

## ON THE CORE OF DOUBLE SEQUENCES

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ABSTRACT. In this paper, we define the core of double sequences via barriers and disks. We show that these definitions are equivalent and give an inequality related to the  $P$ -core of bounded double sequences.

### 1. INTRODUCTION

The concept of the core of a sequence was introduced firstly by Knopp. So, this type core was called the Knopp core. Let  $x = (x_k)$  be a sequence in  $\mathbb{C}$ , the set of all complex numbers, and  $R_k$  be the least convex closed region of complex plane containing  $x_k, x_{k+1}, x_{k+2}, \dots$ . The Knopp core of  $x$  (K-core of  $x$  or core of  $x$ ) is defined by the intersection of all  $R_k$  ( $k = 1, 2, \dots$ ). In the real case the  $K$ -core of  $x$  is reduced to the closed interval  $[\liminf x, \limsup x]$ . If  $A$  is a nonnegative regular matrix, then the core of  $x$ , is contained the core of  $Ax$ , provided that  $Ax$  exists (see [2]). Rhoades [16] gave a slight generalization of Knopp's core theorem. An alternative definition of the Knopp core is given by Shcherbakov [19] via barriers.

By using the definitions of limit inferior, limit superior and the core of a double sequence with the notion of the regularity of four dimensional matrices, Patterson [11, 12] gave some results on core of double sequences. Mursaleen [8] and Mursaleen and Edely [9] defined the almost strong regularity of matrices for double sequences and have applied these matrices to establish a core theorem and introduced the  $M$ -core for double sequences and determined the four dimensional matrices transforming every bounded double sequence  $x = (x_{jk})$  into one whose core is a subset of the  $M$ -core of  $x$ . Çakan and Altay [3] investigated the statistical core for double sequences and studied an inequality related to the statistical and  $P$ -cores of bounded double sequences. Recently, Gökhan, Çolak and Mursaleen [4] generalized the Pringsheim core for bounded double sequences and gave some core theorems via matrix classes. Boos, Legier and Zeller [1] introduced

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and investigated the notion of  $\epsilon$ -convergence of double sequence, which is essentially weaker than the Pringsheim convergence. Quite recently, Sever and Talo [18] introduced the concepts of  $\epsilon$ -limit superior and inferior for real double sequences and defined  $\epsilon$ -core for double sequences.

## 2. DEFINITIONS AND NOTATION

By  $\Omega$ , we denote the set of all complex valued double sequences, i.e.,

$$\Omega = \{x = (x_{mn}) : x_{mn} \in \mathbb{C} \text{ for all } m, n \in \mathbb{N}\},$$

which is a vector space with co-ordinatewise addition and scalar multiplication of double sequences, where  $\mathbb{N}$  and  $\mathbb{C}$  denote the set of positive integers and the complex field, respectively. Any vector subspace of  $\Omega$  is called a double sequence space. The space  $\mathcal{M}_u$  of all bounded double sequences is defined by  $\mathcal{M}_u = \{x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty\}$  which is a Banach space with the norm  $\|\cdot\|_\infty$ . Consider a sequence  $x = (x_{mn}) \in \Omega$ . If for every  $\epsilon > 0$  there exists  $n_0 = n_0(\epsilon) \in \mathbb{N}$  such that  $|x_{mn} - \ell| < \epsilon$  for all  $m, n > n_0$  then we call that the double sequence  $x$  is convergent in Pringsheim's sense to the limit  $\ell$  and write  $P - \lim_{m,n} x_{mn} = \ell$ . By  $\mathcal{C}_p$ , we denote the space of all convergent double sequences in the Pringsheim's sense (see [15]). It is well-known that there are such sequences in the space  $\mathcal{C}_p$  but not in the space  $\mathcal{M}_u$ . So, we may mention the space  $\mathcal{C}_{bp}$  of the double sequences that are both convergent in Pringsheim's sense and bounded, i.e.,  $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$ . By  $\mathcal{C}_{bp0}$ , we denote the space of the double sequences which are both convergent to zero in the Pringsheim's sense and bounded. There is more than one type of convergence for double sequences, so we denote convergence by  $v$ -convergence for  $v \in \{p, bp\}$ .

