SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.10.2.02
Vol.10 (23) (2014), 155–160

J–CONJUGATE–NORMAL MATRICES

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Abstract. In this paper some properties of J–conjugate-normal matrices are given. In particular, a list of twenty one conditions is given, each of which is equivalent to the matrix A being J–conjugate-normal.

1. introduction

An indefinite inner product in $\mathbb{C}^n$ (where by $\mathbb{C}$ we denote the field of complex numbers) is a sesquilinear form $[x, y]$, $x, y \in \mathbb{C}^n$, defined by an equation

$$[x, y] =< Hx, y >,\ x, y \in \mathbb{C}^n.$$

Here $< \cdot, \cdot >$ is the standard Euclidean inner product in $\mathbb{C}^n$, and $H \in M_n(\mathbb{C})$ is an invertible Hermitian matrix. We denote by $X^{[*]H}$ or, if there is no risk of confusion, by $X^{[*]}$, the adjoint of $X \in M_n(\mathbb{C})$, with respect to H, or, in short, H–*adjoint*; that is $X^{[*]} := H^{-1}X^*H$. Here and throughout the paper, X^* stands for the conjugate transpose of the matrix X. A matrix $X \in M_n(\mathbb{C})$ is called H–*selfadjoint* if $X = X^{[*]}$, H–*skewadjoint* if $X = -X^{[*]}$, and H–*unitary* if X is invertible and $X^{[*]} = X^{-1}$. A more general class of H–*normal* matrices X is defined by the property that X commutes with $X^{[*]}$. (see ([1], ch.4)).

In the particular case $H = J$ where $J = I_r \oplus -I_{n-r}$, we have $[x, y] := y^*Jx, x, y \in \mathbb{C}^n$ and $X^{[*]J} = JX^*J$.

Correspondingly J–*selfadjoint* matrix, J–*skewadjoint* matrix and J–*unitary* matrix is defined.

In [2], H. Fassbender and Kh. D. Ikramov presented a list of about forty conditions on a matrix $A \in M_n(\mathbb{C})$, each of which is equivalent to A being conjugate-normal.

In this paper we introduce J–conjugate-normal matrices, then our purpose is to give some properties of J–conjugate-normal matrices that are

2010 *Mathematics Subject Classification.* 15A24, 15B99.
Key words and phrases. J–conjugate-normal matrix, J–symmetric matrix, J–skew–symmetric matrix, J–nonnegative matrix, H–normal matrix.

similar to a number of conditions in [2] for conjugate-normality. Sixteen of these conditions are equivalent to J–conjugate-normality. Four conditions are necessary, but not sufficient, and one condition is sufficient but not necessary.

Throughout the paper we use notation $X^{[*]}$ instead of $X^{[*]J}$.

2. J–conjugate-normal matrices

Definition 2.1. A matrix $A \in M_n(\mathbb{C})$ is called J–conjugate-normal if

$$AA^{[*]} = \overline{A^{[*]}A}.$$

Definition 2.2. A matrix $A \in M_n(\mathbb{C})$ is called J–symmetric if $A = \overline{A^{[*]}}$, i.e, $A = JA^TJ$. Also a matrix $A \in M_n(\mathbb{C})$ is called J–skewsymmetric if $A = -\overline{A^{[*]}}$, i.e, $A = -JA^TJ$.

For any matrix $A \in M_n(\mathbb{C})$, we can write $A = S_A + K_A$, such that S_A is J–symmetric and K_A is J–skewsymmetric. This decomposition for matrix A is uniquely determined by:

$$S_A = \frac{1}{2}(A + \overline{A^{[*]}}) \text{ and } K_A = \frac{1}{2}(A - \overline{A^{[*]}}). \tag{2.1}$$

We introduce the matrices $A_L = \overline{A}A$ and $A_R = A\overline{A} = \overline{A_L}$.

Theorem 2.3. *If A is J–conjugate-normal then A_L and A_R are J–normal. (Cf. Theorem 3 in [2].)*

Proof. We have

$$A_R{}^{[*]}A_R = (A\overline{A})^{[*]}(A\overline{A}) = (\overline{A^{[*]}}A^{[*]})(A\overline{A}) = \overline{A^{[*]}AA^{[*]}A} = (\overline{A^{[*]}A})^2,$$

and

$$\begin{aligned} A_RA_R{}^{[*]} &= (A\overline{A})(A\overline{A})^{[*]} = A\overline{AA^{[*]}}A^{[*]} = AA^{[*]}AA^{[*]} \\ &= \overline{A^{[*]}AA^{[*]}A} = (\overline{A^{[*]}A})^2. \end{aligned}$$

Thus A_R is J–normal. Hence, $A_L = \overline{A_R}$ is J–normal as well. □

Remark 2.4. The reverse implication of the above theorem is false.

For instance, if $J = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, then the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

is not J–conjugate-normal, because

$$\{AA^{[*]}\}_{12} = -2 \neq 2 = \{\overline{A^{[*]}A}\}_{12},$$

but $A_L = A_R = 2I_2$ are J-normal matrices.

