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A NOTE ON PATA-TYPE CYCLIC CONTRACTIONS

ZORAN KADELBURG AND STOJAN RADENOVIĆ

Abstract. Several fixed point results are obtained for cyclic mappings satisfying contractive conditions of Pata-type. Some of them improve existing results in the literature. An example shows a possible usage of the obtained results.

1. Introduction

A very interesting generalization of Banach Contraction Principle was recently proved by V. Pata [11]. Already several other fixed point results were adapted to use Pata-type conditions (see, e.g., [1, 3, 6]).

On the other hand, Kirk et al. [7] were first to prove fixed point results for so-called cyclic contractions, and this was exploited in a lot of subsequent works (see, e.g., [9, 10]). The importance of this approach is that a standard contraction is always continuous, while a cyclic one need not be.

Cyclic contractions with Pata-type conditions were treated in [1, 2, 4], where Banach-type and Kannan-type results were obtained.

Since there was a gap in the proof of the main result in [1], the authors made a correction in [2], assuming an additional assumption in order for their proof "to work". Using an idea from [4], we will show here that the original result is in fact correct without this additional condition.

Applying similar ideas, we show that the respective results hold also for Chatterjea-type [5] cyclic contractions, as well as for cyclic generalized contractions [13]. An example is given to illustrate a possible usage of the obtained results.

2. Results

Throughout the paper, (X,d) will be a metric space and A_i, $i = 1, 2, \dots, p$, will be its nonempty closed subsets. For $j \in \mathbb{N}$, $j > p$, we will always put $A_j := A_i$, where $j \equiv i \pmod p$ and $1 \le i \le p$. Finally, let $Y := \bigcup_{i=1}^{p} A_i$.

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If a mapping $f : Y \to Y$ satisfies that $f(A_i) \subseteq A_{i+1}$ for $i = 1, 2, \dots, p$, we will say that $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y w.r.t. f.

For a fixed $x_1 \in X$, $\|x\|$ will be used for $d(x, x_1)$, $x \in X$. It will be clear that the results do not depend on the particular choice of x_1.

2.1. **A Pata-Banach-type cyclic fixed point result.** The following theorem was proved in [1].

Theorem 2.1. *Let the metric space* (X, d) *be complete,* $f : Y \to Y$*, with* $Y = \bigcup_{i=1}^{p} A_i$ *being a cyclic representation of* Y *w.r.t.* f*. Let* $\psi : [0, 1] \to [0, +\infty)$ *be an increasing function, continuous at* 0 *with* $\psi(0) = 0$*. Assume that there exist constants* $\Lambda \geq 0$*,* $\alpha \geq 1$ *and* $\beta \in [0, \alpha]$ *such that*

$$d(fx, fy) \leq (1 - \varepsilon)d(x, y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\|]^{\beta} \tag{2.1}$$

holds for all $\varepsilon \in [0, 1]$*,* $x \in A_i$ *and* $y \in A_{i+1}$*. Then* f *has a unique fixed point* x^**, it belongs to* $\bigcap_{i=1}^{p} A_i$ *and the Picard iteration sequence* $\{f^n x_1\}_{n\in\mathbb{N}}$ *converges to* x^* *for any initial point* $x_1 \in Y$*.*

The proof of this result in [1] was not fully correct, so the authors made a new proof in [2], assuming an additional hypothesis, namely

"there exists $y_1 \in Y$ such that the sequence $\{d(y_1, f^n y_1)\}_{n\in\mathbb{N}}$ is bounded."

We will show now that this assumption is in fact superfluous and the result is valid "as it stands".

Proof. Without loss of generality, we can assume that $x_1 \in A_1$. Denote, as usual, $x_n = fx_{n-1}$, $n \geq 2$, and put $c_n = \|x_n\| = d(x_n, x_1)$. It is easy to show that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \leq \cdots \leq d(x_1, x_2) = c_2.$$

The only nontrivial part is the proof that the sequence $\{c_n\}_{n\in\mathbb{N}}$ is bounded, provided that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. It is enough to consider those n (if they exist) for which $|c_n| > 1$.

