SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.11.2.03
Vol.11 (24), No.2, (2015), 165–170

A NOTE ON THE JACOBSON RADICAL OF A GRADED RING

EMIL ILIĆ-GEORGIJEVIĆ

Abstract. We prove that $J(R_e) = R_e \cap J(R)$, where S is a cancellative partial groupoid with idempotent e, $R = \bigoplus_{s\in S} R_s$ an Artinian S-graded ring inducing S, $J(R)$ the Jacobson radical of R and $J(R_e)$ the Jacobson radical of R_e. We also prove that $J(R)$ is nil if $J(R_e)$ is nil under certain assumptions.

1. Introduction

According to [5, 6, 7], if Δ is a partial groupoid, then a ring $R = \bigoplus_{\delta\in\Delta} R_\delta$, where R_δ are additive subgroups of R, is called Δ-*graded* and we say that R *induces* Δ if and only if: *i)* $R_\xi R_\eta \subseteq R_{\xi\eta}$ whenever $\xi\eta$ is defined, *ii)* $R_\xi R_\eta \neq \{0\}$ implies that the product $\xi\eta$ is defined. In that case, Δ is called the *partial groupoid induced by* R [5, 6, 7]. The set $A = \bigcup_{\delta\in\Delta} R_\delta$ is called the *homogeneous part of* R and elements of A are called *homogeneous elements of* R. The unique $\xi \in \Delta$ for which $0 \neq x \in A$ belongs to R_ξ is called the *degree of* x and is denoted by $\delta(x)$. A partial groupoid Δ is called *right cancellative* if $\xi\zeta = \eta\zeta$ implies $\xi = \eta$. The notion of a *left cancellative* and *cancellative partial groupoid* is clear enough. This notion of a graded ring is equivalent to the notion of a graded ring given by M. Krasner in [8] (see also [1, 3, 9]) and in the sequel, by a graded ring we will mean precisely this. In case when Δ is a right (left) cancellative partial groupoid, we will, following M. Krasner, say that a graded ring R is *right (left) regular*. It is clear that the partial operation on Δ can be made into an operation by simply associating the degree to 0 from the set $\Delta \setminus \Delta^*$, where $\Delta^* = \{\delta \in \Delta \mid R_\delta \neq \{0\}\}$, denote it also by 0, and by defining $\xi\eta := 0$ for those $\xi, \eta \in \Delta$ for which $R_\xi R_\eta = \{0\}$ (see e.g. [9]). A. V. Kelarev [6] provided us with the following open problem:

2010 *Mathematics Subject Classification.* 16W50, 16N20.
Key words and phrases. Graded rings and modules, regular anneids and moduloids, Jacobson radical.

Question 1 ([6])**.** *Is it true that, if Δ is a finite cancellative partial groupoid with idempotent e and $R = \bigoplus_{\delta\in\Delta} R_\delta$ is a Δ-graded ring inducing Δ, then $J(R_e) = R_e \cap J(R)$?*

Here we will answer this question affirmatively in case R is Artinian. More precisely, we prove the following theorem.

Theorem 2. *If Δ is a cancellative partial groupoid with idempotent e and $R = \bigoplus_{\delta\in\Delta} R_\delta$ an Artinian Δ-graded ring inducing Δ, then $J(R_e) = R_e \cap J(R)$.*

In this note, we will also consider the question of whether the Jacobson radical of a graded ring is nil if the Jacobson radical of a homogeneous component of an idempotent degree is nil, related to another problem posed by A. V. Kelarev:

Question 3 ([6])**.** *Is it true that, for every groupoid S and every S-graded ring R satisfying a polynomial identity, if the radicals of subrings R_e are nil for all idempotents e in S, then the radical of R is nil?*

Related to this, we deal with one more open problem which can be found in [6], which asks whether a certain power of the Jacobson radical is contained in the homogeneous Jacobson radical of a Δ-graded ring inducing Δ, where Δ is a finite cancellative partial groupoid.

