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ON $\mathcal{I}$ AND $\mathcal{I}^{\mathcal{K}}$-CAUCHY NETS AND COMPLETENESS

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Abstract. In this paper we study the concept of $\mathcal{I}^{\mathcal{K}}$-Cauchy nets which is more general form of $\mathcal{I}^*$-Cauchy nets ([4,5]). We also investigate its relation with the concept of $\mathcal{I}$-Cauchy nets and study the completeness of a uniform space in terms of $\mathcal{I}^{\mathcal{K}}$-Cauchy nets. Subsequently our results extend similar results of Das and Ghosal [4,5].

1. Introduction

The idea of convergence of a real sequence had been extended to statistical convergence by Fast [9] (see also Schoenberg [21]). A lot of investigations have been done on this type of convergence and its topological consequences after the initial works of Fridy [10] and Šalát [11]. Fridy in [10] also defined statistical Cauchy condition. In particular, very recently Di Maio and Kočinac [17] introduced the concept of statistical convergence as also a related notion of s^*-convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence and also offered some applications to selection principle theory, function spaces and hyperspaces. More recent applications of ideal convergence extending these results can be seen in [7].

However if one considers the concept of nets instead of sequences (which undoubtedly play a more important and natural role in topological and uniform spaces [13]) the above approach does not seem to be appropriate because of the absence of any idea of density in arbitrary directed sets. Instead it seems more appropriate to follow the more general approach found in [17]. Namely it is easy to check that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a non-trivial admissible ideal of $\mathbb{N}$. One may consider an arbitrary

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ideal $\mathcal{I}$ of $\mathbb{N}$ and define $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence of sequences as follows:

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X,ρ) is said to be $\mathcal{I}$-convergent to l if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N} : \rho(x_k, l) \geq \varepsilon\} \in \mathcal{I}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X is said to be $\mathcal{I}^*$-convergent to l if there is $M = \{m_1 < m_2 < \cdots < m_k < \dots\} \in \mathcal{F}(\mathcal{I})$(the filter associated with the ideal $\mathcal{I}$) such that $\lim_{k\to\infty} x_{m_k} = l$ [11]. The notions of $\mathcal{I}$ and $\mathcal{I}^*$-convergences of sequences coincide with the usual convergence if $\mathcal{I} = \mathcal{I}_{fin}$, the ideal consisting of finite sets only.

The idea of $\mathcal{I}^*$-convergence has been very recently further extended as follows: Let $\mathcal{K}$ be an ideal of $\mathbb{N}$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X is said to be $\mathcal{I}^{\mathcal{K}}$-convergent [16] to x if there is $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by $y_n = x_n$, if $n \in M$, $y_n = x$, if $n \notin M$ is $\mathcal{K}$-convergent to x. This Definition is only a special case of the Definition in [16] where the $\mathcal{I}^{\mathcal{K}}$-convergence of a function from an arbitrary set was defined and investigated. $\mathcal{I}^{\mathcal{K}}$-convergence of sequences coincide with $\mathcal{I}^*$-convergence if $\mathcal{K} = \mathcal{I}_{fin}$.

In [17] Di Maio and Kocinac asked when a statistically Cauchy sequence in a uniform space is statistically convergent. In terms of nets the problem is when $\mathcal{I}$-Cauchy nets in uniform spaces are $\mathcal{I}$-convergent. In [4] Das and Ghosal proved that if a uniform space with a countable base is complete, then maximal $\mathcal{I}$-Cauchy nets are $\mathcal{I}$-convergent. Subsequently in [5] using the concepts of $\mathcal{I}^*$-Cauchy nets they presented another solution of the problem.

In this paper we introduce the idea of $\mathcal{I}^{\mathcal{K}}$-Cauchy nets, a more general form of $\mathcal{I}^*$-Cauchy nets and consider similar problems. Also following the line of investigation in [5], the relation between $\mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}$-Cauchy conditions is thoroughly investigated.

The following definitions and notations will be needed.

Throughout the paper the pair (X,Γ) will stand for a uniform space which will be written sometimes simply as X. It can be recalled that in a uniform space (X,Γ), for any point $x \in X$ the collection $\{U(x) : U \in \Gamma\}$ (Where $U(x) = \{y \in X : (x,y) \in U\}$) forms a local neighborhood basis at x. The corresponding topology is called the uniform topology on X. By an open set in X we shall always mean an open set in the uniform topology in X.

Throughout $(D,\geq)$ will stand for a directed set and $\mathcal{I},\mathcal{K}$ be two non-trivial ideals of D. Also the symbol $\mathbb{N}$ is reserved for the set of all natural numbers. A net in X will be denoted by $\{s_\alpha : \alpha \in D\}$ or simply by $\{s_\alpha\}$, when there is no confusion about D. Let for $\alpha \in D$, $D_\alpha = \{\beta \in D; \beta \geq \alpha\}$. Then the collection $\mathcal{F}_0 = \{A \subset D : A \supset D_\alpha, \text{ for some } \alpha \in D\}$ forms a filter in D. Let $\mathcal{I}_0 = \{A \subset D : D \setminus A \in \mathcal{F}_0\}$. Then $\mathcal{I}_0$ is a non-trivial ideal of D.

Definition 1.1. ([4], cf. [15]) A non-trivial ideal $\mathcal{I}$ of D will be called D-admissible if $D_\alpha \in \mathcal{F}(\mathcal{I})$ for all $\alpha \in D$.