Suppose that $n \equiv k \pmod{p}$, $1 \leq k \leq p$. Then we have

$$\begin{aligned} c_n = d(x_n, x_1) &\leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-2}, x_{k-1}) + d(x_{k-1}, x_n) \\ &\leq (k-2)\, c_2 + d(fx_{k-2}, fx_{n-1}). \end{aligned}$$

Since x_{k-2} and x_{n-1} belong to adjacently labelled subsets A_{k-2} and $A_{n-1} = A_{k-1}$, we further have

$$c_n \leq (k-2)\, c_2 + (1 - \varepsilon)\, d(x_{k-2}, x_{n-1}) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x_{k-2}\| + \|x_{n-1}\|]^{\beta}.$$

Since

$$\begin{aligned} d\left(x_{k-2}, x_{n-1}\right) &\leq d\left(x_{k-2}, x_1\right)+d\left(x_1, x_n\right)+d\left(x_n, x_{n-1}\right) \\ &\leq d\left(x_1, x_2\right)+d\left(x_2, x_3\right)+\cdots+d\left(x_{k-3}, x_{k-2}\right)+c_n+c_2 \\ &\leq (k-3+1)\, c_2+c_n=(k-2)\, c_2+c_n, \end{aligned}$$

it follows that

$$\begin{aligned} c_n &\leq (k-2)\, c_2+(1-\varepsilon)\left((k-2)\, c_2+c_n\right)+\Lambda \varepsilon^\alpha \psi(\varepsilon)[1+\|x_{k-2}\|+\|x_{n-1}\|]^\beta . \\ &=(k-2)(2-\varepsilon)\, c_2+(1-\varepsilon)\, c_n+\Lambda \varepsilon^\alpha \psi(\varepsilon)[1+\|x_{k-2}\|+\|x_{n-1}\|]^\beta . \end{aligned} \tag{2.2}$$

Now,

$$\begin{aligned} 1+\|x_{k-2}\|+\|x_{n-1}\| &=1+d\left(x_{k-2}, x_1\right)+d\left(x_{n-1}, x_1\right) \\ &\leq 1+(k-3)\, c_2+d\left(x_{n-1}, x_n\right)+d\left(x_n, x_1\right) \\ &\leq 1+(k-2)\, c_2+c_n, \end{aligned}$$

and we get that

$$\begin{aligned} \Lambda\left[1+\|x_{k-2}\|+\|x_{n-1}\|\right]^\beta &\leq \Lambda\left[1+(k-2)\, c_2+c_n\right]^\alpha \\ &=\Lambda\left(1+c_n\right)^\alpha\left[1+\frac{(k-2)\, c_2}{1+c_n}\right]^\alpha \\ &\leq \Lambda\left(1+c_n\right)^\alpha\left[1+(k-2)\, c_2\right]^\alpha \\ &\leq \Lambda c_n^\alpha\left(1+\frac{1}{c_n}\right)^\alpha\left[1+(k-2)\, c_2\right]^\alpha \\ &\leq \Lambda c_n^\alpha 2^\alpha\left[1+(k-2)\, c_2\right]^\alpha \\ &=a \cdot c_n^\alpha, \end{aligned} \tag{2.3}$$

where $a=\Lambda \cdot 2^\alpha\left[1+(k-2)\, c_2\right]^\alpha$.

We obtain from (2.2) and (2.3) that

$$\varepsilon c_n \leq a \varepsilon^\alpha \psi(\varepsilon)\, c_n^\alpha+b,$$

for some $a, b>0$, and it follows that the sequence $\{c_n\}_{n \in \mathbb{N}}$ is bounded, in the same way as in [11].

All other parts of the proof are the same as in [1]. ☐

2.2. **Auxiliary remarks and results.** Suppose that X is a nonempty set, $Y \subseteq X$, $f: Y \rightarrow Y$ and that $Y=\bigcup_{i=1}^p A_i$ is a cyclic representation of Y w.r.t. f. If $F(f)$ denotes the set of fixed points of f, then

$$F(f) \neq \emptyset \Rightarrow \bigcap_{i=1}^p A_i \neq \emptyset .$$

This follows directly from the conditions $f(A_i) \subseteq A_{i+1}$, $i=1,2, \ldots, p$.

If a certain (non-cyclic) fixed point result is known, in order to obtain the respective cyclic-type fixed point result, it is enough to prove that the respective cyclic contractive condition implies that $\bigcap_{i=1}^{p} A_i \neq \emptyset$. Indeed, in this case $(\bigcap_{i=1}^{p} A_i, d)$ is a complete metric space and the restriction of f satisfies the given standard condition.

