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BEST PROXIMITY POINT THEOREMS FOR MULTI–VALUED MAPPINGS IN COMPLETE METRIC SPACES

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Abstract. In this paper the concept of K–cyclic and C–cyclic contraction single-valued maps are extended to multi–valued maps with MT–functions in the frameworks of complete spaces. We show the existence of a best proximity point for such mappings in the setup of complete metric spaces. Our result extends and improves some best proximity point theorems in the literature. An example is given to support the functionality of the result.

1. Introduction:

In 1969, Nadler [12] studied fixed points of contraction multi–valued mappings. Let (X, d) be a metric space and let $P(X)$ denote the family of all nonempty subsets of X. A mapping $T : X \to P(X)$ is called a multi–valued function. If there exists a constant $k \in (0, 1)$ such that $H(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$, then we say that T is a multi–valued contraction mapping. Note that $H(A, B) = \max\{h(A, B), h(B, A)\}$, where $h(A, B) = \sup\{d(a, B) : a \in A\}$.

It is also shown in [20] that a mapping T of X into the family K(X) of all nonempty compact subsets of X has a fixed point if it satisfies $H(Tx, Ty) \leq k(d(x, y))d(x, y)$ for all $x, y \in X$ with $x \neq y$, where k is a function of $(0, \infty)$ to [0, 1) with $\limsup_{r \to t+} k(r) < 1$ for every $t \in (0, \infty)$.

Let A and B be nonempty subsets of a metric space (X, d). Consider a mapping $T : A \cup B \to A \cup B$, T is called a cyclic map if $T(A) \subset B$ and $T(B) \subset A$, $x \in A$ is called a best proximity point of T in A if $d(x, Tx) = d(A, B)$ is satisfied, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

In a recent paper, Erdal Karapinar and Inci Erhan [4] studied some proximity points by using different types cyclic contraction. Furthermore

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[15, 5, 7] examine several variants of contractions for the existence of a best proximity point. Later, Karapinar, E. [6] have derived a best proximity point theorem for Cyclic Mappings. In 2005, Elderd et al. [2] proved the existence of a best proximity point for relatively nonexpansive mappings using the notion of proximal normal structure. In 2006, Eldred and Veeramani [3] proved the following existence theorem. Recently, best proximity point theorems for various types of contractions have been obtained in [1, 8, 9, 10, 14, 16].

Definition 1.1. Let A and B be nonempty subsets of a metric space (X, d). The cyclic (on A and B) multivalued mapping T is said to be cyclic contraction if there exists a constant $k \in (0, 1)$ such that $H(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B)$ for all $x \in A$ and $y \in B$.

Theorem 1.2. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $f : A \cup B \to A \cup B$ is a cyclic contraction,that is, $f(A) \subset B$ and $f(B) \subset A$, and there exists $k \in (0, 1)$ such that $d(fx, fy) \leq kd(x, y) + (1 - k)d(A, B)$ for every $x \in A$, $y \in B$.*

Then there exists a unique best proximity point in A. Further, for each $x \in A$, $\{f^{2n}x\}$ converges to the best proximity point.

Remark. The following properties of the functional H are well-known:

(1) H is a metric on $CB(X)$, where $CB(X)$ is the family of all nonempty bounded closed subsets of X.
(2) $f(X, d)$ is a metric space, $A, B \in P(X)$ and $q > 1$ be given, then for every $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.

An element $x \in X$ is a fixed point of a multi-valued T if $x \in Tx$. We denote by F_T the set of all fixed points of T, i.e., $F_T = \{x \in X : x \in Tx\}$. Theorem 1.3 is a result of [12].

Theorem 1.3. *Let (X, d) be a metric space and $T : X \to CB(X)$ be a multi-valued contraction. Then T has at least one fixed point.*

2. Preliminaries

In this section, we first define in what follows the MT–function which will be used throughout the paper to get new best proximity point theorems.

Definition 2.1. (See [23]) A function $\psi : [0, \infty) \to [0, 1)$ is said to be an MT–function if it satisfies Mizoguchi–Takahashi's condition ($\lim_{s \to t^+} \psi(s) < 1$ for all $t \in [0, \infty)$).

It is obvious that if $\psi : [0, \infty) \to [0, 1)$ is a non-decreasing function or a non-increasing function, then ψ is an MT–function. So, the set of MT–functions is a rich class, but it is worth to mention that there exist functions which are not MT–function.

