

QUASI-ASYMPTOTIC BEHAVIOR AT INFINITY OF TEMPERED OPERATORS

DENNIS NEMZER

ABSTRACT. A subspace of Mikusiński operators, which was introduced by K. Yosida, is used to investigate quasi-asymptotic behavior at infinity in one dimension.

1. INTRODUCTION

Asymptotic behavior of functions as well as generalized functions have been found to be useful in applications in different areas such as differential equations, integral transforms, and quantum physics, to name a few.

Quasi-asymptotic behavior of Schwartz distributions was introduced in the early 1970s by Zav'yalov [12] and investigated by Vladimirov, Drozhzhinov and Zav'yalov (see [10] and references in [5]). More recently, Pilipović, Stanković, Vindas and others have continued the investigation. For an excellent account of quasi-asymptotic behavior of distributions, the reader is referred to [5], which includes an extensive bibliography. See also Vindas ([8], [9]) for structural properties of the quasi-asymptotic behavior of distributions.

Another approach to generalized functions is Mikusiński's operational calculus [3]. The ring of continuous complex-valued functions on the real line which vanish on $(-\infty, 0)$, denoted by $C_+(\mathbb{R})$, with addition and convolution has no zero divisors by Titchmarsh's theorem. The quotient field of $C_+(\mathbb{R})$ is called the field of Mikusiński operators.

Yosida [11] constructed a space $\mathcal{M}$ in order to provide a simplified version for Mikusiński's operational calculus without using Titchmarsh's convolution theorem. Even though the space $\mathcal{M}$ does not span the full space of Mikusiński operators, it contains many of the important operators such as the fractional differential operators, which are useful for applications.

2010 *Mathematics Subject Classification.* 44A40, 46F05, 46F12, 41A60.

Key words and phrases. Final value theorem, Mikusiński operator, quasi-asymptotic behavior, Stieltjes transform, tempered distribution, tempered operator.

In this note, we study the notion of quasi-asymptotic behavior at infinity on a subspace of $\mathcal{M}$ called tempered operators.

This paper is organized as follows. In Section 2, the space of tempered operators is introduced. The notion of quasi-asymptotic behavior at infinity of a tempered operator is investigated in Section 3. As an application, a final value theorem for the Stieltjes transform is presented in Section 4. In the last section, Section 5, the space of tempered operators with a sequential convergence is shown to be isomorphic to the space of tempered distributions supported on $[0, \infty)$ with tempered convergence.

2. THE SPACE OF TEMPERED OPERATORS

Let $C_+(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$ which vanish on the interval $(-\infty, 0)$.

For $f, g \in C_+(\mathbb{R})$, the convolution is given by

$$(f * g)(x) = \int_0^x f(x-t)g(t) dt.$$

The space of tempered functions supported on $[0, \infty)$ will be denoted by $C_+^t(\mathbb{R})$. That is, $f \in C_+^t(\mathbb{R})$ provided $f \in C_+(\mathbb{R})$ and there exists $m \in \mathbb{N}$ such that $f(x)x^{-m}$ is bounded as $x \rightarrow \infty$.

Notice that for $f, g \in C_+^t(\mathbb{R})$, then $f + g, f * g \in C_+^t(\mathbb{R})$.

Let H denote the Heaviside function. That is, $H(x) = 1$ for $x \geq 0$ and zero otherwise. For each $n \in \mathbb{N}$, we denote by H^n the function $H * \cdots * H$, where H is repeated n times.

For $k = 1, 2, \dots$

$$\mathcal{M}_k = \left\{ \frac{f}{H^k} : f \in C_+^t(\mathbb{R}) \right\}.$$

Definition 2.1. Let $k \in \mathbb{N}$ and $W_n, W \in \mathcal{M}_k$, $n \in \mathbb{N}$. The sequence $\{W_n\}$ converges to W in $\mathcal{M}_k$, denoted $W_n \rightarrow W$ in $\mathcal{M}_k$, provided $W_n = \frac{f_n}{H^k}$ and $W = \frac{f}{H^k}$ such that for some $m \in \mathbb{N}$,

$$\sup_{x \geq 0} \left| \frac{f_n(x) - f(x)}{1 + x^m} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The space of tempered operators $\mathcal{M}^\tau$ is defined as a countable union space. That is,

$$\mathcal{M}^\tau = \bigcup_{k=1}^{\infty} \mathcal{M}_k.$$

Two elements of $\mathcal{M}^\tau$ are equal, denoted $\frac{f}{H^n} = \frac{g}{H^m}$, if and only if $H^m * f = H^n * g$.

Clearly, $\mathcal{M}_k \subset \mathcal{M}_{k+1}$, $k \in \mathbb{N}$. Now, let $k \in \mathbb{N}$ and $W_n, W \in \mathcal{M}_k$ ($n \in \mathbb{N}$) such that $W_n \rightarrow W$ in $\mathcal{M}_k$. Then, $W_n = \frac{f_n}{H^k}$ and $W = \frac{f}{H^k}$ with $\frac{f_n(x) - f(x)}{1+x^m} \rightarrow 0$ uniformly as $n \rightarrow \infty$ (for some $m \in \mathbb{N}$).

Now, there exists $M > 0$ such that

$$\left| \frac{(H * (f_n - f))(x)}{1 + x^{m+2}} \right| \leq M \sup_{x \geq 0} \left| \frac{f_n(x) - f(x)}{1 + x^m} \right| \int_0^\infty \frac{dx}{1 + x^2}.$$

Since $W_n = \frac{H * f_n}{H^{k+1}}$ ($n \in \mathbb{N}$) and $W = \frac{H * f}{H^{k+1}}$, the above shows that $W_n \rightarrow W$ in $\mathcal{M}_{k+1}$.

