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PARTIAL SUMS FOR A CERTAIN SUBCLASS OF MEROMORPHIC UNIVALENT FUNCTIONS

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Abstract. In this paper, the class $\Sigma_\lambda(\alpha, \beta, \gamma)$ of univalent meromorphic functions defined using the Ruscheweyh derivative in the punctured unit disk U* is introduced. We study some results concerning the partial sums of meromorphic univalent starlike functions and meromorphic univalent convex functions.

1. Introduction

Let Σ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \;\; (a_k \geq 0), \tag{1.1}$$

which are regular and univalent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U \backslash \{0\}$. Let $g \in \Sigma$, be given by

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \tag{1.2}$$

then the Hadamard product (or convolution) of f and g is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

A function $f \in \Sigma$ is said to be meromorphically starlike of order α if

$$\mathrm{Re}\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \;\; (z \in U;\; 0 \leq \alpha < 1). \tag{1.4}$$

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Denote by $\Sigma^*(\alpha)$ the class of all meromorphically starlike functions of order α. A function $f \in \Sigma$ is said to be meromorphically convex of order α if

$$\operatorname{Re}\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} > \alpha \ (z \in U;\ (0 \leq \alpha < 1)). \tag{1.5}$$

Denote by $\Sigma_k^*(\alpha)$ the class of all meromorphically convex functions of order α. We note that

$$f(z) \in \Sigma_k^*(\alpha) \Longleftrightarrow -zf'(z) \in \Sigma^*(\alpha).$$

The classes $\Sigma^*(\alpha)$ and $\Sigma_k^*(\alpha)$ had been extensively studied by Pommerenke [7], Miller [6] and others.

For $\lambda > -1$, the Ruscheweyh derivative of order λ is denoted by $D^\lambda f$ and is defined for function of the form (1.1) as follows: If

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$

then

$$D^\lambda f(z) = \frac{1}{z(1-z)^{\lambda+1}} * f(z) = z^{-1} + \sum_{k=1}^{\infty} D_k(\lambda) a_k z^k, \quad z \in U^*, \tag{1.6}$$

where

$$D_k(\lambda) = \frac{(\lambda+1)(\lambda+2)\dots(\lambda+k+1)}{(k+1)!}. \tag{1.7}$$

For $\beta \geq 0,\ 0 \leq \alpha < 1, 0 \leq \gamma < \frac{1}{2}$ and $\lambda > -1$, Atshan and Kulkarni [4] and Atshan [3] defined the class $\Sigma_\lambda(\alpha, \beta, \gamma)$ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\operatorname{Re}\left\{\frac{z(D^\lambda f(z))' + \gamma z^2 (D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z) + \gamma z (D^\lambda f(z))'} + \alpha\right\} \geq$$

$$\beta \left| \frac{z(D^\lambda f(z))' + \gamma z^2 (D^\lambda f(z))''}{(1-\gamma)D^\lambda f(z) + \gamma z (D^\lambda f(z))'} + 1 \right| (z \in U). \tag{1.8}$$

We note that:
$\Sigma_0(\alpha, 0, 0) = \Sigma^*(\alpha)$ $(0 \leq \alpha < 1)$ (see Pommerenke [7]).

Also, we note that
$\Sigma_\lambda(\alpha, \beta, 0) = \Sigma_\lambda^*(\alpha, \beta) =$

$$-\operatorname{Re}\left\{\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} + \alpha\right\} \geq \quad \beta \left| \frac{z(D^\lambda f(z))'}{D^\lambda f(z)} + 1 \right| (z \in U). \tag{1.9}$$

For $\beta \geq 0$, $0 \leq \alpha < 1$, and $\lambda > -1$, we denote by $\Sigma_{k,\lambda}^*(\alpha, \beta)$ the subclass of Σ consisting of functions of the form (1.1) and satisfying the analytic criterion:

$$-\operatorname{Re}\left\{1+\frac{z(D^{\lambda}f(z))''}{D^{\lambda}f'(z)}+\alpha\right\}\geq\beta\left|2+\frac{z(D^{\lambda}f(z))''}{D^{\lambda}f'(z)}\right|(z\in U), \qquad (1.10)$$

We note that:
$\Sigma_{k,0}(\alpha,0)=\Sigma_k^*(\alpha)$ $(0\leq\alpha<1)$ (see Pommerenke [7]).
It is easy to observe from (1.9) and (1.10) that

$$f(z)\in\Sigma_{k,\lambda}^*(\beta,\alpha)\Longleftrightarrow -zf'(z)\in\Sigma_{\lambda}^*(\beta,\alpha). \qquad (1.11)$$

In order to prove our results for functions belonging to the class $\Sigma_\lambda(\alpha,\beta,\gamma)$ we shall need the following lemma given by Atshan and Kulkarni [4].