Example 2.2. (See [25]) Let $\psi : [0, \infty) \to [0, 1)$ be defined by

$$\psi(t) = \begin{cases} \frac{sint}{t} & t \in (0, \frac{\pi}{2}], \\ 0 & \text{otherwise.} \end{cases}$$

Since $\lim_{t \to 0^+} \psi(s) = 1$, ψ is not an MT–function.

Very recently, Du [25] first proved some characterizations of MT-functions.

Theorem 2.3. *Let $\psi : [0, \infty) \to [0, 1)$ be a function. Then the following statements are equivalent.*
(a) *psi is an MT-function,*
(b) *For each $t \in [0, \infty)$, there exists $r_t^{(1)}$ and $\varepsilon_t^{(1)} > 0$ such that $\psi(s) < r_t^{(1)}$ for all $s \in (t, t + \varepsilon_t^{(1)})$,*
(c) *For each $t \in [0, \infty)$, there exists $r_t^{(2)}$ and $\varepsilon_t^{(2)} > 0$ such that $\psi(s) < r_t^{(2)}$ for all $s \in (t, t + \varepsilon_t^{(2)})$,*
(d) *For each $t \in [0, \infty)$, there exists $r_t^{(3)}$ and $\varepsilon_t^{(3)} > 0$ such that $\psi(s) < r_t^{(3)}$ for all $s \in (t, t + \varepsilon_t^{(3)})$,*
(e) *For each $t \in [0, \infty)$, there exists $r_t^{(4)}$ and $\varepsilon_t^{(4)} > 0$ such that $\psi(s) < r_t^{(4)}$ for all $s \in (t, t + \varepsilon_t^{(4)})$,*
(f) *For any non-increasing sequence $\{x_n\}_{n \in N}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in N} \psi(x_n) < L$,*
(g) *ψ is a function of contractive factor* [24]*; that is, for any strictly decreasing sequence $\{x_n\}_{n \in N}$ in $[0, \infty)$, we have $0 \leq \sup_{n \in N} \psi(x_n) < 1$.*

Definition 2.4. [26] Let A and B be nonempty subsets of a metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies:
(a) $T(A) \subset B$ and $T(B) \subset A$;
(b) there exists an MT-function $\psi : [0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leq \psi(d(x, y))d(x, y) + (1 - \psi(d(x, y)))d(A, B), \tag{2.1}$$

for all $x \in A$ and $y \in B$. Then T is called an MT–cyclic contraction with respect to ψ on $A \cap B$.

Motivated by the definition of K–cyclic, C–cyclic and MT–functions, we introduce the following concept of generalized cyclic contractions.

Definition 2.5. Let A and B be nonempty subsets of a metric space (X, d). If a mapping $T : A \cup B \to Cl(A) \cup Cl(B)$ is a generalized cyclic MT–KC contractive map if it is satisfies the following conditions:

(a) $T(A) \subset Cl(B)$ and $T(B) \subset Cl(A)$;
(b) There exists an MT–function $\psi : [0, \infty) \to [0, 1)$ such that

$$H(Tx,Ty) \leq \frac{1}{4}\psi(d(x,y))[d(x,Tx)+d(x,Ty)+d(y,Tx)+d(y,Ty) \\ +(1-\psi(d(x,y)))d(A,B), \tag{2.2}$$

for all $x \in A$ and $y \in B$.

3. Main results

In this section, we shall prove the existence and uniqueness convergence theorems for multi-valued mappings to a best proximity points of the non-self-mappings A and B.

Theorem 3.1. *Let (A,B) be a pair of two nonempty closed subsets of a complete metric space (X,d). Suppose that a mapping $T : A \cup B \to Cl(A) \cup Cl(B)$ be a generalized cyclic MT–KC contractive map, then there exits an orbit $\{x_n\}$ of T at x_0 such that*

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = d(A,B).$$

Proof. Let $x_0 \in A$ and $x_1 \in Tx_0 \subset B$. There exists $x_2 \in Tx_1 \subset A$ such that

$$\begin{aligned} d(x_1,x_2) &\leq d(x_1,Tx_1) \leq h(Tx_0,Tx_1) \leq H(Tx_0,Tx_1) \\ &\leq \frac{1}{4}\psi(d(x_0,x_1))[d(x_0,Tx_0)+d(x_0,Tx_1)+d(x_1,Tx_0)+d(x_1,Tx_1)] \\ &\quad +(1-\psi(d(x_0,x_1)))d(A,B), \\ &\leq \frac{1}{4}\psi(d(x_0,x_1))[d(x_0,x_1)+d(x_0,x_2)+d(x_1,x_1)+d(x_1,x_2)] \\ &\quad +(1-\psi(d(x_0,x_1)))d(A,B), \\ &\leq \frac{1}{4}\psi(d(x_0,x_1))[2d(x_0,x_1)+2d(x_1,x_2)]+(1-\psi(d(x_0,x_1)))d(A,B) \end{aligned}$$