Definition 2.2. Let $W_n, W \in \mathcal{M}^\tau$, $n \in \mathbb{N}$. The sequence $\{W_n\}$ converges to W in $\mathcal{M}^\tau$, denoted $\mathcal{M}^\tau\text{-}\lim_{n \rightarrow \infty} W_n = W$, provided there exists $k \in \mathbb{N}$ such that $W_n, W \in \mathcal{M}_k$ for all $n \in \mathbb{N}$, and $W_n \rightarrow W$ in $\mathcal{M}_k$.

Let $\alpha \in \mathbb{C}$ and $\frac{f}{H^k}, \frac{g}{H^n} \in \mathcal{M}^\tau$. Then with scalar multiplication, addition, and multiplication (convolution) as follows, $\mathcal{M}^\tau$ is an algebra with identity $\delta = \frac{H^2}{H^2}$.

- (1) $\alpha \frac{f}{H^k} = \frac{\alpha f}{H^k}$
- (2) $\frac{f}{H^k} + \frac{g}{H^n} = \frac{H^n * f + H^k * g}{H^{k+n}}$
- (3) $\frac{f}{H^k} * \frac{g}{H^n} = \frac{f * g}{H^{k+n}}$.

Let $f \in C_+^t(\mathbb{R})$. Then, $W_f = \frac{H * f}{H} \in \mathcal{M}^\tau$. Therefore, $C_+^t(\mathbb{R})$ can be identified with a subspace of $\mathcal{M}^\tau$.

The generalized derivative and multiplication by “ x ” are defined as follows. Let $W = \frac{f}{H^k} \in \mathcal{M}^\tau$. Then,

- (1) $W^{(n)} = \frac{f}{H^{k+n}}$, for $n = 1, 2, \dots$
- (2) $xW = \frac{x f - k H * f}{H^k}$ ($k \geq 2$) and $x^n W = x(x^{n-1} W)$, $n = 2, 3, \dots$.

Remarks 2.3.

- (1) The above definitions are well-defined.
- (2) Requiring that $k \geq 2$ in the definition of xW is not a limitation.
Indeed, $\frac{f}{H} = \frac{H * f}{H^2}$.
- (3) If $f, f' \in C_+^t(\mathbb{R})$, then $W'_f = W_{f'}$.
- (4) If $f \in C_+^t(\mathbb{R})$, then $xW_f = W_{xf}$.

As the next theorem shows, convolution is a continuous operator on $\mathcal{M}^\tau$.

Theorem 2.4. Let $V \in \mathcal{M}^\tau$. The mapping $\mathcal{M}^\tau \rightarrow \mathcal{M}^\tau$ given by $W \rightarrow W * V$ is continuous.

Proof. Suppose $\mathcal{M}^\tau\text{-}\lim_{n \rightarrow \infty} W_n = W$. Then there exist $k, m \in \mathbb{N}$ and $f_n, f, g \in C_+(\mathbb{R})$ ($n \in \mathbb{N}$) such that $W_n = \frac{f_n}{H^k}$, $W = \frac{f}{H^k}$, and $V = \frac{g}{H^k}$ with

$$\sup_{x \geq 0} \left| \frac{f_n(x) - f(x)}{1 + x^m} \right| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, there exist $M > 0$ and $q \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$|((f_n - f) * g)(x)| \leq M \sup_{x \geq 0} \left| \frac{f_n(x) - f(x)}{1 + x^m} \right| (1 + x^q) \int_0^\infty \frac{dt}{1 + t^2}, \text{ for all } x.$$

Therefore, for all $n \in \mathbb{N}$,

$$\left| \frac{((f_n - f) * g)(x)}{1 + x^q} \right| \leq A \sup_{x \geq 0} \left| \frac{f_n(x) - f(x)}{1 + x^m} \right|, \text{ for all } x \text{ and some } A > 0.$$

Thus, $\mathcal{M}^\tau\text{-}\lim_{n \rightarrow \infty} W_n * V = W * V$. $\square$

Since $W^{(n)} = W * \delta^{(n)}$ for all $W \in \mathcal{M}^\tau$ and $n \in \mathbb{N}$, the previous theorem shows that differentiation is a continuous operation on $\mathcal{M}^\tau$.

Let $W = \frac{f}{H^k} \in \mathcal{M}_k$. For $\lambda > 0$, define $W(\lambda x) = \frac{1}{\lambda^k} \frac{f(\lambda x)}{H^k}$ ($W_\lambda = \frac{1}{\lambda^k} \frac{f_\lambda}{H^k}$).

Definition 2.5. A tempered operator $W = \frac{f}{H^k}$ is said to *vanish on an open interval* (a, b) , denoted $W(x) = 0$ on (a, b) , provided there exists a polynomial p with degree at most $k - 1$ such that $f(x) = p(x)$ for $a < x < b$. The support of a tempered operator W , denoted $\text{supp } W$, is the complement of the largest open set on which W vanishes.

Theorem 2.6. Let $W \in \mathcal{M}_k$ (for some $k \in \mathbb{N}$) such that $\text{supp } W$ is bounded. Then, there exist $\xi \in \mathbb{C}$ and $m \in \mathbb{N} \cup \{0\}$ with $m \leq k - 1$ such that

$$\mathcal{M}^\tau\text{-}\lim_{\lambda \rightarrow \infty} \lambda^{m+1} W(\lambda x) = \xi \delta^{(m)}.$$

Proof. Let $W = \frac{f}{H^k}$ such that $\text{supp } W$ is bounded. Without loss of generality, assume that $k \geq 2$. Now, there exist $a_0, a_1, \dots, a_{k-1} \in \mathbb{C}$ and $b > 0$ such that

$$f(x) = \sum_{j=0}^{k-1} a_j x^j, \text{ for } x > b.$$

Thus,

$$f(x) = g(x) + \sum_{j=0}^{k-1} a_j x^j,$$

where $g \in C(\mathbb{R})$ and $g(x) = 0$, for $x > b$.

