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A NEW HYBRID CYCLIC ALGORITHM FOR TWO FINITE FAMILIES OF STRICTLY ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. The purpose of this paper is to propose a new hybrid cyclic algorithm for two finite families of strictly asymptotically pseudocontractive mappings and to establish a strong convergence theorem to approximate common fixed point. The main result of the paper is an improvement and generalization of the well known corresponding results. It also provides an affirmative answer to an interesting problem raised by Marino and Xu [Weak and strong convergence theorem for κ-strict pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329 (2007), 336-349].

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. It is said to be λ-strictly pseudocontractive, if there exists constant $\lambda \in (0, 1)$, such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda \|(x - Tx) - (y - Ty)\|^2$ for all $x, y \in C$.

A mapping $T : C \to C$ is said to be $\lambda-$strictly asymptotically pseudocontractive [10], if there exist a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\|^2 \leq k_n^2 \|x - y\|^2 + \lambda \|(x - T^n x) - (y - T^n y)\|^2$$

for some $\lambda \in (0, 1)$, for all $x, y \in C$ and $n \geq 1$.

In recent years iterative methods for obtaining fixed point of nonexpansive mappings have been extensively investigated, see [4, 7, 17, 19, 21] and references therein. Iterative methods for strictly pseudocontractive mappings have also been developed sidewise. Following iterative algorithms have been often used for this purpose.

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Parallel algorithm : Generate a sequence $\{x_n\}$ in C by the recursive formula

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n\ , \quad n \geq 0\,,$$

where the initial guess x_0 is taken in C arbitrarily and the real control sequence $\{\alpha_n\}$ is in the interval $(0,1)$. This algorithm is known as Mann's algorithm [11].

Cyclic algorithm : Let N be a positive integer and $\{T_i\}_{i=0}^{N-1}$ be a N-strict pseudocontractive mapping defined on a closed convex subset C of a Hilbert space H.

Define a sequence $\{x_n\}$ cyclically by beginning with an arbitrary x_0 in C and

$$\begin{aligned} x_1 &= \alpha_0 x_0 + (1-\alpha_0)T_0x_0\,, \\ x_2 &= \alpha_1 x_1 + (1-\alpha_1)T_1x_1\,, \\ &\vdots \\ x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1})T_{N-1}x_{N-1}\,, \\ x_{N+1} &= \alpha_N x_N + (1-\alpha_N)T_0x_N\,, \\ &\ddots \end{aligned}$$

In a more compact form, x_{n+1} can be written as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n$$

where $T_{[n]} = T_i$ with $i = n \mod N$, $0 \leq i \leq N-1$. This algorithm is given by Acedo and Xu [1] for strict pseudo-contractions.

However, the convergence of the above algorithms can only be weak in an infinite-dimensional space (see [2] and [3]). So in order to have strong convergence, one must modify these algorithms. Some modifications have recently been obtained (see [7, 8, 13, 18, 20, 21]).

Marino and Xu [12] proposed modification in Mann's algorithm for λ-strict pseudocontractive mapping defined as below:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + (1-\alpha_n)(\lambda-\alpha_n)\,\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C; \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0. \end{cases} \tag{1.1}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H.

They proved that the sequence $\{x_n\}$ converges strongly to $P_F x_0$, where F is the fixed point set of T. They [12] also posed following problem:

Problem 1. How can an algorithm similar to (1.1) be constructed for asymptotically λ-strict pseudocontractive mapping by using the Ishikawa algorithm (see [6]) which has strong convergence without assuming compactness of C ?

In 2007, Thakur [20] extended the idea of Marino and Xu [12] to asymptotically strict pseudo-contractive mappings and proved a strong convergence theorem.

