SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.11.2.11
Vol.11 (24), No.2, (2015), 253–266

NEW NORM INEQUALITIES OF ČEBYŠEV TYPE FOR POWER SERIES IN BANACH ALGEBRAS

S. S. DRAGOMIR, M. V. BOLDEA AND M. MEGAN

Abstract. Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$, a Banach algebra, with $xy = yx$.

In this paper we establish some new upper bounds for the norm of the *Čebyšev type difference*

$$f(\lambda) f(\lambda xy) - f(\lambda x) f(\lambda y)$$

provide that the complex number λ and the vectors $x, y \in \mathcal{B}$ are such that the series in the above expression are convergent. These results complement the earlier resuls obtained by the authors. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

1. Introduction

Let $\mathcal{B}$ be an algebra. An *algebra norm* on $\mathcal{B}$ is a map $\|\cdot\| : \mathcal{B} \to [0,\infty)$ such that $(\mathcal{B}, \|\cdot\|)$ is a normed space, and, further:

$$\|ab\| \le \|a\| \|b\|$$

for any $a, b \in \mathcal{B}$. The normed algebra $(\mathcal{B}, \|\cdot\|)$ is a *Banach algebra* if $\|\cdot\|$ is a *complete norm.*

We assume that the Banach algebra is *unital,* this means that $\mathcal{B}$ has an identity 1 and that $\|1\| = 1$.

Let $\mathcal{B}$ be a unital algebra. An element $a \in \mathcal{B}$ is *invertible* if there exists an element $b \in \mathcal{B}$ with $ab = ba = 1$. The element b is unique; it is called the *inverse* of a and written a^{-1} or $\frac{1}{a}$. The set of invertible elements of $\mathcal{B}$ is denoted by Inv$\mathcal{B}$. If $a, b \in$Inv$\mathcal{B}$ then $ab \in$Inv$\mathcal{B}$ and $(ab)^{-1} = b^{-1}a^{-1}$.

For a unital Banach algebra we also have:

(i) If $a \in \mathcal{B}$ and $\lim_{n\to\infty} \|a^n\|^{1/n} < 1$, then $1 - a \in$Inv$\mathcal{B}$;

2010 *Mathematics Subject Classification.* 47A63, 47A99.

Key words and phrases. Banach algebras, power series, exponential function, resolvent function, norm inequalities.

(ii) $\{a \in \mathcal{B}\colon\ \|1-b\| < 1\} \subset \mathrm{Inv}\mathcal{B}$;
(iii) $\mathrm{Inv}\mathcal{B}$ is an *open subset* of $\mathcal{B}$;
(iv) The map $\mathrm{Inv}\mathcal{B} \ni a \longmapsto a^{-1} \in \mathrm{Inv}\mathcal{B}$ is continuous.

For simplicity, we denote $\lambda 1$, where $\lambda \in \mathbb{C}$ and 1 is the identity of $\mathcal{B}$, by λ. The *resolvent set* of $a \in \mathcal{B}$ is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} :\ \lambda - a \in \mathrm{Inv}\mathcal{B}\};$$

the *spectrum* of a is $\sigma(a)$, the complement of $\rho(a)$ in $\mathbb{C}$, and the *resolvent function* of a is $R_a : \rho(a) \to \mathrm{Inv}\mathcal{B}$, $R_a(\lambda) := (\lambda - a)^{-1}$. For each $\lambda, \gamma \in \rho(a)$ we have the identity

$$R_a(\gamma) - R_a(\lambda) = (\lambda - \gamma) R_a(\lambda) R_a(\gamma).$$

We also have that $\sigma(a) \subset \{\lambda \in \mathbb{C} :\ |\lambda| \le \|a\|\}$. The *spectral radius* of a is defined as $\nu(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$.

