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THE GEOMETRY OF GOLDEN CONJUGATE CONNECTIONS

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Abstract. Properties of Golden conjugate connections are stated by pointing out their relation to product conjugate connections. We define the analogues in Golden geometry of the structural and the virtual tensors from the almost product geometry and express the Golden conjugate connections in terms of these tensors.

Introduction

Besides the almost complex, almost tangent and almost product structures on a differentiable manifold M there naturally arise some other polynomial structures as C^∞-tensor fields J of type $(1,1)$ which are roots of the algebraic equation

$$q(J) := J^n + a_n J^{n-1} + \cdots + a_2 J + a_1 I_{\mathfrak{X}(M)} = 0,$$

where $I_{\mathfrak{X}(M)}$ is the identity map on the Lie algebra of vector fields on M.

In particular, if the structure polynomial is $q(J) = J^2 - J - I_{\mathfrak{X}(M)}$, its solution J will be called *Golden structure* [7]. The name is motivated by the fact that the Golden ratio is precisely the positive root of the quadratic equation $x^2 - x - 1 = 0$ being equal to 1.6180339887.... This equation is usually called the Fibonacci equation, being the characteristic equation associated to the Fibonacci sequence $f_{n+1} = f_n + f_{n-1}$, for every integer $n \geq 1$, with $f_0 = f_1 = 1$.

It was mentioned by M. Crasmareanu and C.-E. Hreţcanu [5], [8] that the Golden structures appear in pairs, namely, if J is a Golden structure on M, then $I_{\mathfrak{X}(M)} - J$ is also a Golden structure. Based on these considerations, it is natural to look for a connection between Golden structures and almost

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product structures. In [5] the same authors proved that any Golden structure induces an almost product structure and any almost product structure determines two Golden structures.

In this setting, we shall study the properties of the conjugate connections (by a Golden structure), express their virtual and structural tensor fields and see their behavior on invariant distributions. Finally, we shall analyze the impact of the duality between the Golden and almost product structures on Golden and product conjugate connections.

Let M be a smooth, n-dimensional manifold and denote by: $C^{\infty}(M)$–the algebra of smooth real functions on M, $\mathfrak{X}(M)$–the Lie algebra of vector fields on M, $T^r_s(M)$–the $C^{\infty}(M)$-module of tensor fields of (r,s)-type on M. Usually X, Y, $Z, \dots$ will be vector fields on M and if $T \to M$ is a vector bundle over M then $\Gamma(T)$ denotes the C^{∞}-module of sections of T, e.g. $\Gamma(TM) = \mathfrak{X}(M)$.

Consider $\mathcal{C}(M)$ the set of all linear connections on M. Since the difference of two linear connections is a tensor field of $(1,2)$-type, it follows that $\mathcal{C}(M)$ is a $C^{\infty}(M)$-affine module associated with the $C^{\infty}(M)$-linear module $T^1_2(M)$.

Recall the concept of Golden (Riemannian) geometry:

Definition 0.1. ([7]) $J \in T^1_1(M)$ is called *Golden structure* on M if it satisfies the equation:

$$J^2 - J - I_{\mathfrak{X}(M)} = 0, \tag{0.1}$$

where $I_{\mathfrak{X}(M)}$ is the identity operator on $\mathfrak{X}(M)$. The pair (M, J) is a *Golden manifold*. Moreover, if a Riemannian metric g on M is compatible with J, that is $g(JX, Y) = g(X, JY)$, for any X, $Y \in \mathfrak{X}(M)$, we call the pair (g, J) a *Golden Riemannian structure* and (M, g, J) a *Golden Riemannian manifold*.

It was shown [5] that the powers of J satisfy:

$$J^n = f_n J + f_{n-1} I_{\mathfrak{X}(M)}, \tag{0.2}$$

where $\{f_n\}_{n \in \mathbb{N}^*}$ is the Fibonacci sequence: $f_0 = f_1 = 1$, $f_2 = 2$, $f_3 = 3$ and so on.