More recently, Qin et al. [16] proposed a modification of the cyclic algorithm for a finite family of asymptotically λ-strictly pseudocontractive mappings as below:

Let C be a closed convex subset of a real Hilbert space H, and for each $1 \leq i \leq N$, T_i be an asymptotically λ-strict pseudocontractive mapping of C into itself with $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$ is nonempty and bounded, let $x_0 \in C$. For $C_1 = C$, $x_1 = P_{C_1} x_0$ define $\{x_n\}$ as follows

$$\begin{cases} y_{n-1} = \alpha_{n-1} x_{n-1} + (1-\alpha_{n-1}) T_{i(n)}^{h(n)} x_{n-1} \\ C_{n-1} = \Big\{ v \in C : \|y_{n-1} - v\|^2 \leq \|x_{n-1} - v\|^2 \\ \qquad\qquad + [(\lambda - \alpha_{n-1})(1-\alpha_{n-1})] \left\| T_{i(n)}^{h(n)} x_{n-1} - x_{n-1} \right\|^2 + \theta_{n-1} \Big\} \\ Q_{n-1} = \{ v \in C ; \langle x_0 - x_{n-1}, x_{n-1} - v \rangle \geq 0 \}, \\ \quad x_n = P_{C_{n-1} \cap Q_{n-1}} x_0 . , \quad n \in \mathbb{N} \end{cases} \tag{1.2}$$

where $\theta_{n-1}(1-\alpha_n)(k_{h(n)}^2 - 1)\rho_{n-1}^2 \to 0$ as $n \to \infty$, where $\rho_{n-1}^2 = \sup\{||x_{n-1} v|| : v \in \mathcal{F}\}$. They further proved that the sequence $\{x_n\}$ converges strongly to $z_0 = P_{F(T)} x_0$. This gives rise to the question that we are concerned with:

Question 1. Is it possible to define a hybrid cyclic algorithm which converges to a common fixed point of two finite families of asymptotically λ-strictly pseudocontractive mappings ?

In 2011, Zhang [22] gave an affirmative solution to the problem 1 by proving the strong convergence of the iterative sequence:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T_1^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)T_2^n x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|z_n - v\|^2 + \theta_n, \\ Q_n = \{v \in C : \|z_n - v\|^2 \leq \|x_n - v\|^2 + \gamma_n, \\ S_n = \{v \in C ; \langle x_n - v, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n \cap S_n} x_0. \end{cases} \tag{1.3}$$

for a pair of asymptotically λ-strictly pseudocontractive mappings, where $\gamma_n = (k_n - 1)(\text{diam } C)^2$ and $\theta_n = [1 + (1-\alpha_n)k_n]\gamma_n + (1-\alpha_n)\lambda \|z_n - x_n\|^2 + 2(1-\alpha_n)\lambda \|z_n - x_n\| (\text{diam } C)$. Here the choice of the control sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are dependent of the pseudocontractive constant λ.

This brings us to the following question:

Question 2. Is it possible to remove the dependency of the control sequence $\{\alpha_n\}$ and $\{\beta_n\}$ on the pseudocontractive constant λ.

Motivated by (1.1) and (1.2), we have proposed a new hybrid cyclic algorithm and establish a strong convergence theorem for two finite family of strictly asymptotically pseudocontractive mappings where the control condition does not depend upon the pseudocontractive constant.

Our result gives an affirmative answer to the Problem 1, Question 1 and Question 2 and extends the results of Acedo and Xu [1], Kim and Xu [9], Marino and Xu [12], Nakajo and Takahashi [13], Qin et al. [16].

2. Preliminaries

In order to prove our main results, we need the following results:

Lemma 2.1. *Let H be a real Hilbert space. Then the following identities holds:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \quad \forall\, x, y \in H.$
(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, $\forall\, t \in [0,1]$, $\forall\, x, y \in H.$

Lemma 2.2. *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if the relation*

$$\langle x - z, z - y\rangle \geq 0 \quad \forall\, y \in C \text{ holds},$$

where P_C is the nearest point projection from H on to C.

Lemma 2.3. [21] *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in \mathbb{R}$. The set*

$$\Big\{ v \in C \; : \; \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a \Big\}$$

is closed and convex.