If a, b are *commuting* elements in $\mathcal{B}$, i.e. $ab = ba$, then

$$\nu(ab) \le \nu(a)\nu(b) \text{ and } \nu(a+b) \le \nu(a) + \nu(b).$$

Let f be an analytic functions on the open disk $D(0,R)$ given by the *power series* $f(\lambda) := \sum_{j=0}^{\infty} \alpha_j \lambda^j$ $(|\lambda| < R)$. If $\nu(a) < R$, then the series $\sum_{j=0}^{\infty} \alpha_j a^j$ converges in the Banach algebra $\mathcal{B}$ because $\sum_{j=0}^{\infty} |\alpha_j| \left\|a^j\right\| < \infty$, and we can define $f(a)$ to be its sum. Clearly $f(a)$ is well defined and there are many examples of important functions on a Banach algebra $\mathcal{B}$ that can be constructed in this way. For instance, the *exponential map* on $\mathcal{B}$ denoted exp and defined as

$$\exp a := \sum_{j=0}^{\infty} \frac{1}{j!} a^j \text{ for each } a \in \mathcal{B}.$$

If $\mathcal{B}$ is not commutative, then many of the familiar properties of the exponential function from the scalar case do not hold. The following key formula is valid, however with the additional hypothesis of commutativity for a and b from $\mathcal{B}$

$$\exp(a+b) = \exp(a)\exp(b).$$

In a general Banach algebra $\mathcal{B}$ it is difficult to determine the elements in the range of the exponential map $\exp(\mathcal{B})$, i.e. the element which have a "*logarithm*". However, it is easy to see that if a is an element in B such that $\|1-a\| < 1$, then a is in $\exp(\mathcal{B})$. That follows from the fact that if we set

$$b = -\sum_{n=1}^{\infty} \frac{1}{n} (1-a)^n,$$

then the series converges absolutely and, as in the scalar case, substituting this series into the series expansion for $\exp(b)$ yields $\exp(b) = a$.

It is known that if x and y are commuting, i.e. $xy = yx$, then the exponential function satisfies the property

$$\exp(x)\exp(y) = \exp(y)\exp(x) = \exp(x+y).$$

Also, if x is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tx)\,dt = x^{-1}\left[\exp(bx) - \exp(ax)\right].$$

Moreover, if x and y are commuting and $y - x$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)x + sy)\,ds &= \int_0^1 \exp(s(y-x))\exp(x)\,ds \\ &= \left(\int_0^1 \exp(s(y-x))\,ds\right)\exp(x) \\ &= (y-x)^{-1}\left[\exp(y-x) - I\right]\exp(x) \\ &= (y-x)^{-1}\left[\exp(y) - \exp(x)\right].\end{aligned}$$

Inequalities for functions of operators in Hilbert spaces may be found in the papers [3], [2] and in the recent monographs [4], [5], [7] and the references therein.

In order to state some earlier results [6] that motivate our current work we need some preparation as follows.

Let α_n be nonzero complex numbers and let

$$R := \frac{1}{\limsup |\alpha_n|^{\frac{1}{n}}}.$$

Clearly $0 \le R \le \infty$, but we consider only the case $0 < R \le \infty$.

Denote by:

$$D(0,R) = \begin{cases} \{\lambda \in \mathbb{C} : |\lambda| < R\}, & \text{if } R < \infty \\ \mathbb{C}, & \text{if } R = \infty, \end{cases}$$

consider the functions:

$$\lambda \mapsto f(\lambda) : D(0,R) \to \mathbb{C}, f(\lambda) := \sum_{n=0}^{\infty} \alpha_n \lambda^n$$

and

$$\lambda \mapsto f_A(\lambda) : D(0,R) \to \mathbb{C}, f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n.$$

Let $\mathcal{B}$ be a unital Banach algebra and 1 its unity. Denote by

$$B(0,R) = \begin{cases} \{x \in \mathcal{B} : \|x\| < R\}, & \text{if } R < \infty \\ \mathcal{B}, & \text{if } R = \infty. \end{cases}$$

We associate to f the map:

$$x \mapsto \widetilde{f}(x) : B(0,R) \to \mathcal{B}, \widetilde{f}(x) := \sum_{n=0}^{\infty} \alpha_n x^n.$$

Obviously, $\widetilde{f}$ is correctly defined because the series $\sum_{n=0}^{\infty} \alpha_n x^n$ is absolutely convergent, since $\sum_{n=0}^{\infty} \|\alpha_n x^n\| \le \sum_{n=0}^{\infty} |\alpha_n| \|x\|^n$.

In addition, we assume that $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n| < \infty$. Let $s_0 := \sum_{n=0}^{\infty} |\alpha_n| < \infty$ and $s_1 := \sum_{n=0}^{\infty} n |\alpha_n| < \infty$.