Case 1. Assume that there exists $\ell \in \{1, 2, \dots, k - 1\}$ such that $a_\ell \neq 0$ and $a_j = 0$, for $j = \ell + 1, \dots, k - 1$.

Then, there exists a positive constant M such that

$$\left| \frac{\lambda^{-\ell} f(\lambda x) - a_\ell x^\ell}{1 + x^{\ell-1}} \right| \leq \frac{M}{\lambda}, \text{ for all } x \geq 0 \text{ and } \lambda > 1. \quad (2.1)$$

Let $m = k - \ell - 1$. Then $0 \leq m \leq k - 1$, and

$$\lambda^{m+1} W(\lambda x) = \frac{\lambda^{m-k+1} f(\lambda x)}{H^k} = \frac{\lambda^{-\ell} f(\lambda x)}{H^k}. \quad (2.2)$$

Thus, by (2.1) and (2.2),

$$\mathcal{M}^\tau\text{-}\lim_{\lambda \rightarrow \infty} \lambda^{m+1} W(\lambda x) = a_\ell \ell! \delta^{(m)}.$$

Case 2. Assume that $a_j = 0$, for $j = 1, 2, \dots, k - 1$.

Subcase 1. Suppose that $\{g(\lambda \cdot)\}$ converges uniformly on $[0, \infty)$ as $\lambda \rightarrow \infty$.

Let $m = k - 1$.

$$\lambda^{m-k+1} f(\lambda x) = g(\lambda x) + a_0 \rightarrow a_0 \text{ uniformly on } [0, \infty) \text{ as } \lambda \rightarrow \infty.$$

Thus,

$$\mathcal{M}^\tau\text{-}\lim_{\lambda \rightarrow \infty} \lambda^{m+1} W(\lambda x) = a_0 \delta^{(m)}.$$

Subcase 2. Suppose that $\{g(\lambda \cdot)\}$ does not converge uniformly on $[0, \infty)$ as $\lambda \rightarrow \infty$.

Now, since g is continuous and $g(x) = 0$ on (b, ∞) , there exists $M_1 > 0$ such that $|g(\lambda x)| \leq M_1$, for all λ and $x \geq 0$.

Thus,

$$\lambda^{-k+1} f(\lambda x) = \lambda^{-k+1} |g(\lambda x) + a_0| \leq \frac{M_2}{\lambda^{k-1}} \rightarrow 0 \text{ as } \lambda \rightarrow \infty,$$

where $M_2 = M_1 + |a_0|$.

Let $m = 0$. Then,

$$\lambda^{m+1} W(\lambda x) = \frac{\lambda^{-k+1} f(\lambda x)}{H^k}.$$

The above yields,

$$\mathcal{M}^\tau\text{-}\lim_{\lambda \rightarrow \infty} \lambda^{m+1} W(\lambda x) = 0 \delta^{(m)}.$$

□

Corollary 2.7. *Let $W \in \mathcal{M}^\tau$. If the support of W is bounded, then*

$$\mathcal{M}^\tau\text{-}\lim_{\lambda \rightarrow \infty} W(\lambda x) = 0.$$

3. QUASI-ASYMPTOTIC BEHAVIOR AT INFINITY

A real-valued function $L(x)$ is slowly varying at infinity [6], if it is positive, measurable on $[a, \infty)$ for some $a > 0$, and such that for each $\lambda > 0$,

$$\lim_{x \rightarrow \infty} \frac{L(\lambda x)}{L(x)} = 1.$$

The following properties are well-known.

- (1) For each $\alpha > 0$, $x^\alpha L(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $x^{-\alpha} L(x) \rightarrow 0$ as $x \rightarrow \infty$.
- (2) Let $0 < a < b < \infty$. Then,
 - i) $\frac{L(\lambda x)}{L(x)} \rightarrow 1$ uniformly on $[a, b]$ as $\lambda \rightarrow \infty$.
 - ii) L is bounded and integrable on $[a, b]$.

Unless otherwise stated $\alpha \in \mathbb{R}$ and L will denote a slowly varying function.

Let $W \in \mathcal{M}^\tau$. Then, W is said to have *quasi-asymptotic behavior at infinity related to $\lambda^\alpha L(\lambda)$* provided there exists $V \in \mathcal{M}^\tau$ such that

$$\mathcal{M}^\tau\text{-}\lim_{\lambda \rightarrow \infty} \frac{W(\lambda x)}{\lambda^\alpha L(\lambda)} = V.$$

This will be denoted $W \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$.

For $\sigma > 0$, let $\ell_\sigma(x) = H(x)x^\sigma$. Then, for $\alpha > -1$, let $\Theta_{\alpha+1} = \frac{1}{\Gamma(\alpha+2)} \frac{\ell_{\alpha+1}}{H}$, and for $\alpha \leq -1$ and $\alpha + n > 0$, $\Theta_{\alpha+1} = \frac{1}{\Gamma(\alpha+n+1)} \frac{\ell_{\alpha+n}}{H^n}$.

Notice, for $n = 0, 1, 2, \dots$, that $\Theta_{-n} = \delta^{(n)}$. In particular, $\Theta_0 = \delta$ and $\Theta_{-1} = \delta'$. Thus for $0 < \beta < 1$, $\Theta_{-\beta}$ can be interpreted as the fractional differential operator of order β .