Considering the inheritance of this kind of structure on submanifolds, Crasmareanu and Hreţcanu proved in [8] that a Golden structure on a Riemannian Golden manifold M induced a Golden structure on every invariant submanifold of M and illustrated this on a product of spheres in a Euclidian space.

Fix now J a Golden structure on M. Then the associated linear connections are:

Definition 0.2. $\nabla \in \mathcal{C}(M)$ is a *J–connection* if J is covariant constant with respect to ∇, namely $\nabla J = 0$. Let $\mathcal{C}_J(M)$ be the set of these connections.

The concept of integrability is defined in the classical manner:

Definition 0.3. The Golden structure J is *integrable* if its Nijenhuis tensor field vanishes, namely $N_J(X,Y) := [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X,Y] = 0$.

Necessary and sufficient conditions for its integrability were given by A. Gezer, N. Cengiz and A. Salimov in [6]. They also proved that if the Levi–Civita connection belongs to $\mathcal{C}_J(M)$ then the Golden structure J is integrable.

In order to find a measure of how far away a linear connection is from being in $\mathcal{C}_J(M)$ we introduce in the next Section the notion of *Golden conjugate connection*. The present paper is concerned with the study of these connections and is the third in a series containing [2] and [3]. In fact, it is a natural continuation of [3] since, as we recall in Section 3, there is a strong relationship between Golden structures and almost product structures on M. An important tool in our work is provided by the pair (structural tensor, virtual tensor) defined for the almost product geometry in [3] and is considered here in the last part of Section 1. From an applied point of view we treat, in Section 2, the J-invariant distributions.

1. Properties of Golden conjugate connection

In what follows, for simplification we will denote by a superscript J the Golden conjugate connection of ∇

$$\nabla^{(J)} := \nabla + J \circ \nabla J \tag{1.1}$$

and hence if $\nabla \in \mathcal{C}_J(M)$ then $\nabla^{(J)} \in \mathcal{C}_J(M)$. A detailed expression of this connection is:

$$\nabla^{(J)}_X Y = \nabla_X Y + J(\nabla_X JY - J(\nabla_X Y)) = J(\nabla_X JY - \nabla_X Y). \tag{1.2}$$

The first properties of the Golden conjugate connection are stated in the next result:

Proposition 1.1. *Let J be a Golden structure, ∇ a linear connection and $\nabla^{(J)}$ the Golden conjugate connection of ∇. Then:*

1. $\nabla^{n(J)}_X Y = (-1)^{n+1} J^n (f_n \cdot \nabla_X JY - f_{n+1} \cdot \nabla_X Y)$*, for* $n \in \mathbb{N}^*$.
2. $\nabla^{n(J)} J = (-1)^n J^{2n} \circ \nabla J$ *for* $n \in \mathbb{N}^*$.
3. $T_{\nabla^{(J)}} = T_\nabla + J(d^\nabla J)$*, where* d^∇ *is the exterior covariant derivative induced by* ∇*, namely* $(d^\nabla J)(X,Y) := (\nabla_X J)Y - (\nabla_Y J)X$.
4. $R_{\nabla^{(J)}}(X,Y,Z) = J(R_\nabla(X,Y,JZ) - R_\nabla(X,Y,Z))$*; it results that if* ∇ *is flat then* $\nabla^{(J)}$ *is so.*
5. *Assume that* (M, g, J) *is a Golden Riemannian manifold. Then:*

$$(\nabla^{(J)}_X g)(Y,Z) = (\nabla_X g)(JY, JZ) - (\nabla_X g)(Y, JZ) - g(Y, (\nabla_X J)Z). \tag{1.3}$$

This implies that if ∇ is a g-metric connection belonging to $\mathcal{C}_J(M)$ then $\nabla^{(J)}$ is also a g-metric connection.