2. Preliminaries

In this note, we will use techniques developed by E. Halberstadt in [2, 3] for *anneids* [8], structures obtained from graded rings by restricting addition and multiplication to their homogeneous parts. For more details on notions presented in this section, one may also consult [9] and references therein (particularly [1] and [3]) or [10, 13, 14]. If A is an anneid, it has a partial addition since the sum of nonzero elements $x, y \in A$ is a homogeneous element if and only if $\delta(x) = \delta(y)$ in which case we call them *addable* and we write $x\#y$. Multiplication is, however, defined everywhere, according to the very definition of a graded ring. An anneid A can be *linearized* to a graded ring $\overline{A}$ whose homogeneous part it is up to A-isomorphism. More precisely, $\overline{A} = \bigoplus_{a\in A^*} A(a)$, where $A(a) = \{x \in A \mid a\#x\}$.

In order to deal with the Jacobson radical, we will also use the notion of a graded module in the sense of M. Krasner [9, 10]. Let $R = \bigoplus_{\delta\in\Delta} R_\delta$ be a graded ring, $M = \bigoplus_{d\in D} M_d$ a commutative graded group, and let M be a right R-module with external multiplication $(a, x) \to ax$ $(a \in R,\ x \in M)$. M is a *right graded R-module* if

$$(\forall\xi \in \Delta)(\forall s \in D)(\exists t \in D)\ M_s R_\xi \subseteq M_t.$$

The degree of an element $0 \neq x \in \bigcup_{d \in D} M_d$ will be denoted by $d(x)$. If a is a homogeneous element of R and x a homogeneous element of M such that $xa \neq 0$, then $d(xa)$ depends only on $d(x)$ and $\delta(a)$. If $d \in D$ and $\delta \in \Delta$, then, if $M_d R_\delta \neq \{0\}$, there exist $x \in M_d$ and $a \in R_\delta$ such that $xa \neq 0$ and then, we define $d\delta$ to be $d(xa)$. *Regular graded modules* are defined similarly to regular graded rings (see e.g. [9]).

If M is a graded R-module, then, we may observe its homogeneous part N with partial addition induced from that of M, and external multiplication $N \times A \to N$, where A is the homogeneous part of R. Then N is called the *right moduloid* over the anneid A (see e.g. [9]). A nonempty subset N' of an A-moduloid N (right) is called a *submoduloid* if it is itself an A-moduloid. A *right (left) ideal* of an anneid A is a submoduloid of A regarded as a right (left) moduloid over itself. A nonempty subset of an anneid which is both left and right ideal is called an *ideal.* A right A-moduloid N is called *regular* [1, 3, 9] if $0 \neq xa \# xb \neq 0$ implies $a \# b$, $a, b \in A$, $x \in N$. An anneid is called *right (left) regular* if it is regular as a right (left) moduloid over itself. An anneid is called *regular* [1, 3, 9] if it is both left and right regular.

Definition 4 ([3])**.** An A-moduloid M is called *irreducible* if MA is nonzero and if M has no proper submoduloids.

Definition 5 ([2, 3])**.** The *Jacobson radical* $J(A)$ of an anneid A is the intersection of annihilators of all irreducible regular A-moduloids. Intersection of annihilators of all irreducible A-moduloids which are not necessarily regular is called the *large Jacobson radical*, and is denoted by $J_l(A)$.

Remark 6. E. Halberstadt [2, 3] proved that $J(A)$ and $J_l(A)$ coincide when A is an Artinian regular anneid, where an anneid is called Artinian if it satisfies the descending chain condition on ideals.

3. Main results

The key result in proving Theorem 2 is the following result due to E. Halberstadt [2, 3].

Theorem 7 ([2, 3])**.** *Let* $R = \bigoplus_{\delta \in \Delta} R_\delta$ *be a regular graded ring and let* e *be an idempotent of* Δ^**. Then* $J(R_e) = R_e \cap J(A)$*, where* $A = \bigcup_{\delta \in \Delta} R_\delta$ *is the corresponding anneid.*

We now set off to prove Theorem 2 by proving its reformulation.