Lemma 2.4. [15] *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $T : C \to C$ be a $\lambda-$strictly asymptotically pseudocontractive mapping with nonempty fixed point set.Then $(I - T)$ is demiclosed at zero, i.e. if $\{x_n\}$ is a sequence in C such that $x \rightharpoonup z$ and $(I - T)x_n \to 0$, then $(I - T)z = 0$.*

Lemma 2.5. [14] *Let H be a Hilbert space, C be a nonempty closed convex subset of H and $T : C \to C$ be a $\lambda-$strictly asymptotically pseudocontractive mapping for some $0 \leq \lambda < 1$. Then T^n satisfies the Lipschitz condition :*

$$\|T^n x - T^n y\| \leq L \|x - y\|$$

for all $x, y \in C$ and for each $n \geq 1$, where $L > 0$ is a constant.

Lemma 2.6. [16] *Let H be a real Hilbert space, C a nonempty subset of T and $T : C \to C$ be a $\lambda-$strictly asymptotically pseudocontractive mapping. Then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.7. *Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, T_i : $C \to C$ be a λ_i-strictly asymptotically pseudocontractive mapping for sone $0 \leq \lambda_i < 1$ with a sequence $\{k_n^{(i)}\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n^{(i)} = 1$, then there exists a constant $\lambda = \max\{\lambda_i \; : \; 1 \leq i \leq N\}$ and a sequence $\{k_n\} = \max\{k_n^{(i)} \; : \; 1 \leq i \leq N\}$ such that*

$$\|T_i^n x - T_i^n y\| \leq k_n^2 \|x - y\|^2 + \lambda \|(I - T_i^n)x - (I - T_i^n)y\|^2$$

for all $1 \leq i \leq N$, where $\lim_{n\to\infty} k_n = 1$.

3. Main results

In what follows, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, \ldots, N\}$, the set of first N natural numbers. $\{\alpha_n\}$ and $\{\beta_n\}$ sequences in $(0, 1)$.

Let $\{S_j : j \in J\}$ be a family of $(\lambda_{(j,s)}, \{k_n^{(j,s)}\})$ strictly asymptotically pseudocontractive mappings of C and $\{T_j : j \in J\}$ be another family of $(\lambda_{(j,t)}, \{k_n^{(j,t)}\})$ strictly asymptotically pseudocontractive mappings of C with nonempty common fixed point set $\mathcal{F} = \mathcal{F}(T) \cap \mathcal{F}(S)$, where $\mathcal{F}(T) = \bigcap_{j=1}^N F(T_j)$, and $\mathcal{F}(S) = \bigcap_{j=1}^N F(S_j)$.

In view of Lemma 2.7, let $\lambda_t = \max\left\{\lambda_{(j,s)} : j \in J\right\}$, $\lambda_s = \max\{\lambda_{(j,t)} : j \in J\}$, $k_n^{(t)} = \max\left\{k_n^{(j,s)} : j \in J\right\}$, $k_n^{(s)} = \max\left\{k_n^{(j,t)} : j \in J\right\}$, $L_t = \max\{L_{(j,s)} : j \in J\}$, $L_s = \max\left\{L_{(j,t)} : j \in J\right\}$, where $L_{(j,s)}$, $L_{(j,t)}$ are Lipschitz constants of S_j, T_j respectively.

Set $\lambda = \max\{\lambda_t, \lambda_s\}$, $L = \max\{L_t, L_s\}$ and $k_n = \max\left\{k_n^{(t)}, k_n^{(s)}\right\}$, then $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$.