With the above assumptions we have that [6]:

Theorem 1. *Let* $\lambda \in \mathbb{C}$ *such that* $\max\{|\lambda|, |\lambda|^2\} < R < \infty$ *and let* $x, y \in \mathcal{B}$ *with* $\|x\|, \|y\| \le 1$ *and* $xy = yx$. *Then:*

(i) *We have*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \le \sqrt{2} \psi \min\{\|x-1\|, \|y-1\|\} f_A\left(|\lambda|^2\right) \quad (1.1)$$

where:

$$\psi^2 := s_0 s_2 - s_1^2. \quad (1.2)$$

(ii) *We also have*

$$\begin{aligned} &\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \\ &\le \sqrt{2} \min\{\|x-1\|, \|y-1\|\} f_A(|\lambda|) \\ &\times \left\{ f_A(|\lambda|) \left[|\lambda| f_A'(|\lambda|) + |\lambda|^2 f_A''(|\lambda|) \right] - \left[|\lambda| f_A'(|\lambda|) \right]^2 \right\}^{1/2}. \end{aligned} \quad (1.3)$$

For other similar results, see [6].

In this paper we establish some new upper bounds for the norm of the *Čebyšev type difference*

$$\widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \quad (1.4)$$

provide that the complex number λ and the vectors x, $y \in \mathcal{B}$ are such that the series in (1.4) are convergent. Applications for some fundamental functions such as the *exponential function* and the *resolvent function* are provided as well.

2. Results

We start with the following result that is of interest in itself.

Lemma 1. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R) \subset \mathbb{C}$, $R > 0$ and $x, y \in \mathcal{B}$ with $xy = yx$.*

If $\|y\| < 1$, $\lambda \in \mathbb{C}$ and $x \in \mathcal{B}$ with $|\lambda| \|x\| < R$, then we have the inequality:

$$\left\| \widetilde{f}(\lambda x) y^k - \widetilde{f}(\lambda xy) \right\| \leq \frac{\|y-1\|}{1-\|y\|} \left[f_A(|\lambda| \|x\|) - |\alpha_k| |\lambda|^k \|x\|^k \right], \qquad (2.1)$$

for any $k \in \mathbb{N}$, $k \geq 0$.

Proof. We have for $m \geq 2$ and $1 \leq k \leq m-1$ that

$$\left(\sum_{j=0}^{m} \alpha_j \lambda^j x^j \right) y^k - \sum_{j=0}^{m} \alpha_j \lambda^j (xy)^j = \left(\sum_{j=0}^{m} \alpha_j \lambda^j x^j \right) y^k - \sum_{j=0}^{m} \alpha_j \lambda^j x^j y^j$$

$$= \sum_{j=0}^{m} \alpha_j \lambda^j x^j \left(y^k - y^j \right) = \sum_{j=0, j \neq k}^{m} \alpha_j \lambda^j x^j \left(y^k - y^j \right) = A. \qquad (2.2)$$

Since

$$y^k - y^j = \sum_{l=j}^{k-1} \left(y^{l-1} - y^l \right) = \sum_{l=j}^{k-1} y^l (y-1),$$

then by taking the norm in (2.2) we get:

$$\|A\| \leq \sum_{j=0, j \neq k}^{m} |\alpha_j| |\lambda|^j \|x\|^j \left\| \sum_{l=j}^{k-1} y^l (y-1) \right\|$$

$$\leq \sum_{j=0, j \neq k}^{m} |\alpha_j| |\lambda|^j \|x\|^j \sum_{l=j}^{k-1} \|y\|^l \|y-1\|$$

$$= \|y-1\| \sum_{j=0, j \neq k}^{m} |\alpha_j| |\lambda|^j \|x\|^j \sum_{l=j}^{k-1} \|y\|^l =: B. \qquad (2.3)$$

Observe that

$$\sum_{l=j}^{k-1} \|y\|^l \leq \sum_{l=0}^{m-1} \|y\|^l$$

and then we have

$$B \leq \|y-1\| \sum_{l=0}^{m-1} \|y\|^l \sum_{j=0, j \neq k}^{m} |\alpha_j| |\lambda|^j \|x\|^j$$

$$= \|y-1\| \sum_{l=0}^{m-1} \|y\|^l \left(\sum_{j=0}^{m} |\alpha_j| |\lambda|^j \|x\|^j - |\alpha_k| |\lambda|^k \|x\|^k \right). \qquad (2.4)$$