A number  $\alpha \in \mathbb{C}$  is said to be a Pringsheim limit point of a double sequence  $(x_{mn})$  if there exist two increasing sequences  $m_1 < m_2 < \dots < m_i < \dots$  and  $n_1 < n_2 < \dots < n_j < \dots$  such that  $P - \lim_{i,j \rightarrow \infty} x_{m_i n_j} = \alpha$ .

**Definition 2.1** ([11]). Let  $P-C_n\{x\}$  be the least closed convex set that includes all points  $x = (x_{kl})$  for  $k, l > n$ ; then the Pringsheim core of the double sequence  $x = (x_{kl})$  is the set

$$P-C\{x\} = \bigcap_{n=1}^{\infty} P-C_n\{x\}.$$

**Definition 2.2** ([2], p.139). Every line  $L$  divides the plane into two half-planes. If a set of the points  $X$  lies entirely in such a half-plane (some or all of the points may lie on the line  $L$ ), we say that  $L$  is a barrier line for  $X$ .

Let  $\lambda$  be the space of double sequences, converging with respect to some linear convergence rule  $v - \lim : \lambda \rightarrow \mathbb{C}$ . The sum of a double series  $\sum_{i,j} x_{ij}$

with respect to this rule is defined by  $v - \sum_{i,j} x_{ij} = v - \lim_{m,n} \sum_{i=1}^m \sum_{j=1}^n x_{ij}$ . Let  $\lambda, \mu$  be two spaces of double sequences, converging with respect to the linear convergence rules  $v_1 - \lim$  and  $v_2 - \lim$ , respectively, and  $A = (a_{mnkl})$  also be a four dimensional matrix of complex numbers. Define the set

$$\lambda_A^{(v_2)} = \left\{ (x_{kl}) \in \Omega : Ax = \left( v_2 - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and } Ax \in \lambda \right\}. \tag{2.1}$$

Then, we say, with the notation of (2.1), that  $A$  maps the space  $\lambda$  into the space  $\mu$  if  $\mu \subset \lambda_A^{(v_2)}$  and denote the set of all four dimensional matrices, mapping the space  $\lambda$  into the space  $\mu$ , by  $(\lambda : \mu)$ . It is trivial that for any matrix  $A \in (\lambda : \mu)$ ,  $(a_{mnkl})_{k,l \in \mathbb{N}}$  is in the  $\beta(v_2)$ -dual  $\lambda^{\beta(v_2)}$  of the space  $\lambda$  for all  $m, n \in \mathbb{N}$ . An infinite matrix  $A$  is said to be  $\mathcal{C}_v$ -conservative if  $\mathcal{C}_v \subset (\mathcal{C}_v)_A$ .

For more details on double sequences and 4-dimensional matrices, we refer to [5, 6, 10, 13, 14, 20–22].

A matrix  $A$  is said to be *RH-regular* if it maps every bounded convergent sequence into a convergent sequence with the same limit.

**Lemma 2.3** ([5, 17]). *The necessary and sufficient conditions for  $A$  to be RH-regular are:*

$$\begin{aligned} P - \lim_{m,n} a_{mnkl} &= 0, \text{ for each } k, l \in \mathbb{N}, \\ P - \lim_{m,n} \sum_{k,l}^{\infty, \infty} a_{mnkl} &= 1, \\ P - \lim_{m,n} \sum_k^{\infty} |a_{mnkl}| &= 0 \text{ for each } l \in \mathbb{N}, \\ P - \lim_{m,n} \sum_l^{\infty} |a_{mnkl}| &= 0 \text{ for each } k \in \mathbb{N}, \\ \sum_{k,l}^{\infty, \infty} |a_{mnkl}| &\text{ is convergence and} \end{aligned}$$

there exist positive numbers  $A$  and  $B$  such that  $\sum_{k,l > B} |a_{mnkl}| < A$ .