We now define a new hybrid cyclic algorithm as below :
Let $x_0 \in C$ be chosen arbitrarily. For $C_1 = C = D_1$ and $x_1 = P_{C_1 \cap D_1} x_0$ define $\{x_n\}$ by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) T_{i(n)}^{h(n)} z_n, \\ z_n = \beta_n x_n + (1-\beta_n) S_{i(n)}^{h(n)} x_n, \\ C_{n+1} = \Big\{ v \in C_n : \|y_n - v\|^2 \le \|x_n - v\|^2 + \xi_n \\ \qquad\qquad + (1-\alpha_n)(\lambda - \alpha_n) \left\| x_n - T_{i(n)}^{h(n)} z_n \right\|^2 \Big\}, \\ D_{n+1} = \Big\{ v \in D_n : \|z_n - v\|^2 \le \|x_n - v\|^2 + \theta_n \\ \qquad\qquad + (1-\beta_n)(\lambda - \beta_n) \left\| x_n - S_{i(n)}^{h(n)} x_n \right\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1} \cap D_{n+1}} x_0. \end{cases} \tag{3.1}$$

where $\theta_n = (k_n^2 - 1)(\text{diam } C)^2 \to 0$ as $n \to \infty$; $\xi_n = \theta_n + (1-\alpha_n)(\lambda + k_n^2)\left[\|z_n - x_n\|^2 + 2\|z_n - x_n\| \text{diam } C\right]$.

Here, for each $n \ge 1$, we can write $n = (h(n)-1)N + i(n)$, where $i(n) \in J$ and $h(n) \ge 1$ is a positive integer with $h(n) \to \infty$ as $n \to \infty$.

We now prove that the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x^* \in P_{\mathcal{F}} x_0$.

Theorem 3.1. *Let H be a real Hilbert space, C be a nonempty weakly compact convex subset of H. Let $\{T_j, S_j : j \in J\}$ be two family of $(\lambda_{(j,t)}, \{k_n^{(j,t)}\})$ and $(\lambda_{(j,s)}, \{k_n^{(j,s)}\})$ strictly asymptotically pseudocontractive mappings respectively, with $\mathcal{F} = \mathcal{F}(T) \cap \mathcal{F}(S) \neq \emptyset$.*

Assume that $0 < \alpha_n \le a < 1$ and $0 < \beta_n \le b < 1$, where a and b are two positive constants. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a $x^* = P_{\mathcal{F}} x_0$.

Proof. The proof is divided into five steps.

Step 1. We show that $\{x_n\}$ is well defined and $F \subset C_n \cap D_n$ for every $n \geq 1$.

Since $C_1 = C = D_1$ by Lemma 2.3, we see that C_n and D_n are closed and convex for every $n \geq 1$.

Next we show that $\mathcal{F} \subset C_n \cap D_n$ for all $n \geq 1$. For $p \in \mathcal{F}$, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \left\|\beta_n(x_n - p) + (1-\beta_n)(S_{i(n)}^{h(n)} x_n - p)\right\|^2 \\
&= \beta_n \|x_n - p\|^2 + (1-\beta_n)\left\|S_{i(n)}^{h(n)} x_n - p\right\|^2 - \beta_n(1-\beta_n)\left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\left[k_n^2 \|x_n - p\|^2 + \lambda \left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2\right] \\
&\quad - \beta_n(1-\beta_n)\left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 \\
&\leq k_n^2 \|x_n - p\|^2 + (1-\beta_n)(\lambda - \beta_n)\left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 \\
&\leq \|x_n - p\|^2 + (1-\beta_n)(\lambda - \beta_n)\left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 + \theta_n\,,
\end{aligned}$$

where $\theta_n = (k_n^2 - 1)\,(\mathrm{diam}\ C)^2$. Hence $p \in D_{n+1}$ and $\mathcal{F} \subset D_{n+1}$. Hence $\mathcal{F} \subset D_n$ for all $n \in \mathbb{N}$.