Utilising the inequalities (2.2)-(2.4) we conclude that

$$\left\| \left(\sum_{j=0}^{m} \alpha_j \lambda^j x^j \right) y^k - \sum_{j=0}^{m} \alpha_j \lambda^j (xy)^j \right\|$$
$$\leq \|y-1\| \sum_{l=0}^{m-1} \|y\|^l \left(\sum_{j=0}^{m} |\alpha_j| |\lambda|^j \|x\|^j - |\alpha_k| |\lambda|^k \|x\|^k \right) \tag{2.5}$$

for any $m \geq 2$ and $1 \leq k \leq m-1$.

Since the series $\sum_{j=0}^{m} \alpha_j \lambda^j x^j$ and $\sum_{j=0}^{m} \alpha_j (\lambda xy)^j$ are convergent in $\mathcal{B}$ and, because $\|y\| < 1$, then $\sum_{l=0}^{\infty} \|y\|^l = \frac{1}{1-\|y\|}$, then by letting $m \to \infty$ in (2.5), we get the desired result (2.1).

If $k = 0$, then

$$\sum_{j=0}^{m} \alpha_j \lambda^j x^j - \sum_{j=0}^{m} \alpha_j \lambda^j (xy)^j = \sum_{j=1}^{m} \alpha_j \lambda^j x^j \left(1 - y^j\right) =: C.$$

Since

$$1 - y^j = (1-y)\left(1 + y + ... + y^{j-1}\right), \ j \geq 1$$

then

$$\left\|1 - y^j\right\| \leq \|y-1\| \sum_{l=0}^{j-1} \|y\|^l \leq \|y-1\| \sum_{l=0}^{m-1} \|y\|^l$$

and then

$$\|C\| \leq \|y-1\| \sum_{l=0}^{m-1} \|y\|^l \sum_{j=1}^{m} |\alpha_j| |\lambda|^j \|x\|^j$$
$$= \|y-1\| \sum_{l=0}^{m-1} \|y\|^l \left(\sum_{j=0}^{m} |\alpha_j| |\lambda|^j \|x\|^j - |\alpha_0| \right). \tag{2.6}$$

Letting $m \to \infty$ in (2.6), we also obtain the inequality (2.1) for $k = 0$. This proves the lemma. □

Corollary 1. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbb{C}$, $R > 0$ and $x \in \mathcal{B}$.*

If $\|x\| < 1$, $\lambda \in \mathbb{C}$ with $|\lambda| \|x\| < R$, then we have the inequality:

$$\left\| \widetilde{f}(\lambda x) x^k - \widetilde{f}\left(\lambda x^2\right) \right\| \leq \frac{\|x-1\|}{1-\|x\|} \left[f_A(|\lambda| \|x\|) - |\alpha_k| |\lambda|^k \|x\|^k \right], \tag{2.7}$$

for any $k \in \mathbb{N}$, $k \geq 0$.

We can state the following result.

Theorem 2. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ and $x,y\in\mathcal{B}$ with $xy=yx$. If $\lambda,\mu\in\mathbb{C}$ are such that $|\mu|,|\lambda|\,\|x\|<R$ and $\|y\|\le 1$ then:*

$$\left\|\widetilde{f}(\lambda x)\,\widetilde{f}(\mu y)-\widetilde{f}(\mu\cdot 1)\,\widetilde{f}(\lambda xy)\right\| \le \frac{\|y-1\|}{1-\|y\|}\left[f_A(|\lambda|\,\|x\|)\,f_A(|\mu|)-f_{A^2}(|\lambda|\,|\mu|\,\|x\|)\right], \quad (2.8)$$

where $f_{A^2}(\lambda):=\sum_{n=0}^{\infty}|\alpha_n|^2\lambda^n$.