Examples 3.1.

- (1) Let $W_p = \frac{H * p_+}{H}$, where p is a polynomial with degree m and $p_+(x) = H(x)p(x)$. Then, W_p has quasi-asymptotic behavior at infinity related to λ^m .
- (2) Let $W \in \mathcal{M}^\tau$ such that W has bounded support. By Theorem 2.6, W has quasi-asymptotic behavior at infinity related to $\lambda^{-(m+1)}$.
- (3) For all $\alpha \in \mathbb{R}$, $\Theta_{\alpha+1}$ has quasi-asymptotic behavior at infinity related to λ^α .
- (4) $H(x)e^{iax} \stackrel{q}{\sim} ia^{-1}\delta$ at infinity related to λ^{-1} ($a \neq 0$).

(For $H(x)e^{iax} \stackrel{q}{\sim} ia^{-1}\delta$ with quasi-asymptotics in $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions, see [7], p.524.)

Let $f(x) = H(x)e^{iax}$. Then $W_f = \frac{H^2 * f}{H^2} = \frac{g}{H^2}$, where $g(x) = -a^{-2}e^{iax} + ia^{-1}x + a^{-2}$, for $x \geq 0$, and $ia^{-1}\delta = ia^{-1}\frac{H^2}{H^2}$.

Now with $m = 0$, for all λ and $x \geq 0$,

$$\left| \frac{\lambda^{-1}g(\lambda x) - ia^{-1}x}{1 + x^m} \right| = (2a^2\lambda)^{-1} \left| 1 - e^{ia\lambda x} \right| \leq \frac{1}{a^2\lambda}.$$

Therefore,

$$W_f \stackrel{q}{\sim} ia^{-1}\delta \text{ at infinity related to } \lambda^{-1}.$$

- (5) Let $\beta \geq 0$ and $L(\lambda) = \log^\beta \lambda$ for $\lambda > e$ and $L(\lambda) = 1$ for $0 \leq \lambda < e$. Let $\alpha > -1$ and $f \in L^1_{loc}(\mathbb{R})$ such that $\text{supp } f \subseteq [B, \infty)$, $B > 0$, and $f(x) \sim x^\alpha L(x)$ as $x \rightarrow \infty$ in the usual sense (i.e. $\frac{f(x)}{x^\alpha L(x)} \rightarrow 1$ as $x \rightarrow \infty$). Then, W_f has quasi-asymptotic behavior at infinity related to $\lambda^\alpha L(\lambda)$. More precisely, $W_f \stackrel{q}{\sim} W_g$ at infinity related to $\lambda^\alpha L(\lambda)$, where $g(x) = x^\alpha_+$.

To see this, let $\Phi(\lambda, x) = \frac{x^{\alpha+1}}{1+x^m} \left(\frac{(\alpha+1)(H*f)(\lambda x)}{(\lambda x)^{\alpha+1}L(\lambda x)} \frac{L(\lambda x)}{L(\lambda)} - 1 \right)$ and $m > \alpha + 3$.

Given $\varepsilon > 0$. Let $0 < x_1 < (\varepsilon)^{\frac{1}{\alpha+1}}$.

Case 1. For $0 \leq x < x_1$.

There exists $A > 0$ (independent of ε) such that for all $\lambda \geq x_1$, $|\Phi(\lambda, x)| < A\varepsilon$.

Now there exists $x_0 > x_1$ such that for all $x > x_0$, $\frac{x^{\alpha+1}}{1+x^{m-2}} < \varepsilon$.

Case 2. For $x > x_0$.

There exists $C > 0$ (independent of ε) such that for all λ , $|\Phi(\lambda, x)| < C\varepsilon$.

Case 3. For $x_1 \leq x \leq x_0$.

Using the fact that $\lim_{\lambda \rightarrow \infty} \frac{L(\lambda x)}{L(\lambda)} = 1$ uniformly on $[x_1, x_0]$ and that $\frac{(\alpha+1)(H*f)(x)}{x^{\alpha+1}L(x)} \rightarrow 1$ as $x \rightarrow \infty$, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, $|\Phi(\lambda, x)| < \varepsilon$.

Thus, $W_f \stackrel{q}{\sim} W_g$ at infinity related to $\lambda^\alpha L(\lambda)$ follows.

Properties 3.2. Let $n \in \mathbb{N}$ and $W \in \mathcal{M}^\tau$. If $W \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$, then

- (1) $W^{(n)} \stackrel{q}{\sim} V^{(n)}$ at infinity related to $\lambda^{\alpha-n}L(\lambda)$.
- (2) $x^n W \stackrel{q}{\sim} x^n V$ at infinity related to $\lambda^{\alpha+n}L(\lambda)$.

Proof.

(1) Follows from definitions.