Again for $p \in \mathcal{F}$, if follows from Lemma 2.1 and (3.1), that

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\alpha_n(x_n - p) + (1-\alpha_n)(T_{i(n)}^{h(n)} z_n - p)\right\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\|T_{i(n)}^{h(n)} z_n - p\right\|^2 - \alpha_n(1-\alpha_n)\left\|x_n - T_{i(n)}^{h(n)} z_n\right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left[k_n^2 \|z_n - p\|^2 + \lambda \left\|z_n - T_{i(n)}^{h(n)} z_n\right\|^2\right] \\
&\quad - \alpha_n(1-\alpha_n)\left\|x_n - T_{i(n)}^{h(n)} z_n\right\|^2 \\
&\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left[k_n^2 (\|z_n - x_n\| + \|x_n - p\|)^2 + \lambda \left\|z_n - T_{i(n)}^{h(n)} z_n\right\|^2\right] \\
&\quad - \alpha_n(1-\alpha_n)\left\|x_n - T_{i(n)}^{h(n)} z_n\right\|^2 \\
&\leq k_n^2 \|x_n - p\|^2 + (1-\alpha_n)\lambda \left\|z_n - T_{i(n)}^{h(n)} z_n\right\|^2 - \alpha_n(1-\alpha_n)\left\|x_n - T_{i(n)}^{h(n)} z_n\right\|^2 \\
&\quad + (1-\alpha_n)k_n^2\left[\|z_n - x_n\|^2 + 2\|z_n - x_n\|\ \mathrm{diam}\ C\right]. \qquad (3.2)
\end{aligned}$$

Also

$$\left\|z_n - T_{i(n)}^{h(n)} z_n\right\|^2 \leq \|z_n - x_n\|^2 + \left\|x_n - T_{i(n)}^{h(n)} z_n\right\|^2 + 2\|z_n - x_n\|\ \mathrm{diam}\ C\,.$$

This with (3.2) gives,

$$\|y_n - p\|^2 \le k_n^2 \|x_n - p\|^2 + (1-\alpha_n)\lambda\Big[\|z_n - x_n\|^2 + \Big\|x_n - T_{i(n)}^{h(n)} z_n\Big\|^2$$

$$+ 2\|z_n - x_n\| \text{ diam } C\Big] + (1-\alpha_n)k_n^2\Big[\|z_n - x_n\|^2 + 2\|z_n - x_n\| \text{ diam } C\Big]$$

$$- \alpha_n(1-\alpha_n)\Big\|x_n - T_{i(n)}^{h(n)} z_n\Big\|^2$$

$$\le \|x_n - p\|^2 + (1-\alpha_n)(\lambda + k_n^2)\Big[\|z_n - x_n\|^2 + 2\|z_n - x_n\| \text{ diam } C\Big]$$

$$+ (1-\alpha_n)(\lambda - \alpha_n)\Big\|x_n - T_{i(n)}^{h(n)} z_n\Big\|^2 + \theta_n$$

$$= \|x_n - p\|^2 + (1-\alpha_n)(\lambda - \alpha_n)\Big\|x_n - T_{i(n)}^{h(n)} z_n\Big\|^2 + \xi_n\,,$$

where $\xi_n = \theta_n + (1-\alpha_n)(\lambda + k_n^2)\Big[\|z_n - x_n\|^2 + 2\|z_n - x_n\| \text{ diam } C\Big]$.

This implies that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. Thus $\mathcal{F} \subset C_n \cap D_n$.

Step 2. We show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

From $x_n = P_{C_n \cap D_n} x_0$ and $x_{n+1} = P_{C_{n+1}\cap D_{n+1}} x_0 \in C_{n+1} \cap D_{n+1} \subset C_n \cap D_n$, by Lemma 2.2 we have

$$\begin{aligned} 0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\,\|x_0 - x_{n+1}\| \end{aligned}$$

This implies that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|\,, \quad \text{for all } n \ge 1.$$

hence $\{\|x_0 - x_n\|\}$ is nondecreasing.