Proof. 1. Observe that:

$$\nabla_X^{2(J)}Y = -J^2(\nabla_X JY - 2\nabla_X Y), \quad \nabla_X^{3(J)}Y = J^3(2\nabla_X JY - 3\nabla_X Y)$$

and the result follows by induction.
2. We have:

$$\begin{aligned}(\nabla_X^{(J)}J)Y &:= \nabla_X^{(J)}JY - J(\nabla_X^{(J)}Y)\\ &= J(\nabla_X J^2Y - \nabla_X JY) - J^2(\nabla_X JY - \nabla_X Y)\\ &= -\nabla_X JY - J(\nabla_X JY) + 2J(\nabla_X Y) + \nabla_X Y\\ &= -J^2(\nabla_X JY) + J^3(\nabla_X Y) = -J^2((\nabla_X J)Y).\end{aligned}$$

Computing $\nabla^{2(J)}J$ and $\nabla^{3(J)}J$ we obtain:

$$\nabla^{2(J)}J = -J^2 \circ \nabla^{(J)}J = J^4 \circ \nabla J, \quad \nabla^{3(J)}J = J^4 \circ \nabla^{(J)}J = -J^6 \circ \nabla J$$

and by induction we get the conclusion.
3.

$$\begin{aligned}&T_{\nabla^{(J)}}(X,Y) := \nabla_X^{(J)}Y - \nabla_Y^{(J)}X - [X,Y]\\ &:= J(\nabla_X JY) - J(\nabla_X Y) - J(\nabla_Y JX) + J(\nabla_Y X) - J^2[X,Y] + J[X,Y]\\ &= J(\nabla_X JY - \nabla_Y JX - J[X,Y]) - J(T_\nabla(X,Y))\\ &= J((d^\nabla J)(X,Y)) + J^2(T_\nabla(X,Y)) - J(T_\nabla(X,Y))\\ &= J((d^\nabla J)(X,Y)) + T_\nabla(X,Y).\end{aligned}$$

4.

$$\begin{aligned}R_{\nabla^{(J)}}(X,Y,Z) :&= \nabla_X^{(J)}\nabla_Y^{(J)}Z - \nabla_Y^{(J)}\nabla_X^{(J)}Z - \nabla_{[X,Y]}^{(J)}Z\\ &= J(\nabla_X J^2(\nabla_Y JZ)) - J(\nabla_X J(\nabla_Y JZ)) - J(\nabla_X J^2(\nabla_Y Z))\\ &+ J(\nabla_X J(\nabla_Y Z)) - J(\nabla_Y J^2(\nabla_X JZ)) + J(\nabla_Y J(\nabla_X JZ))\\ &+ J(\nabla_Y J^2(\nabla_X Z)) - J(\nabla_Y J(\nabla_X Z)) - J(\nabla_{[X,Y]}JZ) + J(\nabla_{[X,Y]}Z)\\ &= J(\nabla_X\nabla_Y JZ - \nabla_Y\nabla_X JZ - \nabla_{[X,Y]}JZ)\\ &\qquad - J(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z)\end{aligned}$$

which yields the claimed formula.
5. Using $g(JX,Y) = g(X,JY)$ and $g(JX,JY) = g(X,Y) + g(JX,Y)$ we

obtain:

$$\begin{aligned}(\nabla^{(J)}_X g)(Y,Z) &:= X(g(Y,Z)) - g(\nabla^{(J)}_X Y, Z) - g(Y, \nabla^{(J)}_X Z)\\ &= X(g(Y,Z)) - g(J(\nabla_X JY), Z) + g(J(\nabla_X Y), Z)\\ &\qquad - g(Y, J(\nabla_X JZ)) + g(Y, J(\nabla_X Z))\\ &= X(g(JY,JZ)) - X(g(Y,JZ)) - g(\nabla_X JY, JZ) + g(\nabla_X Y, JZ)\\ &\qquad - g(JY, \nabla_X JZ) + g(JY, \nabla_X Z)\\ &:= (\nabla_X g)(JY,JZ) - (\nabla_X g)(Y,JZ) - g(Y,\nabla_X JZ) + g(Y, J(\nabla_X Z))\\ &= (\nabla_X g)(JY,JZ) - (\nabla_X g)(Y,JZ) - g(Y,(\nabla_X J)Z).\end{aligned}$$

□

More generally, let $f \in Diff(M)$ be an automorphism of the G-structure defined by J, i.e. $f_* \circ J = J \circ f_*$. If f is an affine transformation for ∇, namely $f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$, then f is also affine transformation for $\nabla^{(J)}$.

