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# INEQUALITIES BETWEEN RECIPROCALS OF MEANS

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Abstract. The subject of this paper is linear combinations of reciprocal values of classical bivariate means and their behavior for translated values of the arguments. Based on the known asymptotic expansions some optimal parameters were obtained.

## 1. Introduction

In this paper we shall deal with various inequalities between reciprocals of classical means. Before proceeding further, we should fix notation. Let us denote

$$N(s,t)=\frac{s^2+t^2}{s+t}, \qquad Q(s,t)=\sqrt{\frac{s^2+t^2}{2}}$$

$$A(s,t)=\frac{s+t}{2}, \qquad I(s,t)=\frac{1}{e}\left(\frac{t^t}{s^s}\right)^{\frac{1}{t-s}},$$

$$He(s,t)=\frac{s+\sqrt{st}+t}{3}, \qquad L(s,t)=\frac{t-s}{\log t-\log s},$$

$$G(s,t)=\sqrt{st}, \qquad H(s,t)=\frac{2st}{s+t}.$$

Names of these means are contraharmonic mean $N$, quadratic mean $Q$, arithmetic mean $A$, identric mean $I$, Heronian mean $He$, logarithmic mean $L$, geometric mean $G$, and harmonic mean $H$.

We should mention a few examples of what has already been proved. Let us start with the example from the paper [16]

$$\frac{3}{L(s,t)}<\frac{1}{G(s,t)}+\frac{2}{H(s,t)} \tag{1.1}$$

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proved by J. Sándor. The same author together with E. Neuman in [15] showed the improved inequality

$$\frac{3}{L(s,t)} < \frac{1}{G(s,t)} + \frac{4}{A(s,t)+G(s,t)}. \tag{1.2}$$

Later on, C. P. Chen [7] gave a similar but weaker bound with arithmetic and harmonic mean:

$$\frac{3}{L(s,t)} < \frac{2}{A(s,t)} + \frac{1}{H(s,t)}. \tag{1.3}$$

In [15] the following inequality was also proved

$$\frac{3}{P(s,t)} < \frac{1}{A(s,t)} + \frac{4}{A(s,t)+G(s,t)}, \tag{1.4}$$

which sharpens Seiffert's result from [20]:

$$\frac{3}{P(s,t)} < \frac{1}{G(s,t)} + \frac{2}{A(s,t)}. \tag{1.5}$$

$P$ stands for the first Seiffert mean. More inequalities of this type can be found in [18]. Furthermore, the author in [13] finds that the double inequality

$$\frac{\alpha}{G(s,t)} + \frac{1-\alpha}{A(s,t)} < \frac{1}{P(s,t)} < \frac{\beta}{G(s,t)} + \frac{1-\beta}{A(s,t)}$$

holds for all $s,t>0$, with $s \neq t$ if and only if $\alpha = 0$ and $\beta \geq \frac{1}{3}$. Seiffert means were also the subject of study in [21] where some inequalities of this type were proved. We should also mention a recent paper, written by J. Sándor [19], which contains some new inequalities involving logarithmic and first Seiffert means. For some other interesting results see [17].

The purpose of this paper is to present a systematic approach to the inequalities of such type. Our technique is based on the asymptotic expansion of the mean in question. It turns out that every mean has its own asymptotic expansion. Manipulations with such expansions are well defined which enables us to obtain a so called *asymptotic inequality* between means, which is necessary condition for the corresponding inequality between means to be true. Similar analysis was made in our recent papers [9, 10, 11], where we considered linear combinations between means.

## 2. Asymptotic inequalities between means

We shall repeat the following definition from [9]:

**Definition 2.1.** Let $M_1$ and $M_2$ be two means, and

$$M_1(x+s,x+t) - M_2(x+s,x+t) = c_k(t,s)x^{-k+1} + \mathcal{O}(x^{-k}). \tag{2.1}$$

If $c_k(s,t) > 0$ for all $s$ and $t$, $s \neq t$, then we call mean $M_1$ *asymptotically* greater than mean $M_2$, and write

$$M_1 \succ M_2.$$

Of course, it is equivalent to say $M_2$ is asymptotically smaller than mean $M_1$ and write

$$M_2 \prec M_1.$$

Now it is easy to establish the relationship between asymptotic inequality and proper inequality.