(2) Let $W = \frac{f}{H^k}$ and $V = \frac{g}{H^k}$. Since $W \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$,

$$\frac{\frac{f(\lambda x)}{\lambda^{\alpha+k}L(\lambda)} - g(x)}{1 + x^m} \rightarrow 0 \text{ uniformly as } \lambda \rightarrow \infty. \quad (3.1)$$

Now, there exist positive constants M , M_1 , and M_2 such that for $x \geq 0$

$$\left| x \left(\frac{\frac{f(\lambda x)}{\lambda^{\alpha+k}L(\lambda)} - g(x)}{1+x^{m+2}} \right) \right| \leq M \left| \frac{\frac{f(\lambda x)}{\lambda^{\alpha+k}L(\lambda)} - g(x)}{1+x^m} \right| \quad (3.2)$$

and,

$$\begin{aligned} \left| \frac{H * \left(\frac{f(\lambda x)}{\lambda^{\alpha+k}L(\lambda)} - g(x) \right)}{1+x^{m+2}} \right| &\leq \int_0^x \left| \frac{\frac{f(\lambda t)}{\lambda^{\alpha+k}L(\lambda)} - g(t)}{1+t^{m+2}} \right| dt \\ &\leq M_1 \int_0^1 \left| \frac{\frac{f(\lambda t)}{\lambda^{\alpha+k}L(\lambda)} - g(t)}{1+t^m} \right| dt + M_2 \int_1^\infty \frac{1}{t^2} \left| \frac{\frac{f(\lambda t)}{\lambda^{\alpha+k}L(\lambda)} - g(t)}{1+t^m} \right| dt. \end{aligned} \quad (3.3)$$

By (3.1), (3.2), and (3.3), it follows

$$\frac{\frac{(xf)(\lambda x) - k(H*f)(\lambda x)}{\lambda^{(\alpha+1)+k}L(\lambda)} - (xg(x) - (kH * g)(x))}{1+x^{m+2}} \rightarrow 0 \text{ uniformly as } \lambda \rightarrow \infty.$$

That is,

$$xW \stackrel{q}{\sim} xV \text{ at infinity related to } \lambda^{\alpha+1}L(\lambda).$$

By using an inductive argument, the result follows. $\square$

Theorem 3.3. *Let $U, W, V_1, V_2 \in \mathcal{M}^\tau$ such that $W \stackrel{q}{\sim} V_1$ at infinity related to $\lambda^\alpha L_1(\lambda)$ and $U \stackrel{q}{\sim} V_2$ at infinity related to $\lambda^\beta L_2(\lambda)$. Then, $W * U \stackrel{q}{\sim} V_1 * V_2$ at infinity related to $\lambda^{\alpha+\beta+1}(L_1 L_2)(\lambda)$.*

Proof. Let $m \in \mathbb{N}$, $W = \frac{f}{H^k}$, $U = \frac{g}{H^k}$, $V_1 = \frac{h_1}{H^k}$, and $V_2 = \frac{h_2}{H^k}$ such that

$$\frac{\frac{f(\lambda x)}{\lambda^{\alpha+k}L_1(\lambda)} - h_1(x)}{1+x^m} \rightarrow 0 \text{ uniformly as } \lambda \rightarrow \infty \quad (3.4)$$

and,

$$\frac{\frac{g(\lambda x)}{\lambda^{\beta+k}L_2(\lambda)} - h_2(x)}{1+x^m} \rightarrow 0 \text{ uniformly as } \lambda \rightarrow \infty. \quad (3.5)$$

Notice that

$$\begin{aligned} &\frac{(f * g)(\lambda x)}{\lambda^{\alpha+\beta+2k+1}L_1(\lambda)L_2(\lambda)} - (h_1 * h_2)(x) \\ &= \left[\left(\frac{f\lambda}{\lambda^{\alpha+k}L_1(\lambda)} - h_1 \right) * \frac{g\lambda}{\lambda^{\beta+k}L_2(\lambda)} \right] (x) + \left[\left(\frac{g\lambda}{\lambda^{\beta+k}L_2(\lambda)} - h_2 \right) * h_1 \right] (x). \end{aligned} \quad (3.6)$$

Now, there exist $n \in \mathbb{N}$ and positive constants A , B and λ_0 such that for all $\lambda > \lambda_0$ and $x \geq 0$,

$$\begin{aligned} \left| (1+x^n)^{-1} \left[\left(\frac{f\lambda}{\lambda^{\alpha+k}L_1(\lambda)} - h_1 \right) * \frac{g\lambda}{\lambda^{\beta+k}L_2(\lambda)} \right] (x) \right| \\ \leq A \sup_{x \geq 0} \left| \frac{f(\lambda x)}{\lambda^{\alpha+k}L_1(\lambda)} - h_1(x) \right| \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} \left| (1+x^n)^{-1} \left[\left(\frac{g\lambda}{\lambda^{\beta+k}L_2(\lambda)} - h_2 \right) * h_1 \right] (x) \right| \\ \leq B \sup_{x \geq 0} \left| \frac{g(\lambda x)}{\lambda^{\beta+k}L_2(\lambda)} - h_2(x) \right|. \end{aligned} \quad (3.8)$$

Multiplying both sides of (3.6) by $(1+x^n)^{-1}$ and using (3.4), (3.5), (3.7), and (3.8), the conclusion follows. $\square$

As the next theorem shows, quasi-asymptotic behavior at infinity is a local property provided $\alpha > -1$.

Theorem 3.4. *Let $W, V, U \in \mathcal{M}^\tau$ and $\alpha > -1$. If $W \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$ and $W(x) = U(x)$ on (b, ∞) for some $b > 0$ (i.e. $W - U$ vanishes on (b, ∞)), then $U \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$.*

Proof. Let $W = \frac{f}{H^k}$, $U = \frac{g}{H^k}$, and $V = \frac{h}{H^k}$. Since $W(x) = U(x)$ on (b, ∞) ,

$$f(x) = g(x) + p(x) \text{ for } x > b, \quad (3.9)$$

where p is a polynomial with $\deg p \leq k-1$.