Since $x_n = P_{C_n\cap D_n} x_0$, by Lemma 2.2 we have

$$\langle x_0 - x_n, x_n - y\rangle \ge 0 \quad \text{for all } y \in C_n \cap D_n\,. \tag{3.3}$$

As $\mathcal{F} \subset C_n \cap D_n$, from (3.3) we have

$$\langle x_0 - x_n, x_n - z\rangle \ge 0 \quad \text{for all } z \in \mathcal{F}\,,\ \ n \ge 1\,. \tag{3.4}$$

So, for $z \in \mathcal{F}$, we have

$$\begin{aligned} 0 &\le \langle x_o - x_n, x_n - z\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - z\rangle \\ &= -\langle x_0 - x_n, x_0 - x_n\rangle + \langle x_0 - x_n, x_0 - z\rangle \\ &= -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \, \|x_0 - z\| \, . \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \le \|x_0 - x_n\| \, \|x_0 - z\| \, ,$$

hence

$$\|x_0 - x_n\| \le \|x_0 - z\| \, , \quad \text{for all } z \in \mathcal{F} \text{ and } n \ge 1.$$

This implies $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Step 3. We prove that $\{x_n\}$ is a Cauchy sequence and $\lim\limits_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

For any positive integer $m \ge n$, we have $x_m = P_{C_m \cap D_m} x_0 \subset C_m \cap D_m \subset C_n \cap D_n$. By Lemma 2.2, we have

$$\langle x_n - x_0, x_m - x_n\rangle \ge 0. \tag{3.5}$$

It follows that

$$\begin{aligned} \|x_m - x_n\|^2 &= \|(x_m - x_0) - (x_0 - x_n)\|^2 \\ &= \|x_m - x_0\|^2 + \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_m - x_0\rangle \\ &= \|x_m - x_0\|^2 + \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_m - x_n + x_n - x_0\rangle \\ &= \|x_m - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_m - x_n\rangle \\ &\le \|x_m - x_0\|^2 - \|x_n - x_0\|^2 \, . \end{aligned}$$

Taking limit superior as $m, n \to \infty$, we get that

$$\limsup_{m,n\to\infty} \|x_m - x_n\| = 0 \, ,$$

i.e.,

$$\lim_{m,n\to\infty} \|x_m - x_n\| = 0 \, . \tag{3.6}$$

Hence $\{x_n\}$ is a Cauchy sequence. Putting $m = n+1$ into (3.6), we get that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0 \, . \tag{3.7}$$

Step 4. We prove that

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0 = \lim_{n\to\infty} \|x_n - S_j x_n\| \, , \quad \text{for every } j \in J \, .$$

By the fact that $x_{n+1} \in C_{n+1} \cap D_{n+1} \subset C_n \cap D_n \subset D_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (1-\beta_n)(\lambda-\beta_n) \left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 + \theta_n \,. \quad (3.8)$$

Moreover, since $z_n = \beta_n x_n + (1-\beta_n) S_{i(n)}^{h(n)} x_n$, we get

$$\begin{aligned} \|z_n - x_n\| &= \left\|\beta_n x_n + (1-\beta_n) S_{i(n)}^{h(n)} x_n - x_n\right\| \\ &= (1-\beta_n) \left\|S_{i(n)}^{h(n)} x_n - x_n\right\| \,. \end{aligned} \quad (3.9)$$

From (3.9) and (3.8), we have

$$\begin{aligned} (1-\beta_n)^2 & \left\|S_{i(n)}^{h(n)} x_n - x_n\right\|^2 = \|z_n - x_n\|^2 \\ &= \|z_n - x_{n+1} + x_{n+1} - x_n\|^2 \\ &= \|z_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 + 2\langle z_n - x_{n+1}, x_{n+1} - x_n\rangle \\ &\leq \|x_n - x_{n+1}\|^2 + (1-\beta_n)(\lambda-\beta_n)\left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 + \theta_n \\ &\qquad + \|x_n - x_{n+1}\|^2 + 2\|z_n - x_{n+1}\|\,\|x_{n+1} - x_n\| \\ &= (1-\beta_n)(\lambda-\beta_n)\left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 + \theta_n \\ &\qquad + 2\|x_n - x_{n+1}\|\left(\|x_n - x_{n+1}\| + \|z_n - x_{n+1}\|\right) \,. \end{aligned}$$