Proof. Utilising Lemma 1 we have:

$$\begin{aligned}
&\left\|\widetilde{f}(\lambda x)\left(\sum_{k=0}^{p}\alpha_k\mu^k y^k\right)-\left(\sum_{k=0}^{p}\alpha_k\mu^k\right)\widetilde{f}(\lambda xy)\right\| \\
&=\left\|\sum_{k=0}^{p}\alpha_k\mu^k\left(\widetilde{f}(\lambda x)\,y^k-\widetilde{f}(\lambda xy)\right)\right\| \\
&\le\sum_{k=0}^{p}|\alpha_k|\,|\mu|^k\left\|\widetilde{f}(\lambda x)\,y^k-\widetilde{f}(\lambda xy)\right\| \\
&\le\sum_{k=0}^{p}\frac{\|y-1\|}{1-\|y\|}\left[f_A(|\lambda|\,\|x\|)-|\alpha_k|\,|\lambda|^k\,\|x\|^k\right]|\alpha_k|\,|\mu|^k \\
&=\frac{\|y-1\|}{1-\|y\|}\left[f_A(|\lambda|\,\|x\|)\sum_{k=0}^{p}|\alpha_k|\,|\mu|^k-\sum_{k=0}^{p}|\alpha_k|\,|\lambda|^k\,|\mu|^k\,\|x\|^k\right] \qquad (2.9)
\end{aligned}$$

for any $p\ge 0$.

Since all the series that are involved in the inequality from above are convergent, then by letting $p\to\infty$ we get the desired result (2.8). □

Corollary 2. *Let $f(\lambda)=\sum_{n=0}^{\infty}\alpha_n\lambda^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0,R)\subset\mathbb{C}$, $R>0$ and $x\in\mathcal{B}$. If $\lambda,\mu\in\mathbb{C}$ are such that $|\mu|,|\lambda|\,\|x\|<R$ and $\|x\|<1$ then:*

$$\left\|\widetilde{f}(\lambda x)\,\widetilde{f}(\mu x)-\widetilde{f}(\mu\cdot 1)\,\widetilde{f}\left(\lambda x^2\right)\right\| \le \frac{\|x-1\|}{1-\|x\|}\left[f_A(|\lambda|\,\|x\|)\,f_A(|\mu|)-f_{A^2}(|\lambda|\,|\mu|\,\|x\|)\right]. \quad (2.10)$$

Remark 1. If $\mu = \lambda$, then we get the inequality for the Čebyšev functional:

$$\left\| \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) - \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) \right\| \leq \frac{\|y-1\|}{1-\|y\|} \left[f_A(|\lambda| \|x\|) f_A(|\lambda|) - f_{A^2}\left(|\lambda|^2 \|x\| \right) \right], \quad (2.11)$$

provided that $x, y \in \mathcal{B}$ with $xy = yx$, $\lambda \in \mathbb{C}$ are such that $|\lambda|, |\lambda| \|x\| < R$ and $\|y\| < 1$.

From (2.10) we have

$$\left\| \left[\widetilde{f}(\lambda x) \right]^2 - \widetilde{f}(\lambda \cdot 1) \widetilde{f}\left(\lambda x^2 \right) \right\| \leq \frac{\|x-1\|}{1-\|x\|} \left[f_A(|\lambda| \|x\|) f_A(|\mu|) - f_{A^2}(|\lambda| |\mu| \|x\|) \right]. \quad (2.12)$$

We can state now the second result:

Theorem 3. *Let $f(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n$ be a power series that is convergent on the open disk $D(0, R)$, with $R > 0$. If $x, y \in \mathcal{B}$ with $xy = yx$ and $\|y\|, \|y\| \leq 1$, then we have the inequalities:*

$$\left\| \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) - \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) \right\| \leq \frac{\sqrt{2}}{2} \|x-1\| \|y-1\| f_A(|\lambda|) \left[f_A(|\lambda|) g_A(|\lambda|) - h_A^2(|\lambda|) \right]^{\frac{1}{2}}, \quad (2.13)$$

where

$$f_A(\lambda) := \sum_{n=0}^{\infty} |\alpha_n| \lambda^n, \; g_A(\lambda) := \sum_{n=0}^{\infty} n^4 |\alpha_n| \lambda^n, \; h_A(\lambda) := \sum_{n=0}^{\infty} n^2 |\alpha_n| \lambda^n$$

and $\lambda \in D(0, R)$.

