

A DECOMPOSITION OF δ -OPEN FUNCTIONS

MIGUEL CALDAS AND SAEID JAFARI

ABSTRACT. In 2008, M. Caldas and G. Navalagi [1] introduced a new class of generalized open functions called weakly δ -open functions. By introducing a new type of open functions called relatively weakly δ -open together with weakly δ -open functions, we establish a new decomposition of δ -open functions.

1. INTRODUCTION

General Topology has shown its fruitfulness in both the pure and applied directions. In data mining [13], computational topology for geometric design and molecular design [12], computer-aided geometric design and engineering design (briefly CAGD), digital topology, information systems, noncommutative geometry and its application to particle physics [10], one can observe the influence made in these realms of applied research by general topological spaces, properties and structures. Rosen and Peters [14] have used topology as a body of mathematics that could unify diverse areas of CAGD and engineering design research. They have presented several examples of the application of topology to CAGD and design.

The concept of openness is fundamental with respect to the investigation of general topological spaces.

In 1968, Veličko [17] introduced the notion of δ -open sets, which are stronger than open sets, in order to investigate the characterization of H -closed spaces and showed that τ_δ (=the collection of all δ -open sets) is a topology on X such that $\tau_\delta \subset \tau$ and so τ_δ equal with the semi-regularization topology τ_s . In 1985, M. Mršević et al [11] introduce and studied the class of δ -open functions, also in 1985, D.A.Rose [15] and D.A.Rose with D.S. Janković [16] have defined the notions of weakly open.

Recently Caldas and Navalagi [1] introduced a new class of functions called weakly δ -open functions by utilizing the notion of δ -open sets. This

2010 *Mathematics Subject Classification.* 54A40, 54C10, 54D10; Secondary: 54C08.

Key words and phrases. Topological space, δ -open sets, δ -open functions, weak δ -continuity, weakly δ -open functions.

class of functions contains the class of δ -open functions [3]. They showed that the class of weakly δ -open functions does not necessarily imply the class of δ -open functions.

The object of the present note is to obtain a necessary and sufficient condition for a weakly δ -open function to be δ -open. We believe that the notions given in this paper have a broad applications. For example one can introduce several new notions by changing open sets in the well-known notions of pure and applied mathematics by δ -open sets or closed sets by δ -closed sets.

The study of generalized open sets, generalized openness and properties of them have been found to be useful in computer science and digital topology [8,9]. Also the fuzzy topological version of the notions and results introduced in this paper are very important since Professor El- Naschie has recently shown in [6,7] that the notion of fuzzy topology may be relevant to quantum particle physics in connection with string theory and ε^∞ theory.

Finally we may mention that the present work may be relevant to fractal and Cantorian physics [4-6,8].

We assume throughout the present note that (X, τ) and (Y, σ) (or X and Y) denote topological spaces in which no separation axioms are assumed unless explicitly stated. Recall that, a set A is called regular open (resp. regular closed) if $A = \text{Int}(\text{Cl}(A))$ (resp. $A = \text{Cl}(\text{Int}(A))$). A point $x \in X$ is called a δ -cluster [17] of A if $A \cap U \neq \emptyset$ for each regular open set U containing x . The set of all δ -cluster point of A is called the δ -closure of A and is denoted by $\text{Cl}_\delta(A)$. Hence, a subset A is called δ -closed if $\text{Cl}_\delta(A) = A$. The complement of a δ -closed set is called δ -open. The δ -interior of A denoted by $\text{Int}_\delta(A)$ is defined as $\text{Int}_\delta(A) = \{x \in X : \text{for some open subset } U \text{ of } X, x \in U \subset \text{Int}(\text{Cl}(U)) \subset A\}$. The family of all δ -open (resp. δ -closed) sets of a space X is denoted by $\delta O(X, \tau)$ or $\delta O(X)$ (resp. $\delta C(X, \tau)$ or $\delta C(X)$).

Lemma 1.1 ([2]). *Let (X, τ) be a topological space. Intersection of arbitrary of δ -closed sets in X is δ -closed.*

Lemma 1.2 ([2]). *Let A be a subset of a topological space (X, τ) . Then $\text{Cl}_\delta(A) = \bigcap \{F \in \delta C(X) : A \subset F\}$.*

Corollary 1.3 ([2]). *$\text{Cl}_\delta(A)$ is δ -closed for a subset A in a topological space (X, τ) .*

Proof. It is obvious from the above lemmas. □

2. A NEW DECOMPOSITION OF δ -OPEN FUNCTIONS

The following definition was given by Caldas and Navalagi in [1] as a natural dual to the weak δ -continuity

A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be weakly δ -open [1] if for each $x \in X$ and each open set U of X containing x , there exists a δ -open set V containing $f(x)$ such that $V \subset f(Cl(U))$. Alternatively, $f : (X, \tau) \rightarrow (Y, \sigma)$ is weakly δ -open if and only if $f(U) \subset Int_{\delta}(f(Cl(U)))$ for every $U \in \tau$.

Recall that a function is called δ -open [3] if the image of every open set is δ -open and weakly open ([15], [16]) if $f(U) \subset Int(f(Cl(U)))$ for each open subset U of X . Clearly, every δ -open function is weakly δ -open and every weakly δ -open function is weakly open, but the converses are not generally true as shown in [1].

