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ABSOLUTE TAUBERIAN CONSTANTS FOR LOWER TRIANGULAR MATRICES

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Abstract. In a recent paper [1] the authors obtained absolute Tauberian constants for the H-J generalized Hausdorff transformations, which generalized the corresponding results, obtained earlier by Sherif, for ordinary Hausdorff matrices. In this paper we obtain absolute Tauberian constants for regular lower triangular matrices with row sums one. As corollaries we obtain the corresponding results for factorable and weighted mean matrices.

1. Introduction

A principal contributor to the study of absolute Tauberian constants with respect to different summability methods has been Sherif, who began this work in the 1970's. She obtained Tauberian constants for a variety of methods including Cesaro [4], Hausdorff [5], and quasi-Hausdorff [2]. In a recent paper [1] the authors obtained absolute Tauberian constants for the H-J generalized Hausdorff transformations. In this paper absolute Tauberian constants are obtained for any regular lower triangular matrix with row sums one. We begin by defining terms and notation.

Let A be an infinite matrix, $\sum_{k=0}^{\infty} a_k$ an infinite series with partial sums s_n, and define $b_n = \sum_{k=0}^{\infty} a_{nk} a_k$. Sherif [5] established an inequality of the form

$$\sum_{n=0}^{\infty} |b_n - a_n| \leq K \sum_{n=0}^{\infty} |\Delta(na_n)|,$$

where K is an absolute Tauberian constant, A is a Hausdorff matrix satisfying certain properties, and Δ is the forward difference operator defined by $\Delta\mu_n = \mu_n - \mu_{n+1}$ for any sequence $\{\mu_n\}$.

In this paper we obtain the corresponding inequality when A is a regular lower triangular matrix with row sums one.

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A matrix $A = (a_{nk})$ is said to be regular if it is limit-preserving over c, the space of convergent sequences. Necessary and sufficient conditions for a matrix to be regular are the well-known Silverman-Töeplitz conditions:

(i) $\|A\|_\infty := \sup_n \sum_{k=0}^\infty |a_{nk}| < \infty$,
(ii) $\lim t_n = 1$, where $t_n := \sum_{k=0}^\infty a_{nk}$, and
(iii) $\lim_n a_{nk} = 0$ for each k.

For any series $\sum_{k=0}^\infty a_k$, with partial sums s_n, A a regular lower triangular matrix with row sums one, define

$$u_n = \sum_{k=0}^n a_{nk} s_k.$$

For $n \geq 1$, since $a_{n-1,n} = 0$, we may write

$$\begin{aligned}
b_n := u_n - u_{n-1} &= \sum_{k=0}^n (a_{nk} - a_{n-1,k}) s_k \\
&= \sum_{k=0}^n (a_{nk} - a_{n-1,k}) \sum_{i=0}^k a_i = \sum_{i=0}^n a_i \sum_{k=i}^n (a_{nk} - a_{n-1,k}) \\
&= a_0 \sum_{k=0}^n (a_{nk} - a_{n-1,k}) + \sum_{i=1}^n a_i \sum_{k=i}^n (a_{nk} - a_{n-1,k}) \\
&= a_0(t_n - t_{n-1}) + \sum_{i=1}^n \Big(-\frac{1}{i} \sum_{\nu=0}^{i-1} \Delta(\nu a_\nu) \Big) \\
&\qquad \Big(t_n - \sum_{k=0}^{i-1} a_{nk} - t_{n-1} + \sum_{i=0}^{i-1} a_{n-1,k} \Big).
\end{aligned}$$

Since A has row sums one,

$$b_n = \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \Big[\sum_{i=\nu+1}^n \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) \Big]. \tag{1}$$

2. Results

The following Lemma will be needed in the proof of Theorem 1.

Lemma 1. [[4], p. 235, Lemma 2.1] *Let $\{g_n\}$ be a sequence, A an infinite matrix with $A_n := \sum_{\nu=0}^\infty \alpha_{n\nu} g_\nu$. Suppose that*

$$\sum_{n=0}^\infty |\alpha_{n\nu}| \quad \textit{is bounded.}$$

Define

$$K = \sup_{\nu} \sum_{n=0}^{\infty} |\alpha_{n\nu}|.$$

Then

$$\sum_{n=0}^{\infty} |A_n| \leq K \sum_{\nu=0}^{\infty} |g_\nu|, \tag{2}$$

and this constant is the best possible in the sense that (2) *becomes false if* K *is replaced by any smaller constant.*

Theorem 1. *Let* A *be a regular lower triangular matrix with row sums one. If*

$$K := \sup_{\nu \geq 0} \sum_{n=\nu+1}^{\infty} \left| \sum_{i=\nu+1}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) + \frac{1}{n} \right| < \infty, \tag{3}$$

then

$$\sum_{n=0}^{\infty} |b_n - a_n| \leq K \sum_{n=1}^{\infty} \left| \Delta(na_n) \right|.$$

Proof. Any sequence $\{a_n\}$ has the representation

$$a_n = \begin{cases} a_0, & n = 0, \\ -\dfrac{1}{n} \displaystyle\sum_{\nu=0}^{n-1} \Delta(\nu a_\nu), & n \geq 1. \end{cases}$$

Note that, since A has row sums one, $b_0 - a_0 = (t_0 - 1)a_0 = 0$, and, for $n > 0$, using (1),

$$b_n - a_n = \sum_{\nu=0}^{n-1} \Delta(\nu a_\nu) \left[\sum_{i=\nu+1}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) + \frac{1}{n} \right]. \tag{4}$$

Equation (4) represents a transformation of the type considered in Lemma 1; i.e., for $n \geq 1$,

$$\alpha_{n\nu} = \begin{cases} \displaystyle\sum_{i=\nu+1}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) + \frac{1}{n}, & 0 \leq \nu \leq n-1, \\ 0, & \nu \geq n. \end{cases}$$

From Lemma 1, K is as defined in (3), and the result follows. $\square$

Lower triangular matrices with row sums one enjoy a very interesting property, which will now be investigated.