### 3. MAIN RESULT

In this article, besides Knopp core, we introduce the definition of core of double sequences in the sense of Shcherbakov [19] and show their equivalence. Also, we prove two theorems about core of double sequence via characteristic values of conservative matrices. These theorems are analogous to those of Rhoades and Maddox in [7, 16].

**Theorem 3.1.** *Let  $D$  be the set of all  $P$ -limit points of  $x = (x_{kl}) \in \Omega$ . Then,*

$$D \subseteq P\text{-core}\{x\}$$

*Proof.* Let  $\alpha$  be a  $P$ -limit point of  $x = (x_{kl})$ . Then, there exists an increasing sequence of integers  $(k_i, l_j)$  such that

$$P\text{-}\lim_{i,j} x_{k_i l_j} = \alpha.$$

Now, choose any  $n \in \mathbb{N}$  such that the points

$$\begin{matrix} x_{k_p, l_q} & x_{k_p, l_{q+1}} & x_{k_p, l_{q+2}} & \cdots \\ x_{k_{p+1}, l_q} & x_{k_{p+1}, l_{q+1}} & x_{k_{p+1}, l_{q+2}} & \cdots \\ x_{k_{p+2}, l_q} & x_{k_{p+2}, l_{q+1}} & x_{k_{p+2}, l_{q+2}} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

are in  $P-C_n\{x\}$  for  $k_p, l_q > n$ . Since  $P-C_n\{x\}$  is closed, it contains the limit points of the sequence in itself. Hence,  $\alpha$  is in  $P-C_n\{x\}$ , since  $n$  is arbitrary.  $\square$

Now, we give an alternative definition of the  $P$ -core $\{x\}$ .

- Definition 3.2.** a) If the set  $S$  containing the elements of  $x = (x_{kl})$  has no barrier lines, then  $P$ -core $\{x\}$  is the whole plane.  
 b) If the set  $S$  containing the elements of  $x = (x_{kl})$  has barrier lines, then  $P$ -core $\{x\}$  is the intersection of all half-planes containing the set  $D$  of the limit points of  $x = (x_{kl})$ .

We can extend the definition of the core of sequences introduced by Shcherbakov [19] for double sequences as it follows:

**Definition 3.3.** Let  $x = (x_{kl})$  be a bounded double sequence. For  $z \in \mathbb{C}$ , let

$$B_x(z) = \left\{ w \in \mathbb{C} : |w - z| \leq P\text{-}\limsup_{k,l} |x_{kl} - z| \right\}.$$

Then

$$P-C\{x\} = \bigcap_{z \in \mathbb{C}} B_x(z).$$

**Theorem 3.4.** *The Definition 2.1 and Definition 3.2 are equivalent for bounded double sequences.*

*Proof.* Let  $x = (x_{kl}) \in \Omega$ ,  $E$  be the  $P$ -core $\{x\}$  with respect to the Definition 2.1,  $D$  be the set of all  $P$ -limit points of  $x = (x_{kl})$  and  $F$  be the intersection of all half-planes which contains  $D$ . It is known by the definition that  $D \subseteq E$  and it is also obvious that  $D \subseteq F$ . We need to show that  $E = F$ . Suppose that an  $a \notin E$ . Then,  $a \notin P-C_n\{x\}$  for some fixed value of  $n$ . We can draw a barrier line  $L$  separating  $a$  from  $P-C_n\{x\}$ . Since  $P-C_n\{x\}$  is closed, so  $D \subset P-C_n\{x\}$  and so,  $L$  separates  $a$  from  $D$ . Hence,  $a \notin F$ . This means that

$$F \subset E. \tag{3.1}$$

Now, draw a half-plane  $H$  containing  $D$  and call the barrier line  $L$ . All except a finite number of the elements of  $x = (x_{kl})$  lie on the same side of