It follows that

$$\begin{aligned} (1-\beta_n)(1-\lambda) & \left\|x_n - S_{i(n)}^{h(n)} x_n\right\|^2 \\ &\leq 2\|x_n - x_{n+1}\|\left(\|x_n - x_{n+1}\| + \|z_n - x_{n+1}\|\right) + \theta_n \,. \end{aligned}$$

Since $\{x_n\}$ is Cauchy and hence bounded and $T_{i(n)}, S_{i(n)}$ are Lipschitz continuous, so $\{y_n\}$ and $\{z_n\}$ are bounded, also $\beta_n < 1$ for all n and $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$, therefore, we have

$$\lim_{n\to\infty} \left\|x_n - S_{i(n)}^{h(n)} x_n\right\| = 0 \,. \quad (3.10)$$

Also by (3.8) and (3.9), we have

$$\|z_n - x_{n+1}\| \to 0 \quad \text{and} \quad \|z_n - x_n\| \to 0 \quad \text{as } n \to \infty \,. \quad (3.11)$$

and hence $\xi_n \to 0$ as $n \to \infty$.

Similarly we can show that

$$\lim_{n\to\infty} \left\|T_{i(n)}^{h(n)} z_n - x_n\right\| = 0 \,. \quad (3.12)$$

Also,

$$\begin{aligned}\left\|x_n - T_{i(n)}^{h(n)} x_n\right\| &\le \left\|x_n - T_{i(n)}^{h(n)} z_n\right\| + \left\|T_{i(n)}^{h(n)} z_n - T_{i(n)}^{h(n)} x_n\right\| \\ &\le \left\|x_n - T_{i(n)}^{h(n)} z_n\right\| + L\left\|z_n - x_n\right\| \\ &\to 0 \quad \text{as} \quad n \to \infty\,. \end{aligned} \tag{3.13}$$

Since, for any positive integer $n \ge N$, we can write $n = (h(n) - 1)\,N + i(n)$ where $i(n) \in J$, and observe that

$$\begin{aligned}\left\|x_n - T_n x_n\right\| &\le \left\|x_n - T_{i(n)}^{h(n)} x_n\right\| + \left\|T_{i(n)}^{h(n)} x_n - T_n x_n\right\| \\ &= \left\|x_n - T_{i(n)}^{h(n)} x_n\right\| + \left\|T_{i(n)}^{h(n)} x_n - T_{i(n)} x_n\right\| \\ &\le \left\|x_n - T_{i(n)}^{h(n)} x_n\right\| + L\left\|T_{i(n)}^{h(n)-1} x_n - x_n\right\| \\ &\le \left\|x_n - T_{i(n)}^{h(n)} x_n\right\| + L\left(\left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n-N)}^{h(n)-1} x_{n-N}\right\|\right. \\ &\quad \left. + \left\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{n-N}\right\| + \left\|x_{n-N} - x_n\right\|\right)\,. \end{aligned} \tag{3.14}$$

Since, for each $n > N$, $i(n) = (n - N) \mod N$. Again since $n = (h(n) - 1)N + i(n)$, we have

$$h(n - N) = h(n) - 1\,, \text{ and } i(n - N) = i(n).$$

We observe that

$$\begin{aligned}\left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n-N)}^{h(n)-1} x_{n-N}\right\| &= \left\|T_{i(n)}^{h(n)-1} x_n - T_{i(n)}^{h(n)-1} x_{n-N}\right\| \\ &\le L\left\|x_n - x_{n-N}\right\|\,, \end{aligned} \tag{3.15}$$

and

$$\left\|T_{i(n-N)}^{h(n)-1} x_{n-N} - x_{n-N}\right\| = \left\|T_{i(n)}^{h(n-N)} x_{n-N} - x_{n-N}\right\|\,. \tag{3.16}$$