Assertions similar to the following lemma were used (explicitly or implicitly) in the course of proofs of several fixed point result in various papers (see, e.g., [12]). Here we will use the following result in proofs of cyclic-type case (see also the proof of [8, Theorem 3]).

Lemma 2.1. *Let (X, d) be a metric space, $f : Y \to Y$ be a mapping, where $Y = \bigcup_{i=1}^{p} A_i$ is a cyclic representation of Y w.r.t. f. Assume that $\{x_n\}$ is a sequence in X, where $x_{n+1} = fx_n$, $x_1 \in A_1$, and*

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{2.4}$$

If $\{x_n\}$ is not a Cauchy sequence then there exist $\delta > 0$ and two sequences $\{m\left(k\right)\}$ and $\{n\left(k\right)\}$ of positive integers such that the following sequences tend to δ when $k \to \infty$:

$$\begin{aligned} &d\left(x_{m(k)-j(k)}, x_{n(k)}\right), \qquad &&d\left(x_{m(k)-j(k)+1}, x_{n(k)}\right), \\ &d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right), \qquad &&d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right), \end{aligned}$$

where $j\left(k\right) \in \{1, 2, \dots, p\}$ is chosen so that $n\left(k\right) - m\left(k\right) + j\left(k\right) \equiv 1 \pmod{p}$, for each $k \in \mathbb{N}$.

Proof. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\delta > 0$ and sequences $\{m\left(k\right)\}$ and $\{n\left(k\right)\}$ of positive integers such that

$$n\left(k\right) > m\left(k\right) > k, \quad d\left(x_{m(k)}, x_{n(k)-1}\right) < \delta, \quad d\left(x_{m(k)}, x_{n(k)}\right) \geq \delta \tag{2.5}$$

for all positive integers k. Then, using (2.5) and the triangle inequality, we get

$$\begin{aligned} \delta &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right) \\ &< \delta + d\left(x_{n(k)-1}, x_{n(k)}\right). \end{aligned}$$

Thus we have

$$\delta \leq d\left(x_{m(k)}, x_{n(k)}\right) < \delta + d\left(x_{n(k)-1}, x_{n(k)}\right).$$

Passing to the limit as $k \to \infty$ in the above inequality and using (2.4), we obtain

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \delta. \tag{2.6}$$

Note that, by the way $j(k)$ were chosen, $x_{m(k)-j(k)}$ and $x_{n(k)}$ (for k large enough, $m\left(k\right) > j\left(k\right)$) lie in adjacently labelled sets A_i and A_{i+1} for certain $i \in \{1, 2, \ldots, p\}$. This will be used further in the proof of Theorem 2.2.

Using the triangle inequality, we get

$$\begin{aligned}\left|d\left(x_{m(k)-j(k)}, x_{n(k)}\right) - d\left(x_{m(k)}, x_{n(k)}\right)\right| &\leq d\left(x_{m(k)-j(k)}, x_{n(k)}\right) \\ \leq \sum_{\nu=0}^{j(k)-1} & d\left(x_{m(k)-j(k)+\nu}, x_{m(k)-j(k)+\nu+1}\right) \\ \leq \sum_{\nu=0}^{p-1} & d\left(x_{m(k)-j(k)+\nu}, x_{m(k)-j(k)+\nu+1}\right) \to 0 \text{ as } k \to \infty\end{aligned}$$

(from (2.4)), which, by (2.6), implies that

$$\lim_{k\to\infty} d\left(x_{m(k)-j(k)}, x_{n(k)}\right) = \delta.$$

Using (2.4), we have that

$$\lim_{k\to\infty} d\left(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}\right) = 0 \text{ and } \lim_{k\to\infty} d\left(x_{n(k)+1}, x_{n(k)}\right) = 0. \quad (2.7)$$

Again using the triangle inequality, we get

$$\left|d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right) - d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right| \leq d\left(x_{n(k)+1}, x_{n(k)}\right).$$

Passing to the limit as $k \to \infty$ in the above inequality, and using (2.7), we get

$$\lim_{k\to\infty} d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right) = \delta.$$