A natural generalization of the case $\nabla \in \mathcal{C}_J(M)$ is given by:

Proposition 1.2. *Let ∇ be a symmetric linear connection. Assume that $\nabla J = \eta \otimes J^n$ for $n \in \mathbb{N}$, where η is a 1-form. Then $\nabla^{(J)} = \nabla + \eta \otimes J^{n+1}$ and it is a semi-symmetric connection.*

Proof. We obtain $\nabla^{(J)} = \nabla + \eta \otimes J^{n+1}$ and $T_{\nabla^{(J)}} = \eta \otimes J^{n+1} - J^{n+1} \otimes \eta$ which express the conclusion. □

Definition 1.3. The linear connection ∇ is *special Golden connection* if it is torsion free and $d^{\nabla} J = 0$. Let $\mathcal{C}_G(M)$ be the set of these connections. Note that it contains the symmetric J-connections.

Proposition 1.4. *If ∇ is special Golden connection then:*

1. *$\nabla^{(J)}$ is special Golden, too.*
2. *The Nijenhuis tensor field of J is $N_J(X,Y) = (\nabla_{JX} J)Y – (\nabla_X J)JY$.*

Proof. 1. It is a consequence of Proposition 1.1.
2. A straightforward computation gives:

$$\begin{aligned}N_J(X,Y) – J(T_\nabla(X,JY) – T_\nabla(Y,JX) – T_\nabla(X,Y)) + T_\nabla(X,Y) + T_\nabla(JX,JY)\\ = (\nabla_{JX} J)Y - (\nabla_{JY} J)X\end{aligned}$$

and the result follows form $d^{\nabla} J = 0$. □

Definition 1.5. We say that (g, J, ∇) is special Golden structure if (g, J) is Golden Riemannian structure and ∇ is special Golden connection.

Remark 1.6. If (g, J, ∇) is special Golden structure and $\nabla_X g \circ (I \times J) = \nabla_X g \circ (J \times I)$ then $g((\nabla_X J)Y, Z) = g(Y, (\nabla_X J)Z)$.

The last subject of this section treats two tensor fields associated with a Golden structure. The paper [3] introduces the structural and virtual tensor fields of an almost product structure. Turning to our framework, let us consider for a pair (∇, J) the tensor fields of $(1,2)$–type:
1) *the structural tensor field*

$$C_{\nabla}^{J}(X,Y) := \frac{1}{2}[(\nabla_{JX}J)Y + (\nabla_X J)JY] \tag{1.4}$$

2) *the virtual tensor field*

$$B_{\nabla}^{J}(X,Y) := \frac{1}{2}[(\nabla_{JX}J)Y - (\nabla_X J)JY]. \tag{1.5}$$

It results that:

$$C_{\nabla^{(J)}}^{J} = -J^2 \circ C_{\nabla}^{J}, \quad B_{\nabla^{(J)}}^{J} = -J^2 \circ B_{\nabla}^{J}. \tag{1.6}$$

Also

$$C_{\nabla}^{J}(JX,JY) = C_{\nabla}^{J}(X,Y) + (\nabla_{JX}J)JY, \quad B_{\nabla}^{J}(JX,JY) = -B_{\nabla}^{J}(X,Y). \tag{1.7}$$

The importance of these tensor fields for our study is given by the following straightforward relation:

$$\nabla^{(J)} = \nabla + \nabla J - C_{\nabla}^{J} + B_{\nabla}^{J}. \tag{1.8}$$

Example 1.7. If the linear connection ∇ satisfies $\nabla J = \eta \otimes J^n$ for $n \in \mathbb{N}$, where η is a 1-form, then the structural and the virtual tensor fields have the expressions:

$$C_{\nabla}^{J} = \frac{1}{2}[(\eta \circ J) \otimes J^n + \eta \otimes J^{n+1}], \quad B_{\nabla}^{J} = \frac{1}{2}[(\eta \circ J) \otimes J^n - \eta \otimes J^{n+1}],$$

$$C_{\nabla^{(J)}}^{J} = -\frac{1}{2}[(\eta \circ J) \otimes J^{n+2} + \eta \otimes J^{n+3}], \quad B_{\nabla^{(J)}}^{J} = -\frac{1}{2}[(\eta \circ J) \otimes J^{n+2} - \eta \otimes J^{n+3}].$$