Substituting (3.15), (3.16) in (3.14), we obtain

$$\begin{aligned}\left\|x_n - T_n x_n\right\| &\le \left\|x_n - T_{i(n)}^{h(n)} x_n\right\| \\ &\quad + L\left((1 + L)\left\|x_n - x_{n-N}\right\| + \left\|T_{i(n-N)}^{h(n-N)} x_{n-N} - x_{n-N}\right\|\right)\,. \end{aligned} \tag{3.17}$$

Also, by (3.7), we have

$$\lim_{n\to\infty} \left\|x_n - x_{n+j}\right\| = 0\,, \quad \text{for all } j \ge 1. \tag{3.18}$$

It follows from (3.13), (3.18) and (3.17) that

$$\lim_{n\to\infty} \left\|x_n - T_n x_n\right\| = 0\,.$$

We also have

$$\begin{aligned}\|x_n - T_{n+j}x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \\ &\leq (1+L)\,\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| \\ &\to 0 \quad \text{as } n \to \infty \text{ for any } j \in J\,,\end{aligned}$$

which gives that

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0\,, \quad \text{for all } j \in J\,. \tag{3.19}$$

Similarly, we can show that

$$\lim_{n\to\infty} \|x_n - S_j x_n\| = 0\,, \quad \text{for all } j \in J\,.$$

Step 5. We prove that $\{x_n\}$ converges strongly to a $x^* = P_{\mathcal{F}}x_0$.

Due to the assumption of weakly compactness on C, we may extract a weakly convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges to x^*. Hence from (3.19), we have

$$\lim_{n\to\infty} \|x_{n_k} - T_j x_{n_k}\| = 0\,, \quad \text{for all } j \in J\,.$$

By Lemma 2.4, we have that $(I-T_j)$ is demiclosed at zero, i.e. $(I-T_j)x^* = 0$, so that $x^* \in F(T_j)$. By the arbitrariness of $j \in J$, we get that $x^* \in \mathcal{F}(T) = \bigcap_{j=1}^{N} F(T_j)$

Similarly, we can see that $x^* \in \mathcal{F}(S) = \cap_{j=1}^{N} F(S_j)$, and hence $x^* \in \mathcal{F}$. Since $x_n = P_{C_n \cap D_n}x_0$, by Lemma 2.2 we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \quad \text{for all } y \in C_n \cap D_n\,. \tag{3.20}$$

As $\mathcal{F} \subset C_n \cap D_n$, from (3.20) we have

$$\langle x_0 - x_n, x_n - z\rangle \geq 0 \quad \text{for all } z \in \mathcal{F}\,, \quad n \geq 1\,. \tag{3.21}$$

Taking the limit in (3.21), we have

$$\langle x_0 - x^*, x^* - z\rangle \geq 0\,, \quad \forall z \in \mathcal{F}.$$

In view of Lemma 2.2, we see that

$$x^* \in P_{\mathcal{F}}x_0\,.$$

This completes the proof. □

Remark 1. Theorem 3.1 is significant generalization and extension of the results of Acedo and Xu [1], Kim and Xu [9], Marino and Xu [12], Nakajo and Takahashi [13], Qin et al. [16], Zhang [22] in several aspects as:

1. Theorem 3.1 is an extension and improvement of the result of Zhang [22] to two finite families of mappings, also the dependency of control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ upon the psueodocontractive coefficient λ is removed. Hence provide an affirmative answer to Question 1 and Question 2.
2. Theorem 3.1 provides an affirmative answer to the Problem 1 as compactness on C is relaxed to weak compactness.
3. Also, our theorem extends the result of Qin et al. [16] to two finite family of asymptotically λ-strict pseudocontractive mappings thus extending many other results in their references.

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