Similarly, we have

$$\left|d\left(x_{n(k)}, x_{m(k)-j(k)+1}\right) - d\left(x_{m(k)-j(k)}, x_{n(k)}\right)\right| \leq d\left(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}\right).$$

Passing to the limit as $k \to \infty$, and using (2.4) and (2.7), we obtain

$$\lim_{k\to\infty} d\left(x_{n(k)}, x_{m(k)-j(k)+1}\right) = \delta.$$

Similarly, we have

$$\lim_{k\to\infty} d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right) = \delta.$$

□

Remark 2.1. Applying the previous lemma, the proof of Lemma 8 and hence Theorem 3 in [4] can be considerably shortened. The procedure is similar as in the proofs of Theorems 2.2 and 2.3 that follow, so we omit the details.

2.3. **A Pata-Chatterjea-type cyclic fixed point result.** We will consider now the case of Chatterjea-type fixed point result [5] under a Pata-type cyclic condition. Note that the corresponding standard (non-cyclic) result was treated in [6].

Theorem 2.2. *Suppose that all the conditions of Theorem* 2.1 *are fulfilled, except that condition* (2.1) *is replaced by*

$$\begin{aligned} d\left(fx,fy\right) \leq \frac{1-\varepsilon}{2}\left(d\left(x,fy\right)+d\left(fx,y\right)\right) \\ + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x\|+\|y\|+\|fx\|+\|fy\|]^{\beta}. \end{aligned} \tag{2.8}$$

Then f *has a unique fixed point* x^**, it belongs to* $\bigcap_{i=1}^{p} A_i$ *and the Picard iteration sequence* $\{f^n x_1\}_{n\in\mathbb{N}}$ *converges to* x^* *for any initial point* $x_1 \in Y$*.*

Proof. We will prove that, under the condition (2.8), $\bigcap_{i=1}^{p} A_i \neq \emptyset$ holds true.

As usual, take $x_1 \in A_1$ and define $x_{n+1} = fx_n$, $n = 1, 2, \dots$; only the case when $x_{n+1} \neq x_n$ for each $n \in \mathbb{N}$ is of interest.

The proof is done in the following steps:

1. $d\left(x_{n+1}, x_n\right) \downarrow d^* \geq 0$;

2. $c_n = d\left(x_n, x_1\right) = \|x_n\|$ is a bounded real sequence;

3. $d^* = 0$;

4. $\{x_n\}$ is a Cauchy sequence;

5. Completeness of $\bigcup A_{i=1}^{p}$ and the fact that each A_i contains a subsequence of the sequence $\{x_n\}$ imply that its limit belongs to $\bigcap_{i=1}^{p} A_i$, i.e., $\bigcap_{i=1}^{p} A_i \neq \emptyset$;

6. The uniqueness follows easily.

1. It follows putting $\varepsilon = 0$.

2. Let $n \geq 2$; take $k \equiv n \pmod{p}$ with $1 \leq k \leq p$. Then $x_i \in A_i$ for $i = 1, 2, \dots, k-1$, $x_n \in A_n$ and A_{k-1}, A_n are adjacently labelled. Hence, we get

$$\begin{aligned} c_n = d\left(x_n, x_1\right) &\leq d\left(x_1, x_2\right) + d\left(x_2, x_3\right) + \cdots + d\left(x_{k-2}, x_{k-1}\right) + d\left(x_{k-1}, x_n\right) \\ &\leq (k-2)\,c_2 + d\left(fx_{k-2}, fx_{n-1}\right) \\ &\leq (k-2)\,c_2 + \frac{1-\varepsilon}{2}\left(d\left(x_{k-2}, x_n\right) + d\left(x_{k-1}, x_{n-1}\right)\right) + \Lambda^* \\ &\leq (k-2)\,c_2 + \frac{1-\varepsilon}{2}\left(\left[d\left(x_{k-2}, x_1\right) + c_n\right]\right. \\ &\qquad \left. + \left[d\left(x_{k-1}, x_1\right) + c_n + d\left(x_n, x_{n-1}\right)\right]\right) + \Lambda^*, \end{aligned} \tag{2.9}$$

where

$$\Lambda^* = \Lambda\varepsilon^{\alpha}\psi\left(\varepsilon\right)\left[1+\|x_{k-2}\|+\|x_{n-1}\|+\|x_{k-1}\|+\|x_n\|\right]^{\beta}. \tag{2.10}$$