Moreover, if the series $s_0 := \sum_{n=0}^{\infty} |\alpha_n|$, $s_2 := \sum_{n=0}^{\infty} n^2 |\alpha_n|$ and $s_4 := \sum_{n=0}^{\infty} n^4 |\alpha_n|$ are convergent, then we have the inequalities:

$$\left\| \widetilde{f}(\lambda x) \widetilde{f}(\lambda y) - \widetilde{f}(\lambda \cdot 1) \widetilde{f}(\lambda xy) \right\| \leq \frac{\sqrt{2}}{2} \|x-1\| \|y-1\| f_A\left(|\lambda|^2 \right) \left[s_0 s_4 - s_2^2 \right]^{\frac{1}{2}}, \quad (2.14)$$

for any $\lambda \in \mathbb{C}$ with $|\lambda|, |\lambda|^2 < R$.

Proof. We observe that:

$$\begin{aligned}
B_m :&= \sum_{n,j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j \left(x^n - x^j\right)\left(y^n - 1\right) \\
&= \sum_{n,j=0}^{m} \alpha_n \alpha_j \lambda^n \lambda^j \left(x^n y^n - x^j y^n - x^n + x^j\right) \\
&= \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n \\
&\quad - \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n x^n + \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n \\
&= \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n \lambda^n (xy)^n - \sum_{j=0}^{m} \alpha_j \lambda^j x^j \sum_{n=0}^{m} \alpha_n \lambda^n y^n. \qquad (2.15)
\end{aligned}$$

Taking the norm and using the generalized triangle inequality we have:

$$\|B_m\| \leq \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left\|x^n - x^j\right\| \|y^n - 1\| := C_m. \qquad (2.16)$$

Since

$$y^n - 1 = (y-1)\left(y^{n-1} + ... + 1\right)$$

we have for $\|y\| \leq 1$ that

$$\|y^n - 1\| \leq \|y - 1\| \left\|y^{n-1} + \cdots + 1\right\| \leq n \|y - 1\|.$$

If $n > j$, then for $\|x\| \leq 1$

$$\left\|x^n - x^j\right\| = \left\|x^j \left(x^{n-j} - 1\right)\right\| \leq \|x\|^j \left\|x^{n-j} - 1\right\| \leq (n-j) \|x - 1\|.$$

Similarly, if $j > n$ we have

$$\left\|x^n - x^j\right\| \leq (j - n) \|x - 1\|,$$

therefore for any $n, j \in \mathbb{N}$ we have:

$$\left\|x^n - x^j\right\| \leq |n - j| \|x - 1\|, \|x\| \leq 1.$$

Utilising this facts we have

$$\begin{aligned}
C_m &\leq \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j n |n-j| \|x-1\| \|y-1\| \\
&= \|x-1\| \|y-1\| \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j n |n-j|. \qquad (2.17)
\end{aligned}$$

Further, observe that:

$$\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j n |n-j| = \frac{1}{2} \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j |n-j| (n+j)$$
$$= \frac{1}{2} \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left|n^2 - j^2\right|,$$

therefore

$$C_m \le \frac{1}{2} \|x-1\| \|y-1\| \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left|n^2 - j^2\right| := D_m. \tag{2.18}$$

Using Cauchy-Bunyakovsky-Schwarz inequality we have:

$$\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^{\frac{n}{2}} |\lambda|^{\frac{j}{2}} |\lambda|^{\frac{n}{2}} |\lambda|^{\frac{j}{2}} \left|n^2 - j^2\right|$$
$$\le \left(\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \right)^{\frac{1}{2}} \left(\sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left(n^2 - j^2\right)^2 \right)^{\frac{1}{2}}$$
$$= \left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^n \right) (E_m)^{\frac{1}{2}},$$

where

$$E_m := \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left(n^2 - j^2\right)^2$$
$$= \sum_{n,j=0}^{m} |\alpha_n| |\alpha_j| |\lambda|^n |\lambda|^j \left(n^4 - 2n^2 j^2 + j^4\right)^2$$
$$= 2 \left[\sum_{n=0}^{m} |\alpha_n| |\lambda|^n \sum_{n=0}^{m} |\alpha_n| |\lambda|^n n^4 - \left(\sum_{n=0}^{m} |\alpha_n| |\lambda|^n n^2 \right)^2 \right]. \tag{2.19}$$