Also, $W \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$ implies there exists $m \in \mathbb{N}$ such that

$$\frac{\frac{f(\lambda x)}{\lambda^{\alpha+k}L(\lambda)} - h(x)}{1+x^m} \rightarrow 0 \text{ uniformly as } \lambda \rightarrow \infty. \quad (3.10)$$

By (3.9) there exists $F \in C(\mathbb{R})$ with $F(x) = 0$ for $x > b$ such that

$$f - g = F + p \text{ on } (-\infty, \infty). \quad (3.11)$$

Now, let $n = \max\{m, k-1\}$. Then there exist positive constants A and B such that

$$\left| \frac{\frac{F(\lambda x)}{\lambda^{\alpha+k}L(\lambda)}}{1+x^n} \right| \leq \frac{A}{\lambda^{\alpha+k}L(\lambda)}, \quad (3.12)$$

and,

$$\left| \frac{\frac{p(\lambda x)}{\lambda^{\alpha+k}L(\lambda)}}{1+x^n} \right| \leq \frac{B}{\lambda^{\alpha+1}L(\lambda)}, \quad (3.13)$$

for all $x \geq 0$ and $\lambda \geq 1$.

By (3.10), (3.11), (3.12), and (3.13),

$$\frac{\frac{g(\lambda x)}{\lambda^{\alpha+k}L(\lambda)} - h(x)}{1+x^n} \rightarrow 0 \text{ uniformly as } \lambda \rightarrow \infty.$$

Thus $U \stackrel{q}{\sim} V$ at infinity related to $\lambda^\alpha L(\lambda)$. $\square$

Remark 3.5. It is easily verified that $\delta \stackrel{q}{\sim} \delta$ at infinity related to λ^{-1} and $\delta' \stackrel{q}{\sim} \delta'$ at infinity related to λ^{-2} . Since $\delta(x) = \delta'(x)$ on $(0, \infty)$, quasi-asymptotic behavior at infinity is not necessarily a local property for $\alpha \leq -1$.

4. THE STIELTJES TRANSFORM

In this section, as an application, an Abelian theorem of the final type for the Stieltjes transform is given.

Let $r > -1$ and $f \in L^1_{loc}(\mathbb{R})$ such that $\text{supp } f \subseteq [0, \infty)$ and $f(x)x^{-r+\varepsilon}$ is bounded as $x \rightarrow \infty$ for some $\varepsilon > 0$. Then, the Stieltjes transform of index r of f is given by

$$S_z^r f = \int_0^\infty \frac{f(x)}{(x+z)^{r+1}} dx, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

For $r \in \mathbb{R}$ and $k = 1, 2, \dots$

$$\mathcal{M}_k(r) = \left\{ \frac{f}{H^k} : f \in C_+(\mathbb{R}), f(x)x^{-r-k+\varepsilon} \text{ is bdd as } x \rightarrow \infty \text{ for some } \varepsilon > 0 \right\}$$

and

$$\mathcal{M}(r) = \bigcup_{k=1}^{\infty} \mathcal{M}_k(r).$$

Let $r > -1$ and $W \in \mathcal{M}(r)$. That is, $W = \frac{f}{H^k} \in \mathcal{M}_k(r)$, for some $k \in \mathbb{N}$. The Stieltjes transform of index r of W is defined by

$$\Lambda_z^r W = (r+1)_k \int_0^\infty \frac{f(x)}{(x+z)^{r+k+1}} dx, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where $(r+1)_k = \frac{\Gamma(r+k+1)}{\Gamma(r+1)} = (r+1)(r+2)\dots(r+k)$ and Γ is the gamma function.

Notice that $\Lambda_z^r W = (r+1)_k S_z^{r+k} f$.

Properties 4.1. [4]. Let $W = \frac{f}{H^k} \in \mathcal{M}(r)$. Then for $r > -1$ and $z \in \mathbb{C} \setminus (-\infty, 0]$,

- (1) $\Lambda_z^r W$ is an analytic function.
- (2) $\Lambda_z^r \tau_c W = \Lambda_{z+c}^r W$, $c > 0$ and $\tau_c W = \frac{\tau_c f}{H^k}$, $\tau_c f(x) = f(x - c)$.
- (3) $\Lambda_z^r W^{(m)} = (r+1)_m \Lambda_z^{r+m} W$, $m = 1, 2, \dots$
- (4) $\frac{d^m}{dz^m} \Lambda_z^r W = (-1)^m (r+1)_m \Lambda_z^{r+m} W = (-1)^m \Lambda_z^r W^{(m)}$, $m = 1, 2, \dots$
- (5) $\Lambda_z^{r+1}(xW) = \Lambda_z^r W - z \Lambda_z^{r+1} W$.

Recall, for $\sigma > 0$, let $\ell_\sigma(x) = H(x)x^\sigma$. Then, for $\alpha > -1$, let $\Theta_{\alpha+1} = \frac{1}{\Gamma(\alpha+2)} \frac{\ell_{\alpha+1}}{H}$, and for $\alpha \leq -1$ and $\alpha + n > 0$, $\Theta_{\alpha+1} = \frac{1}{\Gamma(\alpha+n+1)} \frac{\ell_{\alpha+n}}{H^n}$.