which implies that

$$\begin{aligned} [1-\frac{1}{2}\psi(d(x_0,x_1))]d(x_1,x_2) &\leq \frac{1}{2}\psi(d(x_0,x_1))d(x_0,x_1) \\ &\quad +(1-\psi(d(x_0,x_1))(d(x_0,x_1)))d(A,B) \end{aligned}$$

$$d(x_1,x_2) \leq \frac{\psi(d(x_0,x_1))}{2-\psi(d(x_0,x_1))}d(x_0,x_1)+[1-\frac{\psi(d(x_0,x_1))}{2-\psi(d(x_0,x_1))}d(A,B)]. \tag{3.1}$$

From (3.1), finally we obtain

$$d(x_1,x_2)-d(A,B) \leq \frac{\psi(d(x_0,x_1))}{2-\psi(d(x_0,x_1))}[d(x_0,x_1)-d(A,B)].$$

Similarly, there exists $x_3 \in Tx_2 \subseteq B$ such that

$$
\begin{aligned}
d(x_2,x_3) &= H(Tx_1,Tx_2)+h = H(Tx_1,Tx_2)\\
&\leq \frac{1}{4}\psi(d(x_1,x_2))[d(x_1,Tx_1)+d(x_1,Tx_2)+d(x_2,Tx_1)+d(x_2,Tx_2)]\\
&\quad +(1-\psi(d(x_1,x_2)))d(A,B),\\
&\leq \frac{1}{4}\psi(d(x_1,x_2))[d(x_1,x_2)+d(x_1,x_3)+d(x_2,x_2)+d(x_2,x_3)]\\
&\quad +(1-\psi(d(x_1,x_2)))d(A,B),\\
&\leq \frac{1}{4}\psi(d(x_1,x_2))[2d(x_1,x_2)+2d(x_2,x_3)]+(1-\psi(d(x_1,x_2)))d(A,B)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&[1-\frac{1}{2}\psi(d(x_1,x_2))]d(x_2,x_3)\\
&\qquad\qquad \leq \frac{1}{2}\psi(d(x_1,x_2))d(x_1,x_2)+(1-\psi(d(x_1,x_2)))d(A,B)
\end{aligned}
$$

$$
d(x_2,x_3) \leq \frac{\psi(d(x_1,x_2))}{2-\psi(d(x_1,x_2))}d(x_1,x_2)+[1-\frac{\psi(d(x_1,x_2))}{2-\psi(d(x_1,x_2))}]d(A,B),
$$

or

$$
d(x_2,x_3)-d(A,B) \leq \frac{\psi(d(x_1,x_2))}{2-\psi(d(x_1,x_2))}[d(x_1,x_2)-d(A,B)].
$$

By induction, we get

$$
d(x_n,x_{n+1})-d(A,B) \leq \frac{\psi(d(x_{n-1},x_n))}{2-\psi(d(x_{n-1},x_n))}[d(x_{n-1},x_n)-d(A,B)]. \quad (3.2)
$$

Since $\psi(t)<1$ for all $t\in[0,\infty)$, where $\frac{\psi(t)}{1-\psi(t)}<1$ for all $t\in[0,\infty)$. By (3.2), we obtain

$$
d(x_n,x_{n+1})-d(A,B) \leq d(x_{n-1},x_n)-d(A,B),
$$

which implies that $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$ for all $n\in N$. Thus the sequence $\{d(x_n,x_{n+1})\}$ is a strictly decreasing in $[0,\infty)$. Since ψ is an MT–function by applying (g) by Theorem 2.3, we get

$$
0 \leq \sup_{n\in N}\psi(d(x_n,x_{n+1})) < 1.
$$

Suppose $k=\sup_{n\in N}\psi(d(x_n,x_{n+1}))$. Then $0\leq k<1$, since $\psi(d(x_n,x_{n+1}))$ $\leq k$, we get

$$
1-\psi(d(x_n,x_{n+1})) \geq 1-k.
$$

Then

$$
\frac{\psi(d(x_n,x_{n+1}))}{1-\psi(d(x_n,x_{n+1}))} \leq \frac{k}{1-k}
$$

for all $n \in N$. Hence

$$0 \leq \sup_{n \in N} \frac{\psi(d(x_n, x_{n+1}))}{1-\psi(d(x_n, x_{n+1}))} \leq \frac{k}{1-k} < 1.$$