$L$  as  $D$ . Otherwise, there would be at least one  $P$ -limit point on the side of  $L$  remote from  $D$ . Consequently, there is an  $m$  such that the points

$$\begin{matrix} x_{m,m} & x_{m,m+1} & x_{m,m+2} & \cdots \\ x_{m+1,m} & x_{m+1,m+1} & x_{m+1,m+2} & \cdots \\ x_{m+2,m} & x_{m+2,m+1} & x_{m+2,m+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

are in  $D$ . Hence,  $P\text{-}C_m\{x\} \subset H$  and so,  $E \subset H$ . Thus,

$$F \supset E. \tag{3.2}$$

On combining these two results, the proof of the theorem is completed.  $\square$

**Theorem 3.5.** *The Definition 3.2 and Definition 3.3 are equivalent for bounded double sequences.*

*Proof.* Let  $x = (x_{kl}) \in \Omega$ ,  $E$  be the  $P$ -core $\{x\}$  with respect to the Definition 3.2,  $D$  be the set of all  $P$ -limit points of  $x = (x_{kl})$  and  $F$  be the intersection of all half-planes those contain  $D$ .

First assume  $w \notin \bigcap_{z \in \mathbb{C}} B_x(z)$ , say  $w \notin B_x(z_0)$  for some  $z_0 \in \mathbb{C}$ . Let  $H$  be the half-plane containing  $B_x(z_0)$  whose boundary line is perpendicular to the line containing  $w$  and  $z_0$  and tangent to the circular boundary of  $B_x(z_0)$ . Since  $B_x(z_0) \subset H$  and  $B_x(z_0)$  contains  $P$ -limit points, then  $H$  is a half-plane which contains  $P$ -limit points. Since  $w \notin H$ , this implies that  $w$  is not in the intersection of half-planes. Hence, the inclusion

$$P\text{-core}\{x\} \subset \bigcap_{z \in \mathbb{C}} B_x(z) \tag{3.3}$$

holds.

Conversely, if  $w$  is not in the intersection of half-planes, then there exists a  $H$  half-plane such that  $w \notin H$ . Let  $L$  be the line through  $w$  that is perpendicular to the boundary of  $H$ , and let  $p$  be the midpoint of the segment of  $L$  between  $w$  and  $H$ . Let  $z_0$  be a point of  $L$  such that  $z_0 \in H$ , and consider the disk

$$B(z_0) = \left\{ y \in \mathbb{C} : |y - z_0| \leq |p - z_0| \right\}.$$

Since  $x$  is bounded and  $x_{kl} \in H$  for  $k, l > n$  for some  $n \in \mathbb{N}$ , we can choose  $z_0$  sufficiently far from  $p$  so that

$$|p - z_0| = P\text{-}\limsup_{k,l} |x_{kl} - z_0|.$$

Thus  $B(z_0)$  is one of the  $B_x(z)$  disks and  $w \notin \bigcap_{z \in \mathbb{C}} B_x(z)$ . Hence, we conclude that

$$P\text{-core}\{x\} \supset \bigcap_{z \in \mathbb{C}} B_x(z) \tag{3.4}$$

holds.

On combining the inclusions (3.3) and (3.4), we have the desired result. □

**Theorem 3.6.** *Let  $A = (a_{mnkl})$  4-dimensional real matrix for which*

$$\chi(A) = P\text{-}\lim_{m,n} \sum_{k,l} a_{mnkl} - \sum_{k,l} P\text{-}\lim_{m,n} a_{mnkl}$$

*is defined. Then the condition*

$$\begin{aligned} &\text{there exists a positive integer } N \text{ such that } a_{mnkl} \geq 0 \text{ (} m, n \in \mathbb{N} \text{)} \\ &\text{for all } \max\{k, l\} \geq N \in \mathbb{N} \end{aligned} \quad (3.5)$$

*is sufficient for*

$$P\text{-}\liminf_{m,n} A_{mn}x \geq \chi(A)l(x) + \sum_{k,l} \alpha_{kl}x_{kl} \quad (3.6)$$

*and*

$$P\text{-}\limsup_{m,n} A_{mn}x \leq \chi(A)L(x) + \sum_{k,l} \alpha_{kl}x_{kl} \quad (3.7)$$

*whenever the series  $\sum_{k,l} \alpha_{kl}x_{kl}$  is convergent. Where*

$$P\text{-}\lim_{m,n} a_{mnkl} = \alpha_{kl}, \quad P\text{-}\liminf_{k,l} x_{kl} = l(x) \quad \text{and} \quad P\text{-}\limsup_{k,l} x_{kl} = L(x).$$

(Note that if  $P\text{-}\liminf_{k,l} x_{kl} = -\infty$ , then (3.6) is true without (3.5) provided that  $\chi(A) > 0$ , and similarly for (3.7) when  $P\text{-}\limsup_{k,l} x_{kl} = \infty$ .)