Making use of (2.15)-(2.19) we get for $\|x\|, \|y\| \le 1$ that:

$$\left\| \sum_{j=0}^{m} \alpha_j \lambda^j \sum_{n=0}^{m} \alpha_n (\lambda x y)^n - \sum_{j=0}^{m} \alpha_j (\lambda x)^j \sum_{n=0}^{m} \alpha_n (\lambda y)^n \right\|$$
$$\le \frac{\sqrt{2}}{2} \|x-1\| \|y-1\| \sum_{n=0}^{m} |\alpha_n| |\lambda|^n$$

$$\times\left[\sum_{n=0}^{m}|\alpha_n||\lambda|^n\sum_{n=0}^{m}n^4|\alpha_n||\lambda|^n-\left(\sum_{n=0}^{m}n^2|\alpha_n||\lambda|^n\right)^2\right]^{\frac{1}{2}}, \tag{2.20}$$

for any $m\in\mathbb{N}$.

Since all the series involved in (2.20) are convergent, then by letting $m\to\infty$ in (2.20) we deduce the desired result:

$$\begin{aligned}&\left\|\widetilde{f}(\lambda\cdot 1)\,\widetilde{f}(\lambda xy)-\widetilde{f}(\lambda x)\,\widetilde{f}(\lambda y)\right\|\\&\qquad\le\frac{\sqrt{2}}{2}\|x-1\|\,\|y-1\|\,f_A(|\lambda|)\left[f_A(|\lambda|)\,g_A(|\lambda|)-h_A^2(|\lambda|)\right]^{\frac{1}{2}}.\end{aligned} \tag{2.21}$$

Using Cauchy-Bunyakovsky-Schwarz inequality we also have:

$$\begin{aligned}&\sum_{n,j=0}^{m}|\alpha_n||\alpha_j||\lambda|^n|\lambda|^j\left|n^2-j^2\right|\\&\le\left(\sum_{n,j=0}^{m}|\alpha_n||\alpha_j||\lambda|^{2n}|\lambda|^{2j}\right)^{\frac{1}{2}}\left(\sum_{n,j=0}^{m}|\alpha_n||\alpha_j|\left|n^2-j^2\right|^2\right)^{\frac{1}{2}}\\&=\left(\sum_{n=0}^{m}|\alpha_n||\lambda|^{2n}\right)\left\{2\left[\sum_{n=0}^{m}|\alpha_n|\sum_{j=0}^{m}j^4|\alpha_j|-\left(\sum_{n=0}^{m}n^2|\alpha_n|\right)^2\right]\right\}^{\frac{1}{2}}.\end{aligned} \tag{2.22}$$

Making use of this inequality we then obtain in a similar way the second part of the theorem. The details are omitted. □

3. Some examples

Consider the function $f:D(0,1)\to\mathbb{C}$ defined by

$$f(\lambda)=(1-\lambda)^{-1}=\sum_{k=0}^{\infty}\lambda^k.$$

Then

$$f_{A^2}(\lambda):=\sum_{n=0}^{\infty}\lambda^n=(1-\lambda)^{-1}$$

and by (2.8), we have for $x,y\in\mathcal{B}$ with $xy=yx$, $\|y\|<1$ and $\lambda,\mu\in\mathbb{C}$ with $|\mu|,|\lambda|\,\|x\|<1$ that

$$\begin{aligned}&\left\|(1-\lambda x)^{-1}(1-\mu y)^{-1}-(1-\mu)^{-1}(1-\lambda xy)^{-1}\right\|\\&\qquad\le\frac{\|y-1\|}{1-\|y\|}\left[(1-|\lambda|\,\|x\|)^{-1}(1-|\mu|)^{-1}-(1-|\lambda|\,|\mu|\,\|x\|)^{-1}\right].\end{aligned} \tag{3.1}$$

In particular, if $|\lambda|, \|x\| < 1$, then

$$\left\|(1-\lambda x)^{-1}(1-\lambda y)^{-1}-(1-\lambda)^{-1}(1-\lambda xy)^{-1}\right\| \leq \frac{\|y-1\|}{1-\|y\|}\left[(1-|\lambda|\,\|x\|)^{-1}(1-|\lambda|)^{-1}-\left(1-|\lambda|^2\|x\|\right)^{-1}\right]. \quad (3.2)$$

We also have for $|\lambda|, \|x\| < 1$ that

$$\left\|(1-\lambda x)^{-2}-(1-\lambda)^{-1}\left(1-\lambda x^2\right)^{-1}\right\| \leq \frac{\|x-1\|}{1-\|x\|}\left[(1-|\lambda|\,\|x\|)^{-1}(1-|\lambda|)^{-1}-\left(1-|\lambda|^2\|x\|\right)^{-1}\right]. \quad (3.3)$$

If we consider the function

$$f(\lambda)=(1+\lambda)^{-1}=\sum_{k=0}^{\infty}(-1)^k\lambda^k,$$

then the inequalities (3.1)-(3.3) also holds with " + " instead of " − " in the left hand side expressions such as $(1-\lambda x)^{-1}$ etc.