Further, we have

$$d(x_{k-2}, x_1) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-3}, x_{k-2}) \leq (k-3)\, c_2$$

and

$$d(x_{k-1}, x_1) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-2}, x_{k-1}) \leq (k-2)\, c_2.$$

Now it follows from (2.9) that

$$\begin{aligned} c_n &\leq (k-2)\, c_2 + \frac{1-\varepsilon}{2} \left((k-3)\, c_2 + c_n + (k-2)\, c_2 + c_n + c_2 \right) + \Lambda^* \\ &= (k-2)\, c_2 + \frac{1-\varepsilon}{2} \left((2k-6)\, c_2 + 2c_n \right) + \Lambda^* \\ &= (k-2)\, c_2 + (1-\varepsilon)(k-3)\, c_2 + (1-\varepsilon)\, c_n + \Lambda^*, \end{aligned}$$

where

$$\begin{aligned} \Lambda^* &= \Lambda \varepsilon^{\alpha} \psi(\varepsilon) \left[1 + \|x_{n-1}\| + \|x_n\| + \|x_{k-2}\| + \|x_{k-1}\| \right]^{\beta} \\ &\leq \Lambda \varepsilon^{\alpha} \psi(\varepsilon) \left[1 + (2k-6)\, c_2 + 2c_n \right]^{\beta} \\ &\leq \Lambda \varepsilon^{\alpha} \psi(\varepsilon) \left(1 + 2c_n \right)^{\alpha} \left(1 + (2k-6)\, c_2 \right)^{\alpha}. \end{aligned}$$

Similarly as in the proof of Theorem 2.1, we get that

$$\varepsilon c_n \leq a \varepsilon^{\alpha} \psi(\varepsilon)\, c_n^{\alpha} + b,$$

for some $a, b > 0$. In the same way as in [11], it follows that the sequence $\{c_n\}_{n \in \mathbb{N}}$ is bounded.

3. Starting from ($\varepsilon \in (0,1]$)

$$\begin{aligned} d(x_{n+1}, x_n) &= d(fx_n, fx_{n-1}) \\ &\leq \frac{1-\varepsilon}{2} \left(d(x_n, x_n) + d(x_{n+1}, x_{n-1}) \right) \\ &\qquad + \Lambda \varepsilon^{\alpha} \psi(\varepsilon) \left[1 + 2\|x_n\| + \|x_{n-1}\| + \|x_{n+1}\| \right]^{\beta} \\ &\leq \frac{1-\varepsilon}{2} \left(d(x_{n+1}, x_n) + d(x_n, x_{n-1}) \right) + K \varepsilon^{\alpha} \psi(\varepsilon), \quad K > 0 \end{aligned}$$

passing to the limit as $n \to \infty$, we get that

$$d^* \leq (1-\varepsilon)\, d^* + K \varepsilon^{\alpha} \psi(\varepsilon),$$

i.e., $d^* \leq K \varepsilon^{\alpha-1} \psi(\varepsilon)$, $\alpha \geq 1$. Hence, $d^* = 0$.

4. We can now apply Lemma 2.1 (using $\varepsilon \in (0,1]$).

Suppose that $\{x_n\}$ is not a Cauchy sequence, and choose $\delta > 0$ and the sequences $\{m(k)\}$, $\{n(k)\}$ and $\{j(k)\}$ as in Lemma 2.1. Putting $x =$

$x_{m(k)-j(k)}$, $y = x_{n(k)}$ in (2.8) (which is allowed), we get that

$$\begin{aligned}&d\left(x_{m(k)-j(k)+1}, x_{n(k)+1}\right)\\&\qquad\leq \frac{1-\varepsilon}{2}\left(d\left(x_{m(k)-j(k)}, x_{n(k)+1}\right)+d\left(x_{m(k)-j(k)+1}, x_{n(k)}\right)\right)+K\varepsilon^{\alpha}\psi\left(\varepsilon\right),\end{aligned}$$

$K > 0$, since the expression in brackets in (2.8) is bounded.

Passing to the limit in the previous relation, as $k \to \infty$, we get that

$$\delta \leq \frac{1-\varepsilon}{2}\left(\delta+\delta\right)+K\varepsilon^{\alpha}\psi\left(\varepsilon\right),$$

i.e., $\varepsilon\delta \leq K\varepsilon^{\alpha}\psi\left(\varepsilon\right)$. It follows that $\delta \leq K\varepsilon^{\alpha-1}\psi\left(\varepsilon\right)$ and hence $\delta = 0$, which is a contradiction.