Lemma 1. [4, Theorem 2.1] *Let the function f be defined by* (1.1)*. Then $f\in\Sigma_\lambda(\alpha,\beta,\gamma)$ if and only if*

$$\sum_{k=1}^{\infty}(1+\gamma k-\gamma)\left[k(1+\beta)+(\beta+\alpha)\right]D_k(\lambda)\,a_k\leq(1-\alpha)(1-2\gamma). \qquad (1.12)$$

where $0\leq\alpha<1,\ \beta\geq0,\ 0\leq\gamma<\frac{1}{2},\ \lambda>-1,$ *and $D_k(\lambda)$ is given by* (1.7).

Taking $\gamma=0$ in Lemma 1, we obtain the following corollary.

Corollary 1. *Let the function f defined by* (1.1)*. Then $f\in\Sigma_\lambda^*(\beta,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left[k(1+\beta)+(\beta+\alpha)\right]D_k(\lambda)\,a_k\leq(1-\alpha). \qquad (1.13)$$

By using Corollary 1 and (1.11), we can prove the following lemma.

Lemma 2. *Let the function f defined by* (1.1)*. Then $f\in\Sigma_{k,\lambda}^*(\beta,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}k\left[k(1+\beta)+(\beta+\alpha)\right]D_k(\lambda)\,a_k\leq(1-\alpha). \qquad (1.14)$$

In this paper, applying the technique used by Silverman [8], we will investigate the ratio of a function of the form (1.1) to its sequence of partial sums $f_n(z)=\frac{1}{z}+\sum_{k=1}^{n}a_kz^k$ when the coefficients of f are sufficiently small to satisfy condition (1.13) or (1.14). More precisely, we will determine sharp lower bounds for

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\},\ \operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\},\ \operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\}\ \text{and}\ \operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\}.$$

In the sequel, we will make use of well-known result that $\operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)}\right\}>0$ $(z\in U)$ if and only if $w(z)=\sum_{k=1}^{\infty}c_kz^k$ satisfies the inequality $|w(z)|\leq|z|$.

Unless otherwise stated, we will assume that f is of the form (1.1) and its sequence of partial sums is denoted by $f_n(z) = \frac{1}{z} + \sum_{k=1}^{n} a_k z^k$.

For the notational convenience we shall henceforth denote

$$\delta_k(\lambda, \beta, \alpha) = [k(1+\beta) + (\beta+\alpha)]\, D_k(\lambda). \tag{1.15}$$

2. Main results

Theorem 1. *If f of the form* (1.1) *satisfies condition* (1.13), *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\delta_{n+1}(\lambda,\beta,\alpha) - (1-\alpha)}{\delta_{n+1}(\lambda,\beta,\alpha)} \quad (z \in U). \tag{2.1}$$

The result is sharp, with extremal function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{\delta_{n+1}(\lambda,\beta,\alpha)} z^{n+1} \quad (n \geq 1). \tag{2.2}$$

Proof. We may write

$$\begin{aligned}
&\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\left[\frac{f(z)}{f_n(z)} - \frac{\delta_{n+1}(\lambda,\beta,\alpha) - (1-\alpha)}{\delta_{n+1}(\lambda,\beta,\alpha)}\right] \\
&\qquad = \frac{1 + \sum\limits_{k=1}^{n} a_k z^{k+1} + \frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha} \sum\limits_{k=n+1}^{\infty} a_k z^{k+1}}{1 + \sum\limits_{k=1}^{n} a_k z^{k+1}} \\
&\qquad = \frac{1 + A(z)}{1 + B(z)}.
\end{aligned}$$

Set $\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)}$, so that $w(z) = \frac{A(z)-B(z)}{2+A(z)+B(z)}$. Then

$$w(z) = \frac{\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha} \sum\limits_{k=n+1}^{\infty} a_k z^{k+1}}{2 + 2\sum\limits_{k=1}^{n} a_k z^{k+1} + \frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha} \sum\limits_{k=n+1}^{\infty} a_k z^{k+1}}$$

and

$$|w(z)| \leq \frac{\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha} \sum\limits_{k=n+1}^{\infty} a_k}{2 - 2\sum\limits_{k=1}^{n} a_k - \frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha} \sum\limits_{k=n+1}^{\infty} a_k}.$$