Theorem 8. *Let* $R = \bigoplus_{\delta \in \Delta} R_\delta$ *be an Artinian regular graded ring. If* $e \in \Delta^*$ *is an idempotent, then* $J(R_e) = R_e \cap J(R)$*.*

Proof. It is clear that $R_e \cap J(R) \subseteq J(R_e)$ (cf. [4]). It is known [3] that $J_l(A) \subseteq J(R)$, which follows from the simple observation that if M is an irreducible R-module, then M may also be viewed as an irreducible A-moduloid. However, since R is Artinian, A is Artinian as well, and according to Remark 6, we have that $J(A) = J_l(A)$. Now, if $x \in J(R_e)$ does not belong to $J(R)$, then, since $J(A) = J_l(A) \subseteq J(R)$, it certainly does not belong to $J(A)$ which yields a contradiction to Theorem 7. □

Whether the Jacobson radical of a G-graded ring R is nil respectively nilpotent when Jacobson radicals of R_e are nil resp. nilpotent, where e is an idempotent, is a question investigated by many authors in cases of G being a group or various types of semigroups (see [6] and references therein). In the case of anneids, it is known that the Jacobson radical $J(A)$ of an Artinian regular anneid is nilpotent [3]. We investigate nilness rather than nilpotency and we will make use of the following result.

Theorem 9 ([2, 3])**.** *Let A be a regular anneid and $a \in J_l(A)$. Then there exists a natural number n such that the degree e of a^n is an idempotent. Moreover, $a^n \in J(R_e)$.*

Lemma 10. *Let A be a regular anneid. If $J(R_e)$ is nil for every idempotent $e \in \Delta^*$, then the large Jacobson radical $J_l(A)$ of an anneid A is nil.*

Proof. Let $a \in J_l(A)$. We know, by Theorem 9, that then there exists a natural number n such that the degree, say e, of a^n is an idempotent and $a^n \in J(R_e)$. However, $J(R_e)$ is by assumption nil, and so there exists a natural number m such that $(a^n)^m = 0$, that is, a is nilpotent, which concludes the proof. □

Lemma 11. *If $M = \bigoplus_{d \in D} M_d$ is a regular graded irreducible R-module, where $R = \bigoplus_{\delta \in \Delta} R_\delta$ is a regular graded ring, then for all $d \in D$ there exists an idempotent $e \in \Delta^*$ such that M_d is an irreducible R_e-module.*

Proof. Since every nonzero homogeneous element of M is a generator of M, it follows that $xA = N$, for every $0 \neq x \in N$ [3], where A is the corresponding anneid of a graded ring R and N the corresponding A-moduloid of a graded R-module M. Then, if $0 \neq x \in N$, since N is regular, there exists $a \in A$ such that $xa = x$ and hence $d(x)\delta(a) = d(xa) = d(x)$ and moreover, $e = \delta(a)$ is an idempotent [3]. We thus have $xR_e = M_{d(x)}$, that is, $M_{d(x)}$ is an irreducible R_e-module. □

Theorem 12. *Let Δ be a nonempty set and $R = \bigoplus_{\delta \in \Delta} R_\delta$ a regular graded PI-ring with finite support of cardinality, say n, with no more than one nonzero idempotent degree, and let $J(A) = J_l(A)$, where A is the corresponding anneid of R. If $J(R_e)$ is nil for an idempotent $e \in \Delta^*$, then the Jacobson radical $J(R)$ of a ring R is nil.*

Proof. It is enough to prove that $(J(R))^n \subseteq \overline{J_l(A)}$, where $A = \bigcup_{\delta\in\Delta} R_\delta$. Indeed, if $x \in J(R)$, then we would have $x^n \in \overline{J_l(A)}$ and $\overline{J_l(A)}$ is nil according to Lemma 10 and [12], Theorem 1.6.36, since R is a PI-ring by assumption. Let M be a regular graded irreducible R-module. Then, according to Lemma 11 and our assumption on the number of nonzero idempotents, there exists the unique nonzero idempotent $e \in \Delta^*$ such that M is the direct sum of irreducible R_e-modules. Now, following the arguments of Corollary 2.9.4 in [11], we obtain that the length of an R-module M is $\leq n$, which yields the desired inclusion, since, by assumption, $J(A) = J_l(A)$, and hence their linearizations also coincide, i.e. $\overline{J(A)} = \overline{J_l(A)}$. □

Graded ring satisfying assumptions of Theorem 12 exists, as the following example shows.