The rest of the proof is standard. □

2.4. **Cyclic generalized contractions of Pata-type.**

Theorem 2.3. *Suppose that all the conditions of Theorem* 2.1 *are fulfilled, except that condition* (2.1) *is replaced by*

$$\begin{aligned}d\left(fx, fy\right) \leq \left(1-\varepsilon\right)\max\left\{d\left(x,y\right), d\left(x,fx\right), d\left(y,fy\right), \frac{d\left(x,fy\right)+d\left(y,fx\right)}{2}\right\}\\+\Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1+\|x\|+\|y\|+\|fx\|+\|fy\|]^{\beta}.\end{aligned}$$

Then f has a unique fixed point x^*, it belongs to $\bigcap_{i=1}^{p} A_i$ and the Picard iteration sequence $\{f^n x_1\}_{n\in\mathbb{N}}$ converges to x^* for any initial point $x_1 \in Y$.

Proof. The proof is similar to the one of Theorem 2.2. Only the proof of boundedness of $\{c_n\}$ needs some modifications.

We have here that

$$\begin{aligned}c_n &= d\left(x_n, x_1\right) \leq d\left(x_1, x_2\right)+d\left(x_2, x_3\right)+\cdots+d\left(x_{k-2}, x_{k-1}\right)+d\left(x_{k-1}, x_n\right)\\&\leq \left(k-2\right)c_2+d\left(fx_{k-2}, fx_{n-1}\right)\\&\leq \left(k-2\right)c_2+\left(1-\varepsilon\right)\max\Big\{d\left(x_{k-2}, x_{n-1}\right), d\left(x_{k-2}, x_{k-1}\right),\\&\qquad d\left(x_{n-1}, x_n\right), \frac{d\left(x_{k-2}, x_n\right)+d\left(x_{k-1}, x_{n-1}\right)}{2}\Big\}+\Lambda^*,\end{aligned}$$

where Λ^* have the same meaning as in (2.10). Now we have

$$d\left(x_{k-2}, x_{k-1}\right) \leq c_2, \quad d\left(x_{n-1}, x_n\right) \leq c_2$$

and

$$\begin{aligned}d\left(x_{k-2}, x_{n-1}\right) &\leq d\left(x_{k-2}, x_1\right)+d\left(x_1, x_n\right)+d\left(x_n, x_{n-1}\right)\\&\leq \left(k-3\right)c_2+c_n+c_2\\&= \left(k-2\right)c_2+c_n,\end{aligned}$$

as well as

$$\begin{aligned}&\frac{d\left(x_{k-2},x_n\right)+d\left(x_{k-1},x_{n-1}\right)}{2}\\&\qquad\le\frac{d\left(x_{k-2},x_1\right)+c_n+d\left(x_{k-1},x_1\right)+d\left(x_1,x_n\right)+d\left(x_n,x_{n-1}\right)}{2}\\&\qquad\le\frac{1}{2}\left(\left(k-3\right)c_2+c_n+\left(k-2\right)c_2+c_n+c_2\right)\\&\qquad=\left(k-3\right)c_2+c_n.\end{aligned}$$

Hence,

$$\begin{aligned}c_n&\le\left(k-2\right)c_2+\left(1-\varepsilon\right)\max\left\{\left(k-2\right)c_2+c_n,c_2,c_2,\left(k-3\right)c_2+c_n\right\}+\Lambda^*\\&\le\left(k-2\right)c_2+\left(1-\varepsilon\right)\left(\left(k-2\right)c_2+c_n\right)+\Lambda\varepsilon^{\alpha}\psi\left(\varepsilon\right)\left[1+(2k-6)c_2+2c_n\right]^{\alpha}\\&=\left(2-\varepsilon\right)\left(k-2\right)c_2+\left(1-\varepsilon\right)c_n+\Lambda\varepsilon^{\alpha}\psi\left(\varepsilon\right)\left[1+(2k-6)c_2+2c_n\right]^{\alpha}.\end{aligned}$$

Now, similarly as in the proof of Theorem 2.1, it follows that

$$\varepsilon c_n\le a\varepsilon^{\alpha}\psi\left(\varepsilon\right)c_n^{\alpha}+b,$$

for some $a,b>0$, and the sequence $\{c_n\}$ is bounded.