*Proof.* Assume that  $l(x) > -\infty$ . To prove (3.5) is sufficient, fix  $\varepsilon > 0$ . There exists an integer  $N_0$  such that  $x_{kl} \geq l(x) - \varepsilon$  for all  $k, l \geq N_0$ .

$$\begin{aligned} A_{mn}x &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl}x_{kl} \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} \left[ x_{kl} + (l(x) - \varepsilon) - (l(x) - \varepsilon) \right] \\ &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} [l(x) - \varepsilon] + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \\ &= [l(x) - \varepsilon] \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} + \sum_{k=1}^{N_0} \sum_{l=1}^{N_0} a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \\ &\quad + \sum_{k=1}^{N_0} \sum_{l=N_0+1}^{\infty} a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \end{aligned}$$

$$+ \sum_{k=N_0+1}^{\infty} \sum_{l=1}^{N_0} a_{mnkl} [x_{kl} - (l(x) - \varepsilon)] + \sum_{k=N_0+1}^{\infty} \sum_{l=N_0+1}^{\infty} a_{mnkl} [x_{kl} - (l(x) - \varepsilon)].$$

Since  $a_{mnkl} \geq 0$  ( $m, n \in \mathbb{N}$ ) for all  $\max\{k, l\} \geq N_0$ , the fifth series is non-negative, and

$$A_{mn}x \geq [l(x) - \varepsilon] \left( \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} - \sum_{k=1}^{N_0} \sum_{l=1}^{N_0} a_{mnkl} \right) + \sum_{k=1}^{N_0} \sum_{l=1}^{N_0} a_{mnkl} x_{kl} + \sum_{k=1}^{N_0} \sum_{l=N_0+1}^{\infty} a_{mnkl} [x_{kl} - (l(x) - \varepsilon)] + \sum_{k=N_0+1}^{\infty} \sum_{l=1}^{N_0} a_{mnkl} [x_{kl} - (l(x) - \varepsilon)].$$

If we take  $P$ -lim inf in this inequality for all  $m, n, k, l \in \mathbb{N}$ , we obtain

$$P\text{-}\liminf_{m,n} A_{mn}x \geq \chi(A)l(x) + \sum_{k,l} \alpha_{kl}x_{kl}.$$

The inequality (3.7) is obtained from (3.6) by considering  $-x$  instead of  $x$ . □

**Theorem 3.7.** *Let  $A = (a_{mnkl})$  be a real four dimensional matrix for which  $\chi(A)$  is defined. Then,*

$$P\text{-}\lim_{m,n} \sum_{k,l} |a_{mnkl}| = P\text{-}\lim_{m,n} \sum_{k,l} a_{mnkl} = t$$

*is sufficient condition for (3.6) and (3.7) to hold for all bounded double sequences  $x = (x_{kl})$  for which the series  $\sum_{k,l} \alpha_{kl}x_{kl}$  is convergent.*

*Proof.* If we write for all  $m, n, k, l \in \mathbb{N}$

$$b_{mnkl} = \frac{|a_{mnkl}| + a_{mnkl}}{2} \text{ and } c_{mnkl} = \frac{|a_{mnkl}| - a_{mnkl}}{2},$$

then

$$a_{mnkl} = b_{mnkl} - c_{mnkl}.$$

By hypothesis, it is clear that

$$P\text{-}\lim_{m,n} \sum_{k,l} b_{mnkl} = t \text{ and } P\text{-}\lim_{m,n} \sum_{k,l} c_{mnkl} = 0.$$