Now $|w(z)| \leq 1$ if and only if

$$2\left(\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\right) \sum_{k=n+1}^{\infty} a_k \leq 2 - 2\sum_{k=1}^{n} a_k,$$

which is equivalent to

$$\sum_{k=1}^{n} a_k + \left(\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\right) \sum_{k=n+1}^{\infty} a_k \leq 1. \tag{2.3}$$

It suffices to show that the left hand side of (2.3) is bounded above by $\sum_{k=1}^{\infty} \left(\frac{\delta_k(\lambda,\beta,\alpha)}{1-\alpha}\right) a_k$, which is equivalent to

$$\sum_{k=1}^{n} \frac{\delta_k(\lambda,\beta,\alpha)-(1-\alpha)}{1-\alpha} a_k + \sum_{k=n+1}^{\infty} \frac{\delta_k(\lambda,\beta,\alpha)-\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha} a_k \geq 0.$$

To see that the function f given by (2.2) gives the sharp result, we observe for $z = re^{\pi i\ /(n+2)}$ that

$$\begin{aligned}\frac{f(z)}{f_n(z)} &= 1 + \frac{1-\alpha}{\delta_{n+1}(\lambda,\beta,\alpha)} z^{n+2} \\ &\to 1 - \frac{1-\alpha}{\delta_{n+1}(\lambda,\beta,\alpha)} \\ &= \frac{\delta_{n+1}(\lambda,\beta,\alpha)-(1-\alpha)}{\delta_{n+1}(\lambda,\beta,\alpha)} \text{ when } r \to 1^-.\end{aligned}$$

Therefore the proof of Theorem 1 is completed. □

Theorem 2. *If f of the form* (1.1) *satisfies condition* (1.14) , *then*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)-(1-\alpha)}{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)} \qquad (z \in U). \tag{2.4}$$

The result is sharp for every n, with extremal function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)} z^{n+1} \qquad (n \geq 1). \tag{2.5}$$

Proof. We may write

$$\begin{aligned}&\frac{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\left[\frac{f(z)}{f_n(z)} - \frac{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)-(1-\alpha)}{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)}\right] \\ &\qquad = \frac{1+\sum_{k=1}^{n} a_k z^{k+1} + \frac{(n+1)\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum_{k=n+1}^{\infty} a_k z^{k+1}}{1+\sum_{k=1}^{n} a_k z^{k+1}} \\ &\qquad = \frac{1+w(z)}{1-w(z)}\end{aligned}$$

where

$$w(z)=\frac{\frac{(n+1)\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_kz^{k+1}}{2+2\sum\limits_{k=1}^{n}a_kz^{k+1}+\frac{(n+1)\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_kz^{k+1}}.$$

Now

$$|w(z)|\leq\frac{\frac{(n+1)\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_k}{2-2\sum\limits_{k=1}^{n}a_k-\frac{(n+1)\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_k}.$$

if

$$\sum_{k=1}^{n}a_k+\left(\frac{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\right)\sum_{k=n+1}^{\infty}a_k\leq1. \tag{2.6}$$

The left hand side of (2.6) is bounded above by

$$\sum_{k=1}^{\infty}\left(\frac{k\delta_k(\lambda,\beta,\alpha)}{1-\alpha}\right)a_k,$$

if

$$\frac{1}{1-\alpha}\left\{\sum_{k=1}^{n}\left[k\delta_k(\lambda,\beta,\alpha)-(1-\alpha)\right]a_k\right.$$
$$\left.+\sum_{k=n+1}^{\infty}[k\delta_k(\lambda,\beta,\alpha)-(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)]a_k\right\}\geq0$$

and the proof of Theorem 2 is completed. □

We next determine bounds for $\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}$.

Theorem 3. (a) *If f of the form* (1.1) *satisfies condition* (1.13) , *then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}\geq\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{(1-\alpha)+\delta_{n+1}(\lambda,\beta,\alpha)}\quad(z\in U). \tag{2.7}$$

(b) *If f of the form* (1.1) *satisfies condition* (1.14) , *then*

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\}\geq\frac{(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)}{(1-\alpha)-(n+1)\,\delta_{n+1}(\lambda,\beta,\alpha)}\quad(z\in U). \tag{2.8}$$

Equalities hold in (a) *and* (b) *for the functions given by* (2.2) *and* (2.5) , *respectively.*