We consider the *modified Bessel function functions of the first kind*

$$I_\nu(\lambda):=\left(\frac{1}{2}\lambda\right)^{\nu}\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\lambda^2\right)^k}{k!\Gamma(\nu+k+1)},\ \lambda\in C$$

where Γ is the *Gamma function* and ν is a real number. An integral formula is

$$I_\nu(\lambda)=\frac{1}{\pi}\int_0^{\pi}e^{\lambda\cos\theta}\cos(\nu\theta)-\frac{\sin(\nu\pi)}{\pi}\int_0^{\infty}e^{-\lambda\cosh t-\nu t}dt,$$

which simplifies for ν an integer n to [1]

$$I_n(\lambda)=\frac{1}{\pi}\int_0^{\pi}e^{\lambda\cos\theta}\cos(n\theta)\,d\theta.$$

For $n=0$ we have

$$I_0(\lambda)=\frac{1}{\pi}\int_0^{\pi}e^{\lambda\cos\theta}d\theta=\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\lambda^2\right)^k}{(k!)^2},\ \lambda\in C.$$

Now, if we consider the exponential function

$$f(\lambda)=\exp(\lambda)=\sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k,$$

then for $\rho>0$ we have

$$f_{A^2}(\rho)=\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\rho^k=I_0(2\sqrt{\rho}).$$

Making use of the inequality (2.8), we have for $x, y \in \mathcal{B}$ with $xy = yx$, $\|y\| < 1$ and $\lambda, \mu \in \mathbb{C}$ that

$$\begin{aligned}\|\exp(\lambda x + \mu y) - \exp(\lambda xy + \mu \cdot 1)\| \\ \leq \frac{\|y-1\|}{1-\|y\|}\left[\exp(|\lambda|\|x\| + |\mu|) - I_0\left(2\sqrt{|\lambda||\mu|\|x\|}\right)\right]. \quad (3.4)\end{aligned}$$

In particular, we have

$$\begin{aligned}\|\exp(\lambda(x+y)) - \exp(\lambda(xy+1))\| \\ \leq \frac{\|y-1\|}{1-\|y\|}\left[\exp(|\lambda|(\|x\|+1)) - I_0\left(2|\lambda|\sqrt{\|x\|}\right)\right]. \quad (3.5)\end{aligned}$$

We also have for $\|x\| < 1$

$$\begin{aligned}\left\|\exp(2\lambda x) - \exp\left(\lambda\left(x^2+1\right)\right)\right\| \\ \leq \frac{\|x-1\|}{1-\|x\|}\left[\exp(|\lambda|(\|x\|+1)) - I_0\left(2|\lambda|\sqrt{\|x\|}\right)\right] \quad (3.6)\end{aligned}$$

for any $\lambda \in \mathbb{C}$. If we take $\lambda = 1$, then we get

$$\begin{aligned}\left\|\exp(2x) - \exp\left(x^2+1\right)\right\| \\ \leq \frac{\|x-1\|}{1-\|x\|}\left[\exp(\|x\|+1) - I_0\left(2\sqrt{\|x\|}\right)\right] \quad (3.7)\end{aligned}$$

for $\|x\| < 1$.

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(Received: February 23, 2015)

S. S. Dragomir
Mathematics
School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001
Australia
School of Computer Science
& Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050
South Africa
sever.dragomir@vu.edu.au
http://rgmia.org/dragomir

M. V. Boldea
Mathematics and Statistics
Banat University of Agricultural
Sciences and Veterinary Medicine Timişoara
119 Calea Aradului, 300645 Timişoara
România

M. Megan
Department of Mathematics
West University of Timişoara
B-dul V. Pârvan 4, 1900-Timişoara
România