonzero semigroup ideal, so by Lemma 2.7(i) we get

$$F_2(x_1 + x_1^{'} - x_1 - x_1^{'}, u_2, u_3, \dots, u_n) = 0 \tag{3.3}$$

for all $x_1, x_1^{'} \in U_1$ such that $x_1 + x_1^{'} \in U_1, u_2 \in U_2, \dots, u_n \in U_n$. Now take $x_1 = rx$ and $x_1^{'} = ry$ where $r \in U_1$ and $x, y \in N$, so that $x_1, x_1^{'}$

and $x_1 + x_1^{'}$ are all in U_1. It follows from relation (3.3) that $F_2(rx + ry - rx - ry, u_2, u_3, \ldots, u_n) = 0$ for all $r \in U_1$ and for all $x, y \in N, u_2 \in U_2, \ldots, u_n \in U_n$. Replacing r by wr, where $w \in U_1$ in the latter relation, we get $F_2(w(rx + ry - rx - ry), u_2, u_3, \ldots, u_n) = 0$ i.e. $D_2(w, u_2, \ldots, u_n)(rx + ry - rx - ry) + wF_2(rx + ry - rx - ry, u_2, u_3, \ldots, u_n) = 0$. This implies that $D_2(w, u_2, \ldots, u_n)r(x+y-x-y) = 0$ i.e. $D_2(U_1, U_2, \ldots, U_n)U_1(x+y-x-y) = \{0\}$ for all $x, y \in N$. Using Lemma 2.6, we obtain that $D_2(U_1, U_2, \ldots, U_n) \neq \{0\}$. Now applying Lemma 2.2, we conclude that $(x + y - x - y) = 0$ for all $x, y \in N$. Hence $(N, +)$ is abelian. □

Corollary 3.3. ([3], Theorem 3.16) *Let F_1 and F_2 be generalized n-derivations of 3-prime near-ring N with associated nonzero n-derivations D_1 and D_2 of N respectively such that $[F_1(N, N, \ldots, N), F_2(N, N, \ldots, N)] = \{0\}$. Then $(N, +)$ is an abelian group.*

The following example shows that the restriction of 3-primeness imposed on the hypotheses of Theorems 3.2 & 3.3 is not superfluous.

Example 3.2. Let Q be the usual ring of real quaternions and $(S_3, +)$ be the symmetric group of degree 3. Let $S = Q \times S_3$. Define multiplication '$*$' in S by $(q_1, p_1) * (q_2, p_2) = (q_1.q_2, 0)$ for all $(q_1, p_1), (q_2, p_2) \in S$, where '.' is the usual multiplication of the ring Q and 0 stands for identity of group $(S_3, +)$. Then it can be easily seen that $(S, +, *)$ is a distributive near-ring, where '+' stands for componentwise addition. Consider $N = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mid x, y, z, 0 \in S \right\}$. It can be easily seen that N is a zero-symmetric left near-ring with regard to matrix addition and matrix multiplication but not a 3-prime near-ring. Define $D_1, D_2 : \underbrace{N \times N \times \cdots \times N}_{n-\text{times}} \longrightarrow N$ respectively as

$$D_1\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & x_1 x_2 \ldots x_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$D_2\left(\begin{pmatrix} 0 & x_1 & y_1 \\ 0 & 0 & z_1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 & y_2 \\ 0 & 0 & z_2 \\ 0 & 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & x_n & y_n \\ 0 & 0 & z_n \\ 0 & 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 & z_1 z_2 \ldots z_n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that D_1 and D_2 are nonzero n-derivations of N. If we take $F_1 = D_2$ and $F_2 = D_1$, then it can be easily verified that F_1 and F_2 are nonzero generalized n-derivations of N with associated nonzero n-derivations D_1 & D_2 of N respectively. Let $U = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid y, 0 \in S \right\}$. It is obvious that U is a nonzero semigroup ideal of N. If we choose $U_1 = U_2 = \cdots = U_n = U$, then we have the following:
(i) $F_1(x_1, x_2, \ldots, x_n)D_2\,(y_1, y_2, \ldots, y_n) = -F_2\,(x_1, x_2, \ldots, x_n)D_1\,(y_1, y_2, \ldots, y_n)$ for all $x_1, y_1 \in U_1; x_2, y_2 \in U_2; \ldots; x_n, y_n \in U_n$ and
(ii) $[F_1(U_1, U_2, \ldots, U_n), F_2(U_1, U_2, \ldots, U_n)] = \{0\}$. However, $(N, +)$ is not abelian.