Proof. We prove (a). The proof of (b) is similar to (a) and will be omitted. We write

$$\frac{(1-\alpha)+\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\left[\frac{f_n(z)}{f(z)}-\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{(1-\alpha)+\delta_{n+1}(\lambda,\beta,\alpha)}\right]$$

$$=\frac{1+\sum\limits_{k=1}^{n}a_k z^{k+1}-\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_k z^{k+1}}{1+\sum\limits_{k=1}^{\infty}a_k z^{k+1}}$$

$$=\frac{1+w(z)}{1-w(z)},$$

where

$$|w(z)|\le\frac{\frac{(1-\alpha)+\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_k}{2-2\sum\limits_{k=1}^{n}a_k-\frac{\delta_{n+1}(\lambda,\beta,\alpha)-(1-\alpha)}{1-\alpha}\sum\limits_{k=n+1}^{\infty}a_k}\le 1.$$

This last inequality is equivalent to

$$\sum_{k=1}^{n}a_k+\left(\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{1-\alpha}\right)\sum_{k=n+1}^{\infty}a_k\le 1. \qquad (2.9)$$

Since the left hand side of (2.9) is bounded above by

$$\sum_{k=1}^{\infty}\left(\frac{\delta_k(\lambda,\beta,\alpha)}{1-\alpha}\right)a_k,$$

the proof is completed. □

We next turn to ratios involving derivatives. The proof of Theorem 4 below follows the pattern of those in Theorem 1 and (a) of Theorem 3 and so the details may be omitted.

Theorem 4. *If f of the form* (1.1) *satisfies condition* (1.13), *then*

$$(a)\,\mathrm{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\}\ge\frac{\delta_{n+1}(\lambda,\beta,\alpha)+(n+1)(1-\alpha)}{\delta_{n+1}(\lambda,\beta,\alpha)}\qquad(z\in U),$$

$$(b)\,\mathrm{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\}\ge\frac{\delta_{n+1}(\lambda,\beta,\alpha)}{\delta_{n+1}(\lambda,\beta,\alpha)-(n+1)(1-\alpha)}\qquad(z\in U;\alpha\ne 0,).$$

The extremal function for the case (a) *is given by* (2.2) *and the extremal function for the case* (b) *is given by* (2.2) *with* $\alpha\ne 0$.

Remark 1. Putting $\beta = 0$ and $\lambda = 0$ in Theorem 2, we obtain the following corollary:

Corollary 2. *If f of the form* (1.1) *satisfies condition* (1.13) (*with* $\beta = 0$ *and* $\lambda = 0$), *that is* $f \in \Sigma^*(\alpha)$, *then*

$$(a)\,\mathrm{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{2(n+1)-n\alpha}{n+1+\alpha} \qquad (z \in U),$$

$$(b)\,\mathrm{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{n+1+\alpha}{\alpha(n+2)} \qquad (z \in U; \alpha \neq 0).$$

The extremal function for the case (a) *is given by* (2.2) (*with* $\beta=0$ *and* $\lambda=0$) *and the extremal function for the case* (b) *is given by* (2.2). *(with* $\beta = 0, \lambda = 0$ *and* $\alpha \neq 0$).

Remark 2. We note that Corollary 2 corrects the result obtained by Cho and Owa [5, Theorem 4].

Theorem 5. *If f of the form* (1.1) *satisfies condition* (1.14), *then*

$$(a)\,\mathrm{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{\delta_{n+1}(\lambda,\beta,\alpha)-(1-\alpha)}{\delta_{n+1}(\lambda,\beta,\alpha)} \qquad (z \in U).$$

$$(b)\,\mathrm{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{\delta_{n+1}(\lambda,\beta,\alpha)}{(1-\alpha)+\delta_{n+1}(\lambda,\beta,\alpha)} \qquad (z \in U).$$

In both cases, the extremal function is given by (2.5).

Proof. It is well known that $f \in \Sigma_k^*(\alpha) \Longleftrightarrow -zf' \in \Sigma^*(\alpha)$. In particular, f satisfies condition (1.14) if and only if $-zf'$ satisfies condition (1.13). Thus, (a) is an immediate consequence of Theorem 1 and (b) follows directly from (a) of Theorem 3. □

Remark 3. Puting $\beta = 0$ and $\lambda = 0$, in the above results, we get the results obtained by Cho and Owa [5].

Remark 4. Puting $\beta = 0$ and $\lambda = 0$, in the above results, we get the results obtained by Aouf and Silverman [2 with $p = 1$].

Remark 5. Puting $\beta = 0$ and $\lambda = 0$, in the above results, we get the results obtained by Aouf and Mostafa [1 with $p = B = 1$ and $A = 2\alpha - 1, 0 \leq \alpha < 1$].

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