Concerning the behavior of $\nabla^{(\cdot)}$ for families of Golden structures, we remark that if J_1 and J_2 are two Golden structures, then:
1) $J_1 + J_2$ is a Golden structure if and only if $J_1 J_2 + J_2 J_1 = -I_{\mathfrak{X}(M)}$; in this case:

$$\nabla_X^{(J_1+J_2)} Y = \nabla_X^{(J_1)} Y + \nabla_X^{(J_2)} Y + J_1(\nabla_X J_2 Y) + J_2(\nabla_X J_1 Y),$$
$$C_{\nabla}^{J_1+J_2} - B_{\nabla}^{J_1+J_2} = (C_{\nabla}^{J_1} + C_{\nabla}^{J_2}) - (B_{\nabla}^{J_1} + B_{\nabla}^{J_2}) - J_1 \circ \nabla J_2 - J_2 \nabla J_1.$$

2) There exists no other nontrivial Golden structure collinear with a given one.

2. Invariant distributions

Let $\mathcal{D} \subset TM$ be a fixed distribution considered as a vector subbundle of TM.

Definition 2.1.

i) $\mathcal{D}$ is called *J–invariant* if $X \in \Gamma(\mathcal{D})$ implies $JX \in \Gamma(\mathcal{D})$.

ii) [4] The linear connection ∇ *restricted to* $\mathcal{D}$, if $Y \in \Gamma(\mathcal{D})$, implies $\nabla_X Y \in \Gamma(\mathcal{D})$ for any $X \in \Gamma(TM)$.

If ∇ is restricted to $\mathcal{D}$ then ∇ may be considered as a connection in the vector bundle $\mathcal{D}$. From this fact, in [1], a connection which restricts to $\mathcal{D}$ is called *adapted to* $\mathcal{D}$.

Proposition 2.2. *If the distribution $\mathcal{D}$ is J-invariant and the linear connection ∇ id restricted to $\mathcal{D}$, then $\nabla^{(J)}$ is also restricted to $\mathcal{D}$.*

Proof. Fix $Y \in \Gamma(\mathcal{D})$. Then $JY \in \Gamma(\mathcal{D})$ and for any $X \in \Gamma(TM)$ we have $\nabla_X Y \in \Gamma(\mathcal{D})$. Therefore, $\nabla^{(J)}_X Y = J(\nabla_X JY) - J(\nabla_X Y) \in \Gamma(\mathcal{D})$. □

A more general notion like restricting to a distribution is that of geodesic invariance [4]. The distribution $\mathcal{D}$ is ∇–*geodesically invariant* if for every geodesic $\gamma : [a, b] \to M$ of ∇ with $\dot{\gamma}(a) \in \mathcal{D}_{\gamma(a)}$ it follows $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for any $t \in [a, b]$. The cited book gives a necessary and sufficient condition for a distribution $\mathcal{D}$ to be ∇–geodesically invariant: for any X and $Y \in \Gamma(\mathcal{D})$, *the symmetric product* $\langle X : Y\rangle := \nabla_X Y + \nabla_Y X$ to belong to $\Gamma(\mathcal{D})$ or equivalently, for any $X \in \Gamma(\mathcal{D})$ to have $\nabla_X X \in \Gamma(\mathcal{D})$.