Lemma 2.1. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is weakly δ -open, then for each $U \in \tau$, we have $f(U) \subset f(Cl(U)) \cap V$, where V is a δ -open subset of Y .*

Proof. Take $V = Int_{\delta}(f(Cl(U)))$. □

Lemma 2.1 suggests the following new generalization of the class of δ -open functions.

Definition 2.2. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be relatively weakly δ -open provided that $f(U)$ is δ -open in $f(Cl(U))$ for every open subset U of X .

If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is δ -open, then for each $U \in \tau$, $f(U) \in \delta O(Y)$ and since $f(U) \subset f(Cl(U))$, it follows that $f(U)$ is also a δ -open subset of $f(Cl(U))$. Therefore the following theorem has been established.

Theorem 2.3. *If $f : (X, \tau) \rightarrow (Y, \sigma)$ is δ -open, then f is relatively weakly δ -open.*

Example 2.4. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Let $f : (X, \tau) \rightarrow (X, \sigma)$ be the identity function. Then f is weakly δ -open, but f is not relatively weakly δ -open, since for $U = \{a\} \in \tau$, $f(Cl(U)) = \{a, b\}$ and $f(U)$ is not a δ -open in $f(Cl(U))$.

The significance of relative weakly δ -open is that it yields a decomposition of δ -open functions with weakly δ -open functions as the other factor.

Theorem 2.5. *A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is δ -open if and only if f is weakly δ -open and relatively weakly δ -open.*

Proof. The necessity is given by Theorem 2.2 and the fact that every δ -open function is weakly δ -open. We prove the sufficiency. Assume f is weakly δ -open and relatively weakly δ -open. Let U be an open set in X . Since f is relatively weakly δ -open, we have $f(U) = f(Cl(U)) \cap V$ where V is a δ -open set of Y . Let $y \in f(U)$. By the fact that f is weakly δ -open, there exists $W \in \delta O(Y)$ containing y such that $W \subset f(Cl(U))$. We can take W to be a subset of V . It follows that $y \in W \subset f(Cl(U)) \cap V = f(U)$ and thus the claim follows. □

Acknowledgment. The authors are grateful for the referee's for their valuable remarks which improved the paper.

REFERENCES

- [1] M. Caldas and G. Navalagi, *On generalizing δ -open functions*, Pro Mathematica, 22 (2008), 103–121.
- [2] E. Ekici, *On δ -semiopen sets and a generalization of functions*, Bol. Soc. Paran. Mat., 23 (2005), 73–84.
- [3] L. L. Herrington, *Some properties preserved by the almost- continuous function*, Boll. Un. Mat. Ital., 10 (1974), 556–568.
- [4] M. S. El-Naschie, O. E. Rossler and I. Prigogine, *Quantum Mechanics, Diffusion and Chaotic Fractals*, Oxford: Elsevier Pergamon; 1995.
- [5] M. S. El-Naschie, *Superstrings, knots and noncommutative geometry in ε^∞ space*, Inst. J. Theor. Phys., 37 (1998), 2935–51.
- [6] M. S. El-Naschie, *On the certification of heterotic strings, M theory and ε^∞ theory*, Chaos Solitons Fractals, (2000), 2397–2408.
- [7] M. S. El-Naschie, *The two-slit experement as the fundation, of E-infinity of high energy physics*, Chaos, Solitons Fractals, 25 (2005), 509–514.
- [8] E. D. Khalimsky, R. Kopperman and P. R. Meyer, *Computer graphics and connected topologies in finite ordered sets*, Topology Appl., 36 (1990), 1–17.
- [9] E. Kovalevsky and R. Kopperman, *Some topology-based image processing algorithms*, Ann NY. Acad. Sci., 728 (1994), 174–182.
- [10] G. Landi, *An introduction to Noncommutative Spaces and Their Geometris*, (lecture notes in physics), Springer-Verlag (New York), 1997.
- [11] M. Mršević, I. L. Reilly and M. K. Vamanamurthy, *On semi-regularization properties*, J. Austral. Math. Soc. (Series A), 38 (1985), 40–54.
- [12] E. L. F. Moore and T. J. Peters, *Computational Topology for Geometric Design and Molecular Design*, Mathematics in Industry-Challenges and Frontiers 2003, Editors: D. R. Ferguson and T. J. Peters, SIAM 2005.
- [13] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*, In: System theory, knowledge engineering and problem solving, Vol. 9., Dordrecht. Kluwer; 1991.
- [14] D. W. Rosen and T. J. Peters, *The role of topology in engineering design research*, Research in Engineering Design 2 (1996), 81–98.
- [15] D. A. Rose, *On weak openness and almost openness*, Internat. J. Math.& Math. Sci., 7 (1984), 35–40.
- [16] D. A. Rose and D. S. Janković., *Weakly closed functions and Hausdorff spaces*, Math. Nachr., 130 (1987), 105–110.
- [17] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., 78 (1968), 103–118.

(Received: September 16, 2014)

(Revised: November 20, 2014)

M. Caldas
 Departamento de Matemática Aplicada,
 Universidade Federal Fluminense,
 Rua Mario Santos Braga, s/n
 24020-140, Niteroi, RJ Brasil
 gmamccs@vm.uff.br

S. Jafari
College of Vestsjaelland South
Herrestaede 11
4200 Slagelse, Denmark
jafari@stofanet.dk