Call

$$\alpha = \sup_{n \in N} \frac{\psi(d(x_n, x_{n+1}))}{1-\psi(d(x_n, x_{n+1}))}.$$

Then $\alpha \in [0, 1)$. From (3.2), we have

$$\begin{aligned}
d(x_n, x_{n+1}) - d(A, B) &\leq \frac{\psi(d(x_{n-1}, x_n)))}{1-\psi(d(x_{n-1}, x_n))}[d(x_{n-1}, x_n) - d(A, B)] \\
&\leq \alpha[d(x_{n-1}, x_n) - d(A, B)] \\
&\leq \alpha^2[d(x_{n-2}, x_{n-1}) - d(A, B)] \\
&\dots \\
&\leq \alpha^n[d(x_0, x_1) - d(A, B)]. \qquad (3.3)
\end{aligned}$$

Since $\alpha \in [0, 1)$ and taking $n \to \infty$ in (3.3), we have $\lim_{n\to\infty} \alpha^n = 0$, So that, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B).$$

This completes the proof. □

Theorem 3.2. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space (X, d). Suppose that a mapping $T : A \cup B \to Cl(A \cup B)$ be a generalized cyclic MT–KC contractive map. Assume that a sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then the sequence $\{x_n\}$ is bounded.*

Proof. Suppose $x_{2n} \in Tx_{2n-1} \subset B$. Then, there exists $x_{2n+1} \in Tx_{2n} \subset A$ such that

$$\begin{aligned}
&d(x_{2n}, Tx_0) = H(Tx_{2n-1}, Tx_0) \\
&\leq \frac{1}{4}\psi(d(x_{2n-1}, x_0))[d(x_{2n-1}, Tx_{2n-1}) + d(x_{2n-1}, Tx_0) \\
&\quad + d(x_0, Tx_{2n-1}) + d(x_0, Tx_0) + (1-\psi(d(x_{2n-1}, x_0)))d(A, B) \\
&\leq \frac{1}{4}\psi(d(x_{2n-1}, x_0))[d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, Tx_0) \\
&\quad + d(x_0, x_{2n}) + d(x_0, Tx_0)] + (1-\psi(d(x_{2n-1}, x_0)))d(A, B) \\
&\leq \frac{1}{4}\psi(d(x_{2n-1}, x_0))[d(x_{2n-1}, x_{2n}) + d(x_{2n-1}, x_{2n}) + d(x_{2n}, Tx_0) + d(x_0, Tx_0) \\
&\quad + d(x_{2n}, Tx_0) + d(x_0, Tx_0)] + (1-\psi(d(x_{2n-1}, x_0)))d(A, B) \\
&\leq \frac{1}{4}\psi(d(x_{2n-1}, x_0))[2d(x_{2n-1}, x_{2n}) + 2d(x_{2n}, Tx_0) + 2d(x_0, Tx_0)]
\end{aligned}$$

$$
\begin{aligned}
&+(1-\psi(d(x_{2n-1},x_0)))d(A,B)\\
\le\ & \frac{1}{2}\psi(d(x_{2n-1},x_0))[d(x_{2n-1},x_{2n})+d(x_{2n},Tx_0)+d(x_0,Tx_0)]\\
&+(1-\psi(d(x_{2n-1},x_0)))d(A,B)\\
\le\ & \frac{\psi(d(x_{2n-1},x_0))}{2-\psi(d(x_{2n-1},x_0))}[d(x_{2n-1},x_{2n})+d(x_0,Tx_0)]\\
&+(1-\frac{\psi(d(x_{2n-1},x_0))}{2-\psi(d(x_{2n-1},x_0))})d(A,B)\\
\le\ & \frac{\psi(d(x_{2n-1},x_0))}{2-\psi(d(x_{2n-1},x_0))}[d(A,B)+d(x_0,Tx_0)]\\
&+(1-\frac{\psi(d(x_{2n-1},x_0))}{2-\psi(d(x_{2n-1},x_0))})d(A,B)\\
\le\ & \frac{\psi(d(x_{2n-1},x_0))}{2-\psi(d(x_{2n-1},x_0))}d(x_0,Tx_0)+d(A,B).
\end{aligned}
$$