Define

$$S_\nu = \sum_{n=\nu+1}^{\infty} \left[\sum_{i=\nu+1}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) + \frac{1}{n} \right].$$

Then, from (3),

$$\begin{aligned}
S_{\nu-1} &= \sum_{n=\nu}^{\infty} \left[\sum_{i=\nu}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) + \frac{1}{n} \right] \\
&= \frac{1}{\nu} \sum_{k=0}^{\nu-1} (a_{\nu,k} - a_{\nu-1,k}) + \frac{1}{\nu} \\
&\quad + \sum_{n=\nu+1}^{\infty} \left[\frac{1}{\nu} \sum_{k=0}^{\nu-1} (a_{nk} - a_{n-1,k}) + \sum_{i=\nu+1}^{n} \frac{1}{i} \sum_{k=0}^{i-1} (a_{nk} - a_{n-1,k}) + \frac{1}{n} \right],
\end{aligned}$$

Thus

$$\begin{aligned}
S_\nu - S_{\nu-1} &= -\frac{1}{\nu} \sum_{k=0}^{\nu-1} (a_{\nu,k} - a_{\nu-1,k}) - \frac{1}{\nu} \\
&\qquad - \frac{1}{\nu} \sum_{n=\nu+1}^{\infty} \sum_{k=0}^{\nu-1} (a_{nk} - a_{n-1,k}) \\
&= -\frac{1}{\nu} - \frac{1}{\nu} \sum_{k=0}^{\nu-1} (a_{\nu,k} - a_{\nu-1,k}) \\
&\qquad - \frac{1}{\nu} \lim_{m\to\infty} \sum_{n=\nu+1}^{m} \sum_{k=0}^{\nu-1} (a_{nk} - a_{n-1,k}) \\
&= -\frac{1}{\nu} \Big(1 + t_\nu - a_{\nu\nu} - t_{\nu-1} \Big) \\
&\qquad - \frac{1}{\nu} \lim_{m\to\infty} \sum_{k=0}^{\nu-1} \sum_{n=\nu+1}^{m} (a_{nk} - a_{n-1,k}) \\
&= \frac{a_{\nu\nu} - 1}{\nu} - \frac{1}{\nu} \lim_{m\to\infty} \sum_{k=0}^{\nu-1} (a_{mk} - a_{\nu,k}) \\
&= \frac{a_{\nu\nu} - 1}{\nu} - \frac{1}{\nu} (a_{\nu\nu} - 1) = 0.
\end{aligned}$$

Consequently all of the values of S_ν are the same. This fact is illustrated in the proof of Theorem 2.1 of [5], where it is shown (formula (2.20) that each

$$S_\nu = \int_0^1 \frac{|\chi(t)|}{t} dt,$$

which agrees with formula (2.20) of [5], provided that $\chi(t) \geq 0$, where χ is the mass function for the Hausdorff matrix. If the Hausdorff matrix is the Cesàro matrix of order α, for some $\alpha > 0$, then

$$\chi(t) = 1 - (1-t)^\alpha,$$

and $\chi(t) \geq 0$.

If A is the Cesàro matrix of order one, then

$$S_0 = \lim_{m\to\infty} \left[\sum_{i=1}^m \frac{1}{i} \sum_{k=0}^{i-1} \frac{1}{m+1} \right] = \lim_{m\to\infty} \left[\frac{m}{m+1} \right] = 1,$$

If H is the Euler matrix of order a, then

$$\chi(t) = \begin{cases} 0, & 0 \leq t < a, \\ 1, & a \leq t \leq 1. \end{cases}$$

Therefore $S_\nu = \log(1/a)$ for each ν.

A weighted mean matrix is a lower triangular matrix, denoted by $(\overline{N}, p)$, with entries p_k/P_n, where $\{p_k\}$ is a nonnegative sequence with $p_o > 0$, and $P_n := \sum_{k=0}^n p_k$.

Corollary 1. *Let $(\overline{N}, p)$ be a regular weighted mean matrix. Then*

$$K = \lim_{m\to\infty} \sum_{i=1}^m \frac{P_{i-1}}{iP_m},$$

provided that

$$\frac{1}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{i=\nu+1}^n \frac{P_{i-1}}{i} \geq 0.$$

Proof. Note that, from (3), since $a_{nk} = p_k/P_n$,

$$\begin{aligned} &\frac{1}{n} + \sum_{i=\nu+1}^n \frac{1}{i} \sum_{k=0}^{i-1} \left(\frac{p_k}{P_n} - \frac{p_k}{P_{n-1}} \right) \\ &= \frac{1}{n} - \frac{p_n}{P_n P_{n-1}} \sum_{i=\nu+1}^n \frac{P_{i-1}}{i} \end{aligned}$$

provided that $\chi(t) \geq 0$. □

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