We recall that a matrix X is H–normal if $XX^{[*]H} = X^{[*]H}X$, for an invertible matrix H.

To state the next proposition, we associate with each matrix $A \in M_n(\mathbb{C})$ the matrix

$$\widehat{A} = \begin{bmatrix} 0 & A \\ \overline{A} & 0 \end{bmatrix}.$$

If we set

$$H = \begin{bmatrix} J & 0 \\ 0 & J \end{bmatrix}$$

then, we have the following proposition:

Theorem 2.5. *A matrix $A \in M_n(\mathbb{C})$ is J–conjugate-normal if and only if $\widehat{A}$ is H-normal. (Cf. Theorem 4 in [2].)*

Proof. Since

$$\widehat{A}^{[*]H} = \begin{bmatrix} 0 & \overline{A^{[*]}} \\ A^{[*]} & 0 \end{bmatrix},$$

then $\widehat{A}\widehat{A}^{[*]H} = AA^{[*]} \oplus \overline{AA^{[*]}}$ and $\widehat{A}^{[*]H}\widehat{A} = \overline{A^{[*]}A} \oplus A^{[*]}A$. □

In the following theorem im(.) and ker(.) denote the range and the null space of the corresponding matrix.

Theorem 2.6. *If $A \in M_n(\mathbb{C})$ is J–conjugate-normal, then* $\operatorname{im}(A) = \operatorname{im}(\overline{A^{[*]}})$ *and* $\ker(A) = \ker(\overline{A^{[*]}})$. *(Cf. Theorem 5 in [2].)*

Proof. For any matrix A, we have

$$\operatorname{im}(AA^{[*]}) = \operatorname{im}(A) \quad \text{and} \quad \ker(AA^{[*]}) = \ker(A^{[*]})$$

So

$$\operatorname{im}(A) = \operatorname{im}(AA^{[*]}) = \operatorname{im}(\overline{A^{[*]}A}) = \operatorname{im}(\overline{A^{[*]}A}) = \operatorname{im}(\overline{A^{[*]}}).$$

and

$$\ker(A^{[*]}) = \ker(AA^{[*]}) = ker(\overline{A^{[*]}A}) = \ker(\overline{A}) \Rightarrow \ker({A^{[*]}}^{[*]}) = ker(\overline{A^{[*]}})$$

$$\Rightarrow \ker(A) = \ker(\overline{A^{[*]}}).$$

□

A matrix A is said to be J–nonnegative if JA is positive semidefinite, i.e, for any $x \in \mathbb{C}^n$, $x^*JAx \geq 0$ (See [3]).

In the next section some properties of J–conjugate-normal matrices are given.

3. Conditions

Conditions 1–16 are equivalent to A being J–conjugate-normal.
1. We consider Theorem 2.5 as condition 1.
2. A^T is J–conjugate-normal. (Cf. condition 4 in [2].)
3. $\overline{A}$ is J–conjugate-normal. (Cf. condition 5 in [2].)
4. $A^{[*]}$ is J–conjugate-normal. (Cf. condition 6 in [2].)
5. A^{-1} is J–conjugate-normal (for invertible A). (Cf. condition 7 in [2].)
6. $A^{-1}\overline{A^{[*]}}$ is J-unitary (for invertible A). (Cf. condition 8 in [2].)
7. $A = \overline{A^{[*]}A}A^{-1^{[*]}}$ (for invertible A). (Cf. condition 9 in [2].)
8. $(\overline{U^{[*]}}AU)$ is J–conjugate-normal for any (or for some) J–unitary matrix U. (Cf. condition 13 in [2].)

The following eight conditions refer to decomposition of A that is refered in Definition 2.2.
9. $S\overline{K} = K\overline{S}$. (Cf. condition 17 in [2].)
10. $A\overline{S} = S\overline{A}$. (Cf. condition 18 in [2].)
11. $A\overline{S} + SA^{[*]} = 2S\overline{S}$ (or $\overline{S}A + A^{[*]}S = 2\overline{S}S$). (Cf. condition 19 in [2].)
12. $A\overline{K} = K\overline{A}$. (Cf. condition 20 in [2].)
13. $A\overline{K} - KA^{[*]} = 2K\overline{K}$ (or $\overline{K}A - A^{[*]}K = 2\overline{K}K$) (as long as S is nonsingular). (Cf. condition 21 in [2].)
14. $S^{-1}A + A^{[*]}S^{-1} = 2I$ (or $AS^{-1} + \overline{S}^{-1}A^{[*]} = 2I$). (Cf. condition 22 in [2].)
15. $K^{-1}A - A^{[*]}\overline{K}^{-1} = 2I$ (or $AK^{-1} - \overline{K}^{-1}A^{[*]} = 2I$) (as long as K is nonsingular). (Cf. condition 23 in [2].)
16. $S\overline{S} - K\overline{K} = AA^{[*]}$. (Cf. condition 24 in [2].)