Example 13. Let R be a Jacobson PI-ring. Then

$$B = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} \oplus \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$$

is a PI-ring, with respect to matrix addition and multiplication, and is regularly graded in Krasner's sense. Let us denote by A its homogeneous part. According to Halberstadt [2, 3], the Jacobson radical of a regular anneid A is

$$J(A) = \{x \in A \mid xA \cap \overline{A}_e \subseteq J(\overline{A}_e) \text{ for all idempotents } e \in \Delta^*\}.$$

In our case, since $J(R) = R$, and since the grading set of B has exactly one nonzero idempotent, corresponding to $\left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)$, we have $J(A) = A$. On the other hand, the large Jacobson radical $J_l(A)$ of an anneid A equals $J(\overline{A}) \cap A$ [2, 3]. Thus, in our case,

$$\begin{aligned} J_l(A) &= J(B) \cap A \\ &= \begin{pmatrix} J(R) & J(R) \\ 0 & J(R) \end{pmatrix} \cap A = A. \end{aligned}$$

Hence, $J(A) = J_l(A)$. Therefore, all assumptions of Theorem 12 are satisfied, and so, if $J\left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right) = \left(\begin{smallmatrix} R & 0 \\ 0 & R \end{smallmatrix}\right)$ is nil, then $J(B) = B$ is nil.

Acknowledgment. The author would like to express his deepest gratitude to Academician Professor Mirjana Vuković who introduced him to the theory of general graded rings and turned his attention to open questions regarding the Jacobson radical.

References

[1] M. Chadeyras, *Essai d'une théorie noetherienne pour les anneaux commutatifs, dont la graduation est aussi générale que possible*, Mémoires de la S.M.F., 22 (1970), 3–143.

[2] E. Halberstadt, *Le radical d'un anneide régulier*, C. R. Acad. Sci., Paris, Sér. A, Paris, 270 (1970), 361–363.
[3] E. Halberstadt, *Théorie artinienne homogène des anneaux gradués à grades non commutatifs réguliers*, Thèse doct. sci. math., Arch. orig. Cent. Doc. C.N.R.S., no. 5962. Paris: Centre National de la Recherche Scientifique, 182 p. (1971).
[4] A. V. Kelarev, *Radicals of graded rings and applications to semigroup rings*, Comm. Algebra, 20 (3) (1992), 681–700.
[5] A. V. Kelarev, *On groupoid graded rings*, J. Algebra, 178 (1995), 391–399.
[6] A. V. Kelarev, *Ring Constructions and Applications*, Series in Algebra, Vol. 9, World Scientific, 2002.
[7] A. V. Kelarev and A. Plant, *Bergman's lemma for graded rings*, Comm. Algebra, 23 (12) (1995), 4613–4624.
[8] M. Krasner, *Théorie élémentaire des corpoídes commutatifs sans torsion*, Séminaire Krasner, 1953-54, Vol. 2, exposé no. 5, Secrétariat Math. de la Fac. des Sc. Paris 1956.
[9] M. Krasner, *Anneaux gradués généraux*, Colloque d'Algébre Rennes, (1980), 209–308.
[10] M. Krasner and M. Vuković, *Structures paragraduées (groupes, anneaux, modules)*, Queen's Papers in Pure and Applied Mathematics, No. 77, Queen's University, Kingston, Ontario, Canada 1987.
[11] C. Năstăsescu and F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics, 1836, Springer, 2004.
[12] L. H. Rowen, *Polynomial Identities in Ring Theory*, Academic Press, New York, 1980.
[13] M. Vuković, *Structures graduées et paragraduées*, Prepublication de l'Institut Fourier, Université de Grenoble I, 2001, No. 536, pp. 1–40.
[14] M. Vuković and E. Ilić-Georgijević, *Paragraded Structures*, a book in preparation.

(Received: February 6, 2015)
(Revised: June 9, 2015)

University of Sarajevo
Faculty of Civil Engineering
71000 Sarajevo
Bosnia and Herzegovina
emil.ilic.georgijevic@gmail.com