Theorem 3.4. *Let N be a 3-prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N. If $K = \{a \in N \mid [F(u_1, u_2, \ldots, u_n), a] = 0\}$ for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$, where $U_1, U_2, \ldots, U_n$ are nonzero semigroup ideals of N and d stands for the trace of D, then $a \in K$ implies either $a \in Z$ or $d(a) = 0$.*

Proof. Since $a \in K$, by hypothesis we have

$$F(u_1, u_2, \ldots, u_n)a = aF(u_1, u_2, \ldots, u_n) \tag{3.4}$$

for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. Putting au_1 in place of u_1 in the above equation and using Lemma 2.4 we get $D(a, u_2, \ldots, u_n)u_1a + aF(u_1, u_2, \ldots, u_n)a = aD(a, u_2, \ldots, u_n)u_1 + aaF(u_1, u_2, \ldots, u_n)$. Using the identity (3.4), we get $D(a, u_2, \ldots, u_n)u_1a = aD(a, u_2, \ldots, u_n)u_1$. Now putting u_1v_1, where $v_1 \in U_1$ for u_1 in the latter relation and using it again, we have $D(a, u_2, \ldots, u_n)u_1[v_1, a] = 0$. This gives us $D(a, u_2, \ldots, u_n)U_1[a, v_1] = \{0\}$. Application of Lemma 2.2 yields , either $[a, v_1] = 0$ for all $v_1 \in U_1$ or $D(a, u_2, \ldots, u_n) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. If the first case holds, then $av_1 = v_1a$ for all $v_1 \in U_1$. In this case replacing v_1 by v_1r, where $r \in N$ in the latter relation and using it again, we obtain $U_1[a, r] = \{0\}$ i.e. $U_1N[a, r] = \{0\}$. As $U_1 \neq \{0\}$, using the 3-primeness of N, we arrive at $a \in Z$. If the second case holds, then $D(a, u_2, \ldots, u_n) = 0$ for all $u_2 \in U_2, \ldots, u_n \in U_n$. Replacing u_2 by au_2 in the relation $D(a, u_2, \ldots, u_n) = 0$ and using this relation, we get $D(a, a, u_3, \ldots, u_n)u_2 = 0$. This implies that $D(a, a, u_3, \ldots, u_n)NU_2 = \{0\}$. Since $U_2 \neq \{0\}$, the 3-primeness of N yields $D(a, a, u_3, \ldots, u_n) = 0$. Proceeding inductively finally we conclude that $D(a, a, a, \ldots, a) = 0$ or $d(a) = 0$. ☐

Corollary 3.4. ([3], Theorem 3.11) *Let N be a 3-prime near-ring admitting a generalized n-derivation F with associated n-derivation D of N. If $K = \{a \in N \mid [F(N, N, \ldots, N), a] = \{0\}\}$ and d stands for the trace of D, then $a \in K$ implies either $a \in Z$ or $d(a) = 0$.*

The following example justifies the existence of 3-primeness in the hypothesis of the Theorem 3.4.

Example 3.3. Consider N, F_1, D_1 and $U_1 = U_2 = \cdots, = U_n = U$ as discussed in the Example 3.2. Let us choose $a = \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $l = (q, p)$ and $0 \neq q \in Q, p \in S_3$. Then clearly $a \in N$ and it can be easily shown that $[F_1(u_1, u_2, \ldots, u_n), a] = 0$ for all $u_1 \in U_1, u_2 \in U_2, \ldots, u_n \in U_n$. However $a \notin Z$ and $d_1(a) \neq 0$, where d_1 stands for the trace of D_1.

Acknowledgment. The authors are greatly indebted to the referee for his / her valuable comments and helpful suggestions which have improved the paper immensely.

REFERENCES

[1] A. Ali, H. E. Bell and P. Miyan, *Generalized derivations on prime near-rings*, Int. J. Math. Math. Sci., 2013, Article ID 170749, 5 pages.

[2] M. Ashraf and M. A. Siddeeque, *On permuting n-derivations in near-rings*, Commun. Korean Math. Soc., 28 (4) (2013), 697–707.

[3] M. Ashraf and M. A. Siddeeque, *On generalized n-derivations in near-rings*, Palestine J. Math., 3 (Spec. 1) (2014), 468–480.

[4] H. E. Bell, *On derivations in near-rings II*, Kluwer Academic Publishers Dordrecht, 426 (1997), 191–197.

[5] Öznur Gölbasi, *Notes on prime near-rings with generalized derivation*, Southeast Asian Bull. Math., 30 (2006), 49–54.

[6] A. A. M. Kamal and K. H. Al-Shaalan, *Commutativity of near-rings with derivations by using algebraic substructures*, Indian J. Pure Appl. Math., 43 (3) (2012), 211–225.

[7] A. A. M. Kamal and K. H. Al-Shaalan, *Commutativity of rings and near-rings with generalized derivations*, Indian J. Pure Appl. Math., 44 (4) (2013), 473–496.

[8] G. Pilz, *Near-rings*, 2nd Ed., North Holland /American Elsevier, Amsterdam, 1983.

[9] X. K. Wang, *Derivations in prime near-rings*, Proc. Amer. Math. Soc., 121 (2) (1994), 361–366.

(Received: October 13, 2014)
(Revised: September 7, 2015)

Department of Mathematics
Aligarh Muslim University
Aligarh – 202002(India)
mashraf80@hotmail.com
aslamsiddeeque@gmail.com