**Theorem 2.2.** *If $M_1 \geq M_2$, then $M_1 \succ M_2$.*

*Proof.* For $x$ large enough, the sign of the difference $M_1(x+s, x+t) - M_2(x+s, x+t)$ is the same as the sign of the first term in its asymptotic expansion. □

Asymptotic comparison of two means is easy, it is sufficient to know their asymptotic expansions. For example, in the paper [10] the following expansions were obtained (with $\alpha = \frac{s+t}{2}$ and $\beta = \frac{t-s}{2}$):

$$\begin{aligned} Q(x+s, x+t) &= x + \alpha + \frac{\beta^2}{2x} - \frac{\alpha\beta^2}{2x^2} \\ &\qquad + \frac{\beta^2(4\alpha^2 - \beta^2)}{8x^3} + \frac{\alpha\beta^2(4\alpha^2 + 3\beta^2)}{8x^4} + \ldots, \\ G(x+s, x+t) &= x + \alpha - \frac{\beta^2}{2x} + \frac{\alpha\beta^2}{2x^2} \\ &\qquad - \frac{\beta^2(4\alpha^2 + \beta^2)}{8x^3} + \frac{\alpha\beta^2(4\alpha^2 + 3\beta^2)}{8x^4} + \ldots. \end{aligned}$$

Since $A(x+s, x+t) = x + \alpha$, adding the previous two expansions one obtains

$$Q(x+s, x+t) + G(x+s, x+t) = 2A(x+s, x+t) - \frac{\beta^4}{4}x^{-3} + \mathcal{O}(x^{-4}).$$

Analysis of such relations can be made with additional assumption $\alpha = 0$ which simplify these coefficients, see [9] for explanation.

The following list of coefficients obtained in [10] will be useful.

| | $\times x$ | $\times \beta^2 x^{-1}$ | $\times \beta^4 x^{-3}$ | $\times \beta^6 x^{-5}$ |
|---|---|---|---|---|
| $N$ | 1 | 1 | 0 | 0 |
| $Q$ | 1 | $\frac{1}{2}$ | $-\frac{1}{8}$ | $\frac{1}{16}$ |
| $A$ | 1 | 0 | 0 | 0 |

| | $\times x$ | $\times \beta^2 x^{-1}$ | $\times \beta^4 x^{-3}$ | $\times \beta^6 x^{-5}$ |
|---|---|---|---|---|
| $I$ | 1 | $-\frac{1}{6}$ | $-\frac{13}{360}$ | $-\frac{737}{45360}$ |
| $He$ | 1 | $-\frac{1}{6}$ | $-\frac{1}{24}$ | $-\frac{1}{48}$ |
| $L$ | 1 | $-\frac{1}{3}$ | $-\frac{4}{45}$ | $-\frac{44}{945}$ |
| $G$ | 1 | $-\frac{1}{2}$ | $-\frac{1}{8}$ | $-\frac{1}{16}$ |
| $H$ | 1 | $-1$ | 0 | 0 |

In order to consider relations with reciprocal values of means, we need the following lemma. For the proof, see [6].

**Lemma 2.3.** *Let $a_0 = 1$ and $g(x)$ be a function with asymptotic expansion (as $x \to \infty$):*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

*Then for all real $p$ it holds*

$$[g(x)]^p \sim \sum_{n=0}^{\infty} c_n(p) x^{-n}$$

*where $c_0 = 1$ and*

$$c_n = \frac{1}{n} \sum_{k=1}^{n} [k(1+p) - n] a_k c_{n-k}. \tag{2.2}$$

*In particular, for $p = -1$, coefficients of reciprocal value of an asymptotic series are given by:*

$$c_n = -\sum_{k=1}^{n} a_k c_{n-k}. \tag{2.3}$$

Hence, the coefficients of reciprocal value of a mean can be found by recursive formula. Let us apply (2.3) on the means from the previous table. The result is presented in a table of reciprocal values:

| | $\times x$ | $\times \beta^2 x^{-1}$ | $\times \beta^4 x^{-3}$ | $\times \beta^6 x^{-5}$ |
|---|---|---|---|---|
| $1/N$ | 1 | $-1$ | 1 | $-1$ |
| $1/Q$ | 1 | $-\frac{1}{2}$ | $\frac{3}{8}$ | $-\frac{5}{16}$ |
| $1/A$ | 1 | 0 | 0 | 0 |
| $1/I$ | 1 | $\frac{1}{6}$ | $\frac{23}{360}$ | $\frac{1493}{45360}$ |