Conditions 17–20 are necessary, but not sufficient for J–conjugate-normality.
17. We consider Theorem 2.3 as condition 17.
18. $AA^{[*]}A = \overline{A^{[*]}A}A$ (or $AA^{[*]}A = A\overline{AA^{[*]}}$). (Cf. condition 10 in [2].)
19. $A\overline{C} = CA$, where $C = AA^{[*]} - \overline{A^{[*]}A}$. (Cf. condition 11 in [2].)
20. $AA^{[*]} - \overline{A^{[*]}A}$ is J–nonnegative. (Cf. condition 16 in [2].)

Condition 21 is sufficient but is not necessary for J–conjugate-normality.
21. $AB = BA$ implies that $A^{[*]}B = \overline{BA^{[*]}}$. (Cf. condition 12 in [2].)

4. Proofs and comments

Proof of Condition 2. $AA^{[*]} = \overline{A^{[*]}A} \Leftrightarrow AJA^*J = JA^TJ\overline{A} \Leftrightarrow J\overline{A}JA^T = A^*JAJ \Leftrightarrow (A^T)^{[*]}A^T = \overline{A^T(A^T)^{[*]}}$.

Proof of Condition 6.
$(A^{-1}\overline{A^{[*]}})(A^{-1}\overline{A^{[*]}})^{[*]} = I \Leftrightarrow A^{-1}\overline{A^{[*]}A}A^{-1^{[*]}} = I \Leftrightarrow \overline{A^{[*]}A} = AA^{[*]}$.

Proof of Condition 9.
By the decomposition of A that is said in definition 2.2, we see that $A = S + K$ and $A^{[*]} = \overline{S} - \overline{K}$, so
$AA^{[*]} = \overline{A^{[*]}A} \Leftrightarrow (S+K)(\overline{S}-\overline{K}) = (S-K)(\overline{S}+\overline{K}) \Leftrightarrow S\overline{S}-S\overline{K}+K\overline{S}-K\overline{K} = S\overline{S}+S\overline{K}-K\overline{S}-K\overline{K} \Leftrightarrow K\overline{S}-S\overline{K} = S\overline{K}-K\overline{S} \Leftrightarrow 2K\overline{S} = 2S\overline{K} \Leftrightarrow S\overline{K} = K\overline{S}$.

Proof of Condition 10.
Necessity. By Condition 9: $A\overline{S} = (S+K)\overline{S} = S\overline{S} + K\overline{S} = S\overline{S} + S\overline{K} = S(\overline{S} + \overline{K}) = S\overline{A}$.

Sufficiency. $A\overline{S} = S\overline{A} \Rightarrow S\overline{S} + S\overline{K} = S\overline{S} + K\overline{S} \Rightarrow S\overline{K} = K\overline{S}$.
thus by Condition 9, A is J–conjugate-normal.

Proof of Condition 11.
Since S is J–symmetric, and K is J–skewsymmetric, proof is straight forward by applying Condition 9.

Proof of Condition 14.
Necessity. Note that if A is J–conjugate-normal, then by condition 11, $S^{-1}K = \overline{KS}^{-1}$, because: $K = K\overline{SS}^{-1} = S\overline{KS}^{-1} \Rightarrow S^{-1}k = \overline{KS}^{-1}$.
Sufficiency. Apply Condition 9.

To see Conditions 18 and 19 are not sufficient for J–conjugate-normality, consider the following example:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

For Condition 20, consider the example

$$A = \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}, \; J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This matrix is not J–conjugate-normal, but

$$J(AA^{[*]} - \overline{A^{[*]}A}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

therefore $AA^{[*]} - \overline{A^{[*]}A}$ is J–nonnegative.

Proof of condition 19.

First for any matrix $A \in M_n(\mathbb{C})$ we define $[A, A^{[*]}] := AA^{[*]} - A^{[*]}A$. According to Theorem 2.5, since A is J–conjugate-normal, $\widehat{A}$ is H–normal. This means that $\widehat{A}\widehat{A}^{[*]H} = \widehat{A}^{[*]H}\widehat{A}$, but $\widehat{A}\widehat{A}^{[*]H} = AA^{[*]} \oplus \overline{A^{[*]}A}$, and $\widehat{A}^{[*]H}\widehat{A} = \overline{A^{[*]}A} \oplus A^{[*]}A$, thus $[\widehat{A}, \widehat{A}^{[*]}H] = AA^{[*]} - \overline{A^{[*]}A} \oplus \overline{A^{[*]}A} - A^{[*]}A = C \oplus \overline{C}$. On the other hand if $\widehat{A}$ is H–normal then $\widehat{A}$ commute with $[\widehat{A}, \widehat{A}^{[*]H}]$ (see [3]). Hence $A\overline{C} = CA$.

Proof of Condition 21.

Sufficiency is shown by setting $B = A$. To show that necessity does not hold, consider the following example:

$$A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \; J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus A is J–conjugate-normal, but if

$$B = \begin{bmatrix} 2i & 0 \\ 0 & 1-i \end{bmatrix},$$

then $AB = BA$, but $A^{[*]}B \neq \overline{BA^{[*]}}$.

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(Received: August 2, 2013)
(Revised: December 17, 2013)

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