Theorem 4.2. Final Value Theorem. *Let $\alpha \in \mathbb{R}$ and $r > -1$, $r > \alpha$. If $W \in \mathcal{M}(r)$ and $W \stackrel{q}{\sim} \zeta \Theta_{\alpha+1}$ ($\zeta \neq 0$) at infinity related to λ^α , then*

$$\lim_{\substack{z \rightarrow \infty \\ |\arg z| \leq \psi < \pi/2}} \frac{\Gamma(r+1) z^{r-\alpha} \Lambda_z^r W}{\Gamma(r-\alpha)} = \zeta.$$

Proof. Let $n, m \in \mathbb{N}$ such that $\alpha + n > 0$, $W = \frac{f}{H^n} \in \mathcal{M}(r)$, and

$$\sup_{x \geq 0} (1+x^m)^{-1} \left| \frac{f(\lambda x)}{\lambda^{\alpha+n}} - \frac{\xi}{\Gamma(\alpha+n+1)} \ell_{\alpha+n}(x) \right| \rightarrow 0 \text{ as } \lambda \rightarrow \infty. \quad (4.1)$$

Since $W \in \mathcal{M}(r)$, $f \in \mathcal{S}'_+(\mathbb{R})$. By (4.1), it follows that

$$\frac{f(\lambda x)}{\lambda^{\alpha+n}} - \frac{\xi}{\Gamma(\alpha+n+1)} \ell_{\alpha+n}(x) \rightarrow 0 \text{ in } \mathcal{S}'(\mathbb{R}) \text{ as } \lambda \rightarrow \infty.$$

Thus,

$$f \stackrel{q}{\sim} \frac{\xi}{\Gamma(\alpha+n+1)} \ell_{\alpha+n} \text{ at infinity related to } \lambda^{\alpha+n} \text{ (in } \mathcal{S}'(\mathbb{R}) \text{)}.$$

By using the Abelian theorem for $\mathcal{S}'_+(\mathbb{R})$ [7] with $\alpha + n$ for α and $r + n$ for r , it follows that

$$\lim_{\substack{z \rightarrow \infty \\ |\arg z| \leq \psi < \pi/2}} \frac{z^{r-\alpha} \Gamma(r+n+1) S_z^{r+n} f}{\Gamma(r-\alpha)} = \zeta.$$

The result follows by observing

$$\Lambda_z^r W = (r+1)_n S_z^{r+n} f.$$

□

Remark 4.3. By using the results in [7], the Final Value theorem does not require $\alpha > -1$, which is stated as a requirement in several Final Value theorems.

5. TEMPERED DISTRIBUTIONS

In [1], the notion of tempered convergence was introduced and found to be useful in developing distribution theory using the sequential approach.

In this section, we show that the space $\mathcal{M}^\tau$ with the sequential convergence defined in Section 2 is isomorphic to the space of tempered distributions supported on $[0, \infty)$ with tempered convergence.

Let $\mathcal{D}'(\mathbb{R})$ denote the space of distributions on $\mathbb{R}$. The k th tempered derivative of a distribution F is given by $D_t^k F = e^{-x^2/4} D^k(e^{x^2/4} F)$, where D^k is the k th order distributional derivative

A distribution F is said to be *tempered* if there exists a square integrable function f such that

$$D_t^k f = F, \text{ for some } k \in \mathbb{N}.$$

A sequence of tempered distributions $\{F_n\}$ is *tempered convergent* to a distribution F , denoted $F_n \xrightarrow{t} F$, if there exist square integrable functions f_n, f such that

$$D_t^k f_n = F_n, D_t^k f = F, \text{ for some } k \in \mathbb{N}, \text{ and}$$

$$\int_{-\infty}^{\infty} |f_n(x) - f(x)|^2 dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

The space of all tempered distributions will be denoted by $\mathcal{S}'(\mathbb{R})$.

Let $\mathcal{S}'_+(\mathbb{R})$ denote the subspace of $\mathcal{S}'(\mathbb{R})$ for which elements are supported on the interval $[0, \infty)$.

We will use the following theorems to show that the space $\mathcal{M}^\tau$ is isomorphic to $\mathcal{S}'_+(\mathbb{R})$ with tempered convergence.

Theorem 5.1. [1]. *A distribution F is tempered iff there exist $k, r \in \mathbb{N}$ and a continuous function g such that*

$$D^k g = F \text{ and } \frac{g(x)}{1 + |x|^r} \text{ is bounded.}$$

Theorem 5.2. [1]. *$F_n \xrightarrow{t} F$ in $\mathcal{S}'(\mathbb{R})$ iff there exist $k, r \in \mathbb{N}$ and $g_n, g \in C(\mathbb{R})$ such that*

- (1) $D^k g_n = F_n, D^k g = F$.
- (2) $\frac{g_n(x)}{1 + |x|^r}$ is uniformly bounded.
(i.e. there exists $M > 0$ such that for all $n \in \mathbb{N}$, $\left| \frac{g_n(x)}{1 + |x|^r} \right| \leq M$, for all $x \in \mathbb{R}$.)
- (3) $\frac{g_n(x) - g(x)}{1 + |x|^r} \rightarrow 0$ uniformly on $\mathbb{R}$ as $n \rightarrow \infty$.

Define the mapping $\Psi : \mathcal{M}^\tau \rightarrow \mathcal{S}'_+(\mathbb{R})$ by $\Psi\left(\frac{f}{H^k}\right) = D^k f$.

The space $\mathcal{M}^\tau$ is not only algebraically isomorphic to the space $\mathcal{S}'_+(\mathbb{R})$, their convergence structures are also isomorphic.

Let $W, V \in \mathcal{M}^\tau$ and $\alpha, \beta \in \mathbb{C}$. Then, the following are easily verified.

1. $\Psi(\alpha W + \beta V) = \alpha \Psi(W) + \beta \Psi(V)$, and
2. $\Psi(W * V) = \Psi(W) * \Psi(V)$, where convolution on the right-hand side is taken in $\mathcal{D}'(\mathbb{R})$.

It is also routine to show that Ψ is well-defined, injective, and continuous.

To show that Ψ is surjective, let $F \in \mathcal{S}'_+(\mathbb{R})$. By Theorem 5.1, there exist $k, r \in \mathbb{N}$ and $g \in C(\mathbb{R})$ such that

$$D^k g = F \text{ and } \frac{g(x)}{1 + |x|^r} \text{ is bounded.}$$

Since $\text{supp } F \subseteq [0, \infty)$, $D^k g = 0$ on $(-\infty, 0)$.