| | $\times x$ | $\times \beta^2 x^{-1}$ | $\times \beta^4 x^{-3}$ | $\times \beta^6 x^{-5}$ |
|---|---|---|---|---|
| $1/He$ | 1 | $\frac{1}{6}$ | $\frac{5}{72}$ | $\frac{17}{432}$ |
| $1/L$ | 1 | $\frac{1}{3}$ | $\frac{1}{5}$ | $-\frac{1}{7}$ |
| $1/G$ | 1 | $\frac{1}{2}$ | $\frac{3}{8}$ | $\frac{5}{16}$ |
| $1/H$ | 1 | 1 | 1 | 1 |

To see how this table works, let us test it on the inequality (1.1) from the introduction. We shall write it in the form

$$\frac{1}{G} + \frac{2}{H} - \frac{3}{L} > 0.$$

Replacing each part by its asymptotic expansion, one obtain

$$\frac{1}{G} + \frac{2}{H} - \frac{3}{L} = \frac{3\beta^2}{x} + \mathcal{O}(x^{-3}).$$

This value indicates that the inequality is probably true, but its order is only $\mathcal{O}(x^{-1})$ and the coefficients are not optimal. If one wants to obtain a better approximation, one should take coefficients $a$, $b$, $c$, such that

$$M = \frac{a}{G} + \frac{b}{H} + \frac{c}{L} = \mathcal{O}(x^{-3}).$$

Since

$$M = (a+b+c)x + \frac{(3a+6b+2c)\beta^2}{6x} + \frac{(15a+40b+8c)\beta^4}{40x^3} + \mathcal{O}(x^{-5}),$$

it should be $a+b+c=0$ and after that $a+4b=0$. So, we shall take $a=-4$, $b=1$ and $c=3$ to obtain

$$\frac{-4}{G} + \frac{1}{H} + \frac{3}{L} = \frac{\beta^4}{10x^3} + \mathcal{O}(x^{-5}).$$

**Theorem 2.4.** *The following inequality of order $\mathcal{O}(x^{-3})$ between means is valid:*

$$\frac{1}{H} + \frac{3}{L} \geq \frac{4}{G}. \tag{2.4}$$

*The choice of the coefficients is optimal, the true inequality with some other coefficients has lower order.*

*Proof.* We have to prove that for all positive $t$ and $s$ the following is true:

$$\frac{s+t}{2st} + 3\frac{\log t - \log s}{t-s} - \frac{4}{\sqrt{st}} \geq 0.$$

Because of the homogeneity property, it is sufficient to consider the value of this function for the point $(s,t)$ on a curve $\varphi(s,t)=0$ which connects the coordinate axes, see [9] for details. We choose here $t = 1/s$. Because of the symmetry, it is enough to consider the values $0 < s \le 1$, i.e. $t \ge 1$. Therefore, the following has to be proved:

$$\frac{1}{2}\left(t+\frac{1}{t}\right)+6\frac{\log t}{t-\dfrac{1}{t}}-4\ge 0, \qquad t\ge 1,$$

or

$$f(t) := 12\log t + t^2 - \frac{1}{t^2} - 8\left(t-\frac{1}{t}\right)\ge 0, \qquad t\ge 1.$$

We have $f'(t) = 2(1-t)^4/t^3 \ge 0$. Hence, $f$ is increasing and because of $f(1) = 0$, $f$ is positive on $[1,\infty\rangle$. □

In the set consisted of harmonic, geometric, Heronian, arithmetic and contraharmonic mean, all inequalities with three means and optimal coefficients are provable in the similar manner.