The rest of the proof is the same as in Theorems 2.1 and 2.2. ☐

The following is an example to which the preceding results can be applied.

Example 2.1. Let $X=\mathbb{R}$ with the standard metric and let

$$A_1=\left\{\frac{1}{2n}:n\in\mathbb{N}\right\}\cup\{0\},\quad A_2=\left\{-\frac{1}{2n-1}:n\in\mathbb{N}\right\}\cup\{0\},\quad Y=A_1\cup A_2.$$

Define $f:Y\to Y$ by

$$fx=\begin{cases}-\dfrac{x}{x+4}, & x\in A_1,\\[2ex] -\dfrac{x}{4}, & x\in A_2.\end{cases}$$

It is easy to see that $Y=A_1\cup A_2$ is a cyclic representation of Y w.r.t. f (with $p=2$). We will show that f satisfies the contractive condition of Theorem 2.3.

Indeed, let $x\in A_1$ and $y\in A_2$. Then

$$\begin{aligned}d(fx,fy)&=\left|\frac{x}{x+4}-\frac{y}{4}\right|\le\frac{1}{4}(x+|y|)=\frac{1}{4}d(x,y)\\&\le\frac{1}{4}\max\left\{d\left(x,y\right),d\left(x,fx\right),d\left(y,fy\right),\frac{d\left(x,fy\right)+d\left(y,fx\right)}{2}\right\}\\&:=\frac{1}{4}\mathcal{F}(x,y).\end{aligned}$$

The rest of the procedure is the same as for a similar situation in [11]. For arbitrary $\varepsilon \in [0,1]$, write the obtained inequality in the form

$$\begin{aligned} d(fx,fy) &\leq (1-\varepsilon)\mathcal{F}(x,y) + (\tfrac{1}{4}+\varepsilon-1)\mathcal{F}(x,y) \\ &\leq (1-\varepsilon)\mathcal{F}(x,y) + (\tfrac{1}{4}+\varepsilon-1)(\|x\|+\|y\|+\|fx\|+\|fy\|). \end{aligned}$$

We want to prove that there are some $\gamma \geq 0$ and $\Lambda \geq 0$ such that

$$(\tfrac{1}{4}+\varepsilon-1)(\|x\|+\|y\|+\|fx\|+\|fy\|) \leq \Lambda\varepsilon^{1+\gamma}(1+\|x\|+\|y\|+\|fx\|+\|fy\|),$$

holds for each $\varepsilon \in [0,1]$ and all comparable $x, y \in X$. Indeed, this will be the case if one can find $\Lambda \geq 0$ such that

$$\Lambda \geq \frac{\frac{1}{4}+\varepsilon-1}{\varepsilon^{1+\gamma}}$$

holds for some $\gamma \geq 0$ and each $\varepsilon \in [0,1]$. By a routine procedure, it is easy to show that this is the case if we chose γ such that $\frac{\gamma}{1+\gamma} > 1-\frac{1}{4}$ and then

$$\Lambda = \frac{\gamma^\gamma}{(1+\gamma)^{1+\gamma}}\frac{1}{(1-\frac{1}{4})^\gamma}.$$

Hence, we have that, for the chosen γ and Λ,

$$\begin{aligned} d(fx,fy) \leq (1-\varepsilon)\max&\left\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\right\} \\ &+ \Lambda\varepsilon^{1+\gamma}[1+\|x\|+\|y\|+\|fx\|+\|fy\|], \end{aligned}$$

for each $\varepsilon > 0$ and all $x, y \in X$. Thus, the conditions of Theorem 2.3 are fulfilled (with $\alpha = \beta = 1$ and $\psi(\varepsilon) = \varepsilon^\gamma$), and the mapping f has a unique fixed point (which is 0).

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Zoran Kadelburg
University of Belgrade, Faculty of Mathematics
Studentski trg 16
11000 Beograd, Serbia
kadelbur@matf.bg.ac.rs

Stojan Radenović
University of Belgrade
Faculty of Mechanical Engineering
Kraljice Marije 16
11120 Beograd, Serbia
radens@beotel.net