Thus, there exists a polynomial p such that $\deg p \leq k - 1$ and $g(x) = p(x)$, $x < 0$.

Let $f(x) = g(x) - p(x)$, $x \in \mathbb{R}$.

Then, $f \in C(\mathbb{R})$, $\frac{f(x)}{1 + |x|^{r+k}}$ is bounded, $D^k f = D^k(g - p) = F$, and $\text{supp } f \subseteq [0, \infty)$. Therefore, $\frac{f}{H^k} \in \mathcal{M}^\tau$ and $\Psi\left(\frac{f}{H^k}\right) = D^k f = F$. That is, Ψ is surjective.

Now to show that the inverse map is continuous, let $F_n, F \in \mathcal{S}'_+(\mathbb{R})$ ($n \in \mathbb{N}$) such that $F_n \xrightarrow{t} F$. Thus,

$$\text{supp } F_n \subseteq [0, \infty), \quad \text{supp } F \subseteq [0, \infty). \quad (5.1)$$

By Theorem 5.2, there exist $k, r \in \mathbb{N}$ and $g_n, g \in C(\mathbb{R})$ ($n \in \mathbb{N}$) such that

$$D^k g_n = F_n, \quad D^k g = F \quad (5.2)$$

$$\frac{g_n(x)}{1 + |x|^r} \text{ is uniformly bounded, and}$$

$$\frac{g_n(x) - g(x)}{1 + |x|^r} \rightarrow 0 \text{ uniformly on } \mathbb{R} \text{ as } n \rightarrow \infty. \quad (5.3)$$

By (5.1) and (5.2), for all $n \in \mathbb{N}$,

$$D^k g_n = 0 \text{ on } (-\infty, 0) \text{ and } D^k g = 0 \text{ on } (-\infty, 0).$$

Thus, for each $n \in \mathbb{N}$, there exist $\alpha_{n,j} \in \mathbb{C}$ ($j = 0, 1, 2, \dots, k - 1$) such that

$$g_n(x) = p_n(x) = \alpha_{n,0} + \alpha_{n,1}x + \dots + \alpha_{n,k-1}x^{k-1}, \quad x < 0$$

and,

$$\beta_j \in \mathbb{C} \quad (j = 0, 1, 2, \dots, k - 1) \text{ such that}$$

$$g(x) = p(x) = \beta_0 + \beta_1 x + \cdots + \beta_{k-1} x^{k-1}, \quad x < 0.$$

It follows from (5.3), for each $0 \leq j \leq k-1$,

$$\alpha_{n,j} \rightarrow \beta_j \text{ as } n \rightarrow \infty.$$

Now, for each $n \in \mathbb{N}$, define

$$\begin{aligned} f_n(x) &= g_n(x) - p_n(x), \text{ and} \\ f(x) &= g(x) - p(x), \quad x \in \mathbb{R}. \end{aligned}$$

Then, for each $n \in \mathbb{N}$, $f_n, f \in C_+(\mathbb{R})$, $\frac{f_n(x)}{1+|x|^{r+k}}$ and $\frac{f(x)}{1+|x|^{r+k}}$ are bounded.

Also, $\frac{f_n(x)-f(x)}{1+|x|^{r+k}} \rightarrow 0$ uniformly on $\mathbb{R}$ as $n \rightarrow \infty$.

Now, for each $n \in \mathbb{N}$,

$$\begin{aligned} D^k f_n &= D^k(g_n - p_n) = F_n, \text{ and} \\ D^k f &= D^k(g - p) = F. \end{aligned}$$

For each $n \in \mathbb{N}$, let

$$W_n = \frac{f_n}{H^k} \text{ and } W = \frac{f}{H^k}.$$

Then, for each $n \in \mathbb{N}$,

$$W_n, W \in \mathcal{M}^\tau \text{ and } \Psi(W_n) = F_n, \Psi(W) = F, \text{ and}$$

$$\mathcal{M}^\tau\text{-}\lim_{n \rightarrow \infty} W_n = W.$$

Thus, the inverse map is continuous.

Therefore, the algebraic structures as well as the convergence structures of the spaces $\mathcal{M}^\tau$ and $\mathcal{S}'_+(\mathbb{R})$ are isomorphic.

Notice that $\Psi(W_\lambda) = (\Psi(W))_\lambda$. It follows that if $W \in \mathcal{M}^\tau$ has quasi-asymptotic behavior at infinity related to $\lambda^\alpha L(\lambda)$, then, using weak convergence in $\mathcal{S}'(\mathbb{R})$, $\Psi(W)$ has quasi-asymptotic behavior at infinity related to $\lambda^\alpha L(\lambda)$. The converse remains open. That is, using weak convergence in $\mathcal{S}'(\mathbb{R})$, if $F \in \mathcal{S}'_+(\mathbb{R})$ has quasi-asymptotic behavior at infinity related to $\lambda^\alpha L(\lambda)$, is it necessarily true that $\Psi^{-1}(F) \in \mathcal{M}^\tau$ has quasi-asymptotic behavior at infinity related to $\lambda^\alpha L(\lambda)$?

Acknowledgment. The author would like to thank the referee for carefully checking the manuscript and for his/her helpful comments and suggestions.

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(Received: January 7, 2015)

(Revised: March 29, 2015)

Department of Mathematics
 California State University, Stanislaus
 One University Circle
 Turlock, CA 95382
 U.S.A.
 nemzer@comcast.net