**Theorem 2.5.** *Consider the harmonic (H), geometric (G), Heronian (He), arithmetic (A) and contraharmonic mean (N). The following inequalities of order $\mathcal{O}(x^{-3})$ hold (with optimal coefficients):*

$$\frac{2}{H}+\frac{3}{He}\ge\frac{5}{G}, \qquad \frac{1}{H}+\frac{1}{A}\ge\frac{2}{G}, \qquad \frac{3}{H}+\frac{1}{N}\ge\frac{4}{G}, \qquad \frac{1}{H}+\frac{5}{A}\ge\frac{6}{He},$$
$$\frac{7}{H}+\frac{5}{N}\ge\frac{12}{He}, \qquad \frac{1}{H}+\frac{1}{N}\ge\frac{2}{A}, \qquad \frac{1}{G}+\frac{2}{A}\ge\frac{3}{He}, \qquad \frac{7}{G}+\frac{2}{N}\ge\frac{9}{He},$$
$$\frac{2}{G}+\frac{1}{N}\ge\frac{3}{A}, \qquad \frac{6}{He}+\frac{1}{N}\ge\frac{7}{A}.$$

*Proof.* We shall prove the last inequality. Define

$$\varphi(s) = \frac{1/A(s,\frac{1}{s}) - 1/N(s,\frac{1}{s})}{1/He(s,\frac{1}{s}) - 1/N(s,\frac{1}{s})}, \quad s\in\langle 0,1]\,. \tag{2.5}$$

After simplifying, this expression becomes

$$\varphi(s) = \frac{(s+1)^2(s^2+s+1)}{(s^2+1)(2s^2+3s+2)}$$

and further

$$\varphi'(s) = \frac{(1-s)(1+s)(3s^4+8s^3+12s^2+8s+3)}{(1+s^2)^2(2s^2+3s+2)^2}\ge 0.$$

Now for all $s\in\langle 0,1]$ we have $\varphi(s)\le\varphi(1)=\frac{6}{7}$. This is equivalent to

$$\frac{6}{He(s,\frac{1}{s})}+\frac{1}{N(s,\frac{1}{s})}\ge\frac{7}{A(s,\frac{1}{s})}$$

and the property of symmetry and homogeneity of means ensure that inequality continue to hold for all $s,t>0$.

The rest of the inequalities can be proved similarly. □

The table below is a complete analysis for the linear combination of reciprocals of three means. For each combination an analysis similar to the one given above was made. In order to keep the table in a decent size, we compare following six means: quadratic, arithmetic, identric, logarithmic, geometric, and harmonic.

| | $Q$ | $A$ | $I$ | $L$ | $G$ | $H$ | $\times\beta^4/x^3$ | $\geq 0$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | −4 | 3 | | | | 17/30 | + |
| 2 | 2 | −5 | | 3 | | | 27/20 | + |
| 3 | 1 | −2 | | | 1 | | 3/4 | + |
| 4 | 2 | −3 | | | | 1 | 7/4 | + |
| 5 | 1 | | −5 | 4 | | | 77/90 | + |
| 6 | 1 | | −3 | | 2 | | 202/945 | + |
| 7 | 5 | | −9 | | | 4 | 53/10 | + |
| 8 | 1 | | | −6 | 5 | | 11/28 | + |
| 9 | 4 | | | −9 | | 5 | 69/28 | + |
| 10 | 1 | | | −3 | | 2 | 5/4 | + |
| 11 | | 1 | −2 | 1 | | | 13/180 | + |
| 12 | | 2 | −3 | | 1 | | 202/945 | + |
| 13 | | 5 | −6 | | | 1 | 37/60 | + |
| 14 | | 1 | | −3 | 2 | | 3/20 | + |
| 15 | | 2 | | −3 | | 1 | 2/5 | + |
| 16 | | 1 | | | −2 | 1 | 1/4 | + |
| 17 | | | 1 | −2 | 1 | | 7/180 | + |
| 18 | | | 4 | −5 | | 1 | 23/90 | + |
| 19 | | | 3 | | −5 | 2 | 19/60 | + |
| 20 | | | | 3 | −4 | 1 | 1/10 | + |

This table should be read as follows. The signs of the coefficients are chosen such that the whole combination is asymptotically greater than zero. Then the first row reads as:

$$\frac{1}{Q} - \frac{4}{A} + \frac{1}{I} \sim \frac{17\beta^4}{30x^3} \succ 0.$$

The sign + in the last column means that this asymptotic inequality is correct, which can (easily) be verified by CAS. We used Mathematica for this purpose. Of course, this does not mean that these inequalities were proved in a traditional way using calculus. Verification of such inequalities can be a tedious job, at least for those involving identric means.