Therefore, the sequence $\{x_{2n}\}$ is bounded. Similarly, it can be shown that $\{x_{2n+1}\}$ is also bounded. This complete the proof. □

Theorem 3.3. *Let (A,B) be a pair of two nonempty closed subsets of a complete metric space (X,d). Let a mapping $T : Cl(A)\cup Cl(B)\to Cl(A\cup B)$ be a generalized cyclic MT–KC contractive map. Suppose that a sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then x is a best proximity point of T.*

Proof. Suppose the sequence $\{x_{2n_k}\}$ is a subsequence of $\{x_{2n}\}$ converging to some element x in A. Furthermore,

$$
\begin{aligned}
d(A,B) &\le d(x,x_{2n_k-1})\\
&\le d(x,x_{2n_k})+d(x_{2n_k},x_{2n_k-1})\\
&\le d(x,x_{2n_k})+d(A,B).
\end{aligned}
$$

Therefore $d(x,x_{2n_k-1})\to d(A,B)$. In light of the fact that the sequence $\{x_{2n_k}\}$ has a subsequence of $\{x_{2n}\}$ converging to some element x in A. So, because of Theorem 3.1 e have $d(x_{2n_k},x_{2n_k-1})\to d(A,B)$. Since T is a generalized cyclic MT–KC contractive map cyclic contraction, it follows that

$$
\begin{aligned}
&d(A,B)\le d(x_{2n_k},Tx)=H(Tx_{2n_k-1},Tx)\\
&\quad\le \frac{1}{4}\psi(d(x_{2n_k-1},x))[d(x_{2n_k-1},Tx_{2n_k-1})+d(x_{2n_k-1},Tx)\\
&\qquad+d(x,Tx_{2n_k-1})+d(x,Tx)]+(1-\psi(d(x_{2n_k-1},x)))d(A,B)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{4}\psi(d(x_{2n_k-1},x))[d(x_{2n_k-1},x_{2n_k})+d(x_{2n_k-1},Tx)\\
&\quad +d(x,x_{2n_k})+d(x,Tx)]+(1-\psi(d(x_{2n_k-1},x)))d(A,B)\\
&\leq \frac{1}{4}\psi(d(x_{2n_k-1},x))[d(x_{2n_k-1},x_{2n_k})+d(x_{2n_k-1},x_{2n_k})+d(x_{2n_k},Tx)\\
&\quad +d(x,Tx)+d(x_{2n_k},Tx)+d(x,Tx)]+(1-\psi(d(x_{2n_k-1},x)))d(A,B)\\
&\leq \frac{1}{4}\psi(d(x_{2n_k-1},x))[2d(x_{2n_k-1},x_{2n_k})+2d(x_{2n_k-1},Tx)+2d(x,Tx)]\\
&\quad +(1-\psi(d(x_{2n_k-1},x)))d(A,B)\\
&\leq \frac{1}{2}\psi(d(x_{2n_k-1},x))[d(x_{2n_k-1},x_{2n_k})+d(x_{2n_k},Tx)+d(x,Tx)]\\
&\quad +(1-\psi(d(x_{2n_k-1},x)))d(A,B)\\
&\leq \frac{\psi(d(x_{2n_k-1},x))}{2-\psi(d(x_{2n_k-1},x))}[d(x_{2n_k-1},x_{2n_k})+d(x,Tx)]\\
&\quad +(1-\frac{\psi(d(x_{2n_k-1},x))}{2-\psi(d(x_{2n_k-1},x))})d(A,B)\\
&\leq \frac{\psi(d(x_{2n_k-1},x))}{2-\psi(d(x_{2n_k-1},x))}d(x,Tx)+d(A,B).
\end{aligned}
$$

Since ψ is an MT–function by applying (g) by Theorem 2.3, we get

$$0 \leq \sup_{n\in N}\psi(d(x_n,x_{2n_k-1}))<1.$$

and taking $k\to\infty$ in the inequality above, then we obtain

$$d(x,Tx)=d(A,B),$$

that is x is a best proximity point of T. This completes the proof.

□

The following examples illustrate our main results.