Since  $x = (x_{kl})$  is bounded, there exists a number  $K > 0$  such that  $|x_{kl}| < K$  for all  $k, l \in \mathbb{N}$ . For any fixed  $\varepsilon > 0$ , there exist integer  $M, N > q$  such that  $x_{kl} \geq l(x) - \varepsilon$  and  $k, l > N$  and  $m, n > M$ ,

$$\sum_{k,l} c_{mnkl} < \frac{\varepsilon}{K + d + \varepsilon},$$

where  $d = \max(|l|, |L|)$ . Let  $r > \max\{M, N\}$  and write

$$\begin{aligned}
A_{mn}x &= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} x_{kl} \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} \left[ x_{kl} + (l(x) - \varepsilon) - (l(x) - \varepsilon) \right] \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} [l(x) - \varepsilon] + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \\
&= [l(x) - \varepsilon] \sum_{k,l} a_{mnkl} + \sum_{k=1}^r \sum_{l=1}^r a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \\
&\quad + \sum_{k=1}^r \sum_{l=r+1}^{\infty} a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] + \sum_{k=r+1}^{\infty} \sum_{l=1}^r a_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \\
&\quad + \sum_{k=r+1}^{\infty} \sum_{l=r+1}^{\infty} b_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] - \sum_{k=r+1}^{\infty} \sum_{l=r+1}^{\infty} c_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right].
\end{aligned}$$

Since

$$\sum_{k=r+1}^{\infty} \sum_{l=r+1}^{\infty} b_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] \geq 0,$$

$$\sum_{k=r+1}^{\infty} \sum_{l=r+1}^{\infty} c_{mnkl} \left[ x_{kl} - (l(x) - \varepsilon) \right] < \left[ K + |l(x)| + \varepsilon \right] \sum_{k=r+1}^{\infty} \sum_{l=r+1}^{\infty} c_{mnkl} < \varepsilon,$$

$$\sum_{l=r+1}^{\infty} a_{mnkl} < \varepsilon$$

and

$$\sum_{k=r+1}^{\infty} a_{mnkl} < \varepsilon$$

for each  $m, n > r$ , we have

$$P\text{-}\liminf_{m,n} A_{mn}x \geq [l(x) - \varepsilon] \left( t - \sum_{k,l} \alpha_{kl} \right) + \sum_{k,l} \alpha_{kl} x_{kl} + \varepsilon. \quad (3.8)$$

Since the proof of (3.7) is similar to that of (3.6), we omit it.  $\square$

**Definition 3.8.**  $A = (a_{mnkl})$  is called almost positive if  $P\text{-}\lim_{m,n} \sum a_{mnkl}^- = 0$  where  $a_{mnkl}^- = \max\{-a_{mnkl}, 0\}$ .



**Theorem 3.9.** *Let  $B = (b_{mnkl})$  any RH-regular and almost positive matrix. Then, there is no matrix  $A$  such that  $P\text{-lim sup } Ax \leq P\text{-lim inf } Bx$  for  $x \in M_u$ .*

*Proof.* Suppose that, if possible, there exists such a matrix  $A = (a_{mnkl})$ . By Theorem 3.2 [11]

$$P\text{-lim sup } Bx \leq P\text{-lim sup } x$$

and we have

$$P\text{-lim sup } Ax \leq P\text{-lim inf } Bx \leq P\text{-lim sup } Bx \leq P\text{-lim sup } x$$

whence  $A = (a_{mnkl})$  is RH-regular. By the Corollary 3.1 [12] there exists  $z \in M_u$  such that  $P\text{-lim inf } Az \neq P\text{-lim sup } Az$ . Since  $P\text{-lim sup } Ax \leq P\text{-lim sup } Bx$  we get  $P\text{-lim inf } Bx \leq P\text{-lim inf } Ax$ . So,  $P\text{-lim inf } Bz < P\text{-lim sup } Az \leq P\text{-lim inf } Bz$ , a contradiction.

This completes the proof of the theorem.  $\square$

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