## 3. Four means

By the same technique, taking combinations of four means, one can obtain asymptotic inequalities of order $\mathcal{O}(x^{-5})$. As in the previous table, we will give complete relations for selected six means.

| | $Q$ | $A$ | $I$ | $L$ | $G$ | $H$ | $\times\beta^6/x^5$ | $\geq 0$ |
|---|---|---|---|---|---|---|---|---|
| 1 | $-13$ | 154 | $-243$ | 102 | | | 1489/140 | + |
| 2 | $-11$ | 112 | $-135$ | | 34 | | 202/21 | + |
| 3 | $-37$ | 318 | $-315$ | | | 34 | 1267/36 | + |
| 4 | $-1$ | 7 | | $-15$ | 9 | | 55/56 | + |
| 5 | $-16$ | 94 | | $-105$ | | 27 | 17 | + |
| 6 | $-1$ | 5 | | | $-7$ | 3 | 9/8 | + |
| 7 | $-1$ | | 27 | $-48$ | 22 | | 128/105 | + |
| 8 | $-23$ | | 423 | $-477$ | | 77 | 37759/1260 | + |
| 9 | $-19$ | | 225 | | $-318$ | 112 | 1636/63 | + |
| 10 | $-2$ | | | 75 | $-94$ | 21 | 83/28 | + |
| 11 | | $-7$ | 27 | $-33$ | 13 | | 199/840 | + |
| 12 | | $-46$ | 144 | $-111$ | | 13 | 593/315 | + |
| 13 | | $-19$ | 45 | | $-37$ | 11 | 463/504 | + |
| 14 | | $-2$ | | 15 | $-16$ | 3 | 1/7 | + |
| 15 | | | $-18$ | 57 | $-46$ | 7 | 21/1260 | + |

The last column suggests that all of these asymptotic inequalities are in fact true.

The simplest and most elementary inequality is

$$\frac{5}{A} + \frac{3}{H} \geq \frac{1}{Q} + \frac{7}{G}. \tag{3.1}$$

**Theorem 3.1.** *The inequality* (3.1) *holds for all* $s, t > 0$.

*Proof.* We have to prove

$$\frac{10}{s+t}+\frac{3(s+t)}{2st}\geq\sqrt{\frac{2}{s^2+t^2}}+\frac{7}{\sqrt{st}},\qquad\forall s,t>0.$$

By substitution $s+t=2u$, $st=v^2$, this is equivalent with

$$\frac{5v^2-7uv+3u^2}{uv^2}\geq\sqrt{\frac{1}{2u^2-v^2}}.$$

The numerator of the fraction is always positive, so this is equivalent with

$$(2u^2-v^2)(5v^2-7uv+3u^2)^2\geq u^2v^4$$
$$(u-v)^3[18u^3-30u^2v+5v^2+25v^3]\geq 0.$$

Since $u\geq v$, it is sufficient to prove that the expression into brackets is positive. This is true, since it can be written as

$$2(u-v)(9u^2-7uv-2v^2)+(2u^2-5uv+21v^2)v>0.$$

□

We shall prove another one from the 14th row.

**Theorem 3.2.** *The following inequality is valid for all $s,t>0$:*

$$\frac{15}{L}+\frac{3}{H}\geq\frac{2}{A}+\frac{16}{G}. \tag{3.2}$$

*The order of inequality is $\mathcal{O}(x^{-5})$ and the choice of the coefficients is the best possible.*

*Proof.* Let us choose the connection $t=1/s$, $0<s\leq 1$. Then (3.2) is equivalent to

$$30\frac{\log t}{t-\dfrac{1}{t}}+\frac{3}{2}\left(t+\frac{1}{t}\right)\geq\frac{2}{t+\frac{1}{t}}+16,$$

for $t\geq 1$, or

$$f(t):=60\log t+3\left(t^2-\frac{1}{t^2}\right)-\frac{4t}{t^2+1}+32\left(t-\frac{1}{t}\right)\geq 0,\qquad t\geq 1.$$

We have

$$f'(t)=\frac{60}{t}+6t+\frac{6}{t^3}+4\frac{t^2-1}{(t^2+1)^2}+32\left(1+\frac{1}{t^2}\right)>0$$

for $t\geq 1$. Since $f(1)=0$, we conclude that $f$ is positive function on $[1,\infty\rangle$ and the theorem is proved. □

Inequalities with other combinations of means given in the table still need the analytical proof.

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