Example 3.4. Consider the usual metric space $d(x,y)=|x-y|$, for all $x,y\in X$. Let $X=R$. Suppose $A=[0,1]$ and $B=[2,3]$, then $d(A,B)=1$. Define a mapping $T: A\cup B\to Cl(A)\cup Cl(B)$ as follows:

$$Tx=[1,\frac{5-x}{2}] \text{ for all } x\in A \text{ s.t. } Tx=\{[1,a]:a\in[2,2.5]\}$$

and

$$Ty=[0,\frac{4-y}{2}] \text{ for all } y\in B \text{ s.t. } Ty=\{[0,b]:b\in[0.5,1]\}.$$

It is clear that $T(A)\subset Cl(B)$ and $T(B)\subset Cl(A)$. Since,

$$H(Tx,Ty)=\max[\{sup_{x\in A}d(x,Ty)\},\{sup_{y\in B}d(y,Tx)\}].$$

Let us observe that, for $x = 1$ and $y = 2$, then, we have $Tx = [1, 2]$ and $Ty = [0, 1]$. Then, we obtain,

$$\begin{aligned} H(Tx, Ty) &= \max\{[sup(d(1, 0), d(1, 1))], [sup(d(2, 1), d(2, 2))]\} \\ &= \max\{sup[1, 0], sup[1, 0]\} \\ &= \max\{0, 0\} = 0. \end{aligned}$$

Similarly, we find that

$$\begin{aligned} d(x, Tx) &= \sup[d(x, a) : a \in Tx] = \sup[0, 1] = 1, \\ d(y, Ty) &= \sup[d(y, b) : b \in Ty] = \sup[2, 1] = 2, \\ d(x, Ty) &= \sup[d(x, b) : b \in Ty] = \sup[1, 0] = 1, \\ d(y, Tx) &= \sup[d(y, a) : a \in Tx] = \sup[1, 0] = 1. \end{aligned}$$

Let $\psi : [0, \infty) \to [0, 1)$ and the constant function given by $\psi(t) = \frac{t}{2(1+t)}$. Therefore, it is easy to check that

$$\begin{aligned} H(Tx, Ty) \leq & \frac{1}{4}\psi(d(x, y))[d(x, Tx) + d(x, Ty) + d(y, Tx) + d(y, Ty)] \\ & + (1 - \psi(d(x, y)))d(A, B), \\ \leq & \frac{1}{4}.\frac{1}{4}[1 + 2 + 1 + 1] + (1 - \frac{1}{4})1 = \frac{11}{8}. \end{aligned}$$

It is clear that $0 \leq \frac{11}{8}$. Therefore, condition (2.2) is satisfied. So that a mapping T is a generalized cyclic MT–KC contractive map.

4. Application

In this section, we present some applications of the main results.

Corollary 4.1. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space (X, d). Suppose that a mapping $T : A \cup B \to Cl(A) \cup Cl(B)$ satisfying the following condition:*

(a) *$T(A) \subset Cl(B)$ and $T(B) \subset Cl(A)$;*

(b) *there exists an MT–function $\psi : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned} \int_0^{H(Tx,Ty)} \mu(t)dt \leq & \frac{1}{4}(1 - \psi(d(x, y))) \int_0^{[d(x,Tx)+d(x,Ty)+d(y,Tx)+d(y,Ty)]} \mu(t)dt \\ & + (1 - \psi(d(x, y))) \int_0^{d(A,B))} \mu(t)dt, \end{aligned}$$

for each $x \in A$ and $y \in B$, where $\mu : R^+ \to R^+$ is a Lesbesgue–integrable mapping which is summable on each compact subset of R^+, non negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \mu(t)dt < \epsilon$. Then, there exists $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$ for some sequence $\{x_n\}$.

Corollary 4.2. *Let (A, B) be a pair of two nonempty closed subsets of a complete metric space (X, d). Suppose that a mapping $T : A \cup B \to Cl(A) \cup Cl(B)$ satisfying the following conditions:*

(a) *$T(A) \subset Cl(B)$ and $T(B) \subset Cl(A)$;*

(b) *There exists an MT–function $\psi : [0, \infty) \to [0, 1)$ such that*

$$\int_0^{H(Tx,Ty)} \mu(t)dt \leq \frac{1}{4}(1 - \psi(d(x, y))) \int_0^{[d(x,Tx)+d(x,Ty)+d(y,Tx)+d(y,Ty)]} \mu(t)dt + (1 - \psi(d(x, y))) \int_0^{d(A,B))} \mu(t)dt,$$

for each $x \in A$ and $y \in B$, where $\mu : R^+ \to R^+$ is a Lesbesgue–integrable mapping which is summable on each compact subset of R^+, non negative, and such that for each $\epsilon > 0$, $\int_0^\epsilon \mu(t)dt < 0$. Suppose that a sequence $\{x_{2n}\}$ has a subsequence converging to some element x in A. Then, there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.

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