The following result is a direct consequence of definitions:

Proposition 2.3. *If the distribution $\mathcal{D}$ is J-invariant and the linear connection ∇ is restricted to $\mathcal{D}$ then $\mathcal{D}$ is geodesically invariant for $\nabla^{(J)}$.*

3. On the duality of Golden and product conjugate connections

In [5] Crasmareanu and Hreţcanu proved that any Golden structure J induces an almost product structure [9]:

$$E = \frac{1}{\sqrt{5}}(2J - I_{\mathfrak{X}(M)}) \tag{3.1}$$

and any almost product structure E determines two Golden structures:

$$J_{\pm} = \frac{1}{2}(I_{\mathfrak{X}(M)} \pm \sqrt{5}E). \tag{3.2}$$

Then $\nabla E = \frac{2}{\sqrt{5}}\nabla J$ and respectively $\nabla J_{\pm} = \pm\frac{\sqrt{5}}{2}\nabla E$. Hence, ∇ is a J-connection if and only if ∇ is an E-connection.

We are interested in finding the connection between the conjugate connections associated to them.

Proposition 3.1.

i) *If J is a Golden structure on M and E is the almost product structure given by* (3.1) *then* $4\nabla^{(J)} - 5\nabla^{(E)} + \nabla = 2\nabla J$.

ii) *If E is an almost product structure on M and $J_{\pm}$ are the Golden structures given by* (3.2) *then* $4\nabla^{(J_{\pm})} - 5\nabla^{(E)} + \nabla = \pm\sqrt{5}\nabla E$.

iii) *In each of the previous cases we have* $\left(\nabla^{(J)}\right)^{(E)} = \left(\nabla^{(E)}\right)^{(J)}$.

We showed in [3] that for an almost product structure E, the structural and virtual tensor fields satisfy:

$$\nabla^{(E)} = \nabla - C^{E}_{\nabla} + B^{E}_{\nabla}. \tag{3.3}$$

From (1.8) and this relation we obtain:

Corollary 3.2.

i) *If J is a Golden structure on M and E is the almost product structure* (3.1) *then:*

$$5(C^{E}_{\nabla} - B^{E}_{\nabla}) = 4(C^{J}_{\nabla} - B^{J}_{\nabla}) - 2\nabla J. \tag{3.4}$$

ii) *If E is an almost product structure on M and $J_{\pm}$ are the Golden structures given by* (3.2) *then:*

$$4(C^{J_{\pm}}_{\nabla} - B^{J_{\pm}}_{\nabla}) = 5(C^{E}_{\nabla} - B^{E}_{\nabla}) \pm \sqrt{5}\nabla E. \tag{3.5}$$

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References

[1] A. Bejancu; H. R. Farran, *Foliations and geometric structures*, Mathematics and Its Applications 580, Springer, Dordrecht, 2006. MR2190039 (2006j:53034)

[2] A. M. Blaga and M. Crasmareanu, *The geometry of complex conjugate connections*, Hacet. J. Math. Stat., 41 (1) (2012), 119-126, MR2976917.

[3] A. M. Blaga and M. Crasmareanu, *The geometry of product conjugate connections*, An. Stiint. Univ. Al. I. Cuza Iaşi Mat. (N. S.), 59 (1) (2013), 73-84, MR3098381.

[4] F. Bullo and A. D. Lewis, *Geometric control of mechanical systems. Modeling, analysis, and design for simple mechanical control systems*, Texts in Applied Mathematics, 49, Springer-Verlag, New York, 2005. MR2099139 (2005h:70030).

[5] M. Crasmareanu and C.-E. Hreţcanu, *Golden Differential Geometry*, Chaos Solitons Fractals, 38 (5) (2008), 1229-1238, MR2456523 (2009k:53059).

[6] A. Gezer, N. Cengiz and A. Salimov, *On integrability of Golden Riemannian structures*, Turkish J. Math., 37 (2013), 693-703, MR3070945.

[7] C.-E. Hreţcanu and M. Crasmareanu, *On some invariant submanifolds in Riemannian manifold with Golden Structure*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat., 53 (2007), Supl., 199-211. MR2522394 (2011e:53012).

[8] C.-E. Hreţcanu and M. Crasmareanu, *Applications of the golden ratio on Riemannian manifolds*, Turkish J. Math., 33 (2) (2009), 179-191, MR2537561 (2010d:53035).

[9] F. Özdemir and M. Crasmareanu, *Geometrical objects associated to a substructure*, Turkish J. Math., 35 (4) (2011), 717-728, MR2896417.

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