Definition 1.2. ([4], cf. [15]) A net $\{s_\alpha : \alpha \in D\}$ in a uniform space (X, Γ) is said to be $\mathcal{I}$-convergent to $x_0 \in X$ if for any open set U containing x_0, $\{\alpha \in D : s_\alpha \notin U\} \in \mathcal{I}$.

Definition 1.3. ([5], cf.[15]) $y \in X$ is called an $\mathcal{I}$-cluster point of a net $\{s_\alpha : \alpha \in D\}$ if for any open set U containing y, $\{\alpha \in D : s_\alpha \in U\} \notin \mathcal{I}$.

Definition 1.4. ([15]) A net $\{s_\alpha : \alpha \in D\}$ in a uniform space (X, Γ) is said to be $\mathcal{I}^*$-convergent to $x_0 \in X$ if there exists an $M \in \mathcal{F}(\mathcal{I})$ such that M itself is a directed set and the net $\{s_\alpha : \alpha \in M\}$ is convergent to x_0.

Definition 1.5. ([4]) $\{s_\alpha : \alpha \in D\}$ is said to be $\mathcal{I}$-Cauchy if for any $U \in \Gamma$ there exists a $\beta \in D$ such that $\{\alpha \in D : (s_\alpha, s_\beta) \notin U\} \in \mathcal{I}$.

Definition 1.6. ([5]) $\{s_\alpha : \alpha \in D\}$ is said to be $\mathcal{I}^*$-Cauchy if there exists a $M \in \mathcal{F}(\mathcal{I})$ such that M itself is a directed set and the net $\{s_\alpha : \alpha \in M\}$ is Cauchy.

Definition 1.7. ([20]) An admissible ideal $\mathcal{I}$ of subsets from $\mathbb{N}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, A_3, \dots\}$ belonging to $\mathcal{I}$ there exists a countable family of subsets $\{B_1, B_2, B_3, \dots\}$ of $\mathbb{N}$ such that for each j, $A_j \Delta B_j$ is finite and $B = \bigcup_j B_j \in \mathcal{I}$.

Definition 1.8. ([16]) Let $\mathcal{I}$, $\mathcal{K}$ be two admissible ideals of an arbitrary non-empty set S. Then we say that $\mathcal{I}$ satisfies the condition $AP(\mathcal{I}, \mathcal{K})$ if for every countable family of mutually disjoint sets $\{A_1, A_2, A_3, \dots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, B_3, \dots\}$ from $\mathcal{I}$ such that for each j, $A_j \Delta B_j \in \mathcal{K}$ and $B = \bigcup_j B_j \in \mathcal{I}$.

It is known ([11], see also [3]) that the notions of $\mathcal{I}$ and $\mathcal{I}^*$-convergence (Cauchy condition) of sequences are equivalent if and only if the ideal $\mathcal{I}$ satisfies the condition (AP). Further in [16] it was established that the notions of $\mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}$-convergence of sequences are equivalent if and only if $\mathcal{I}$ and $\mathcal{K}$ are two admissible ideals and $\mathcal{I}$ satisfies $AP(\mathcal{I}, \mathcal{K})$ condition.

Definition 1.9. ([15]) A D-admissible ideal $\mathcal{I}$ is said to satisfy the condition (DP) if for every countable family of mutually disjoint sets $\{A_1, A_2, A_3, \dots\}$ belonging to $\mathcal{I}$ there exists a countable family of subsets $\{B_1, B_2, B_3, \dots\}$ of D such that for each j, $A_j \Delta B_j \subset D \setminus D_{\alpha_j}$ for some $\alpha_j \in D$ and $B = \bigcup_j B_j \in \mathcal{I}$.

The condition (DP) is a special case of condition $AP(\mathcal{I}, \mathcal{K})$.

2. Main results

We first introduce our main idea of $\mathcal{I}^{\mathcal{K}}$-Cauchy condition and then we will investigate some basic results.

Let $M \subset D$ and $\mathcal{K}$ be an ideal of D. Let us put $\mathcal{K}\restriction_M = \{A \cap M : A \in \mathcal{K}\}$. Clearly $\mathcal{K}\restriction_M$ is a sub-ideal of $\mathcal{K}$ i.e. $\mathcal{K}\restriction_M$ is an ideal of M (and so of D) and $\mathcal{K}\restriction_M \subset \mathcal{K}$. If $\mathcal{K}$ is D-admissible then $\mathcal{K}\restriction_M$ is M-admissible when M itself is also a directed set. Also if $M \in \mathcal{F}(\mathcal{I})$, when $\mathcal{I}$ is D-admissible then M is directed set because $\mathcal{F}(\mathcal{I}_{\text{fin}}) \subset \mathcal{F}(\mathcal{I})$ then $M \cap A \neq \emptyset$ for every $A \in \mathcal{F}(\mathcal{I}_{\text{fin}})$. This means that M is cofinal in $(D, \geq)$ and consequently it is a directed set. In [16] $\mathcal{I}^{\mathcal{K}}$-convergence of a function from an arbitrary set was defined and investigated. In the following definition we consider a special case of that definition.

Definition 2.1. Let $\mathcal{K}$ be a non-trivial admissible ideal of D. A net $\{s_\alpha : \alpha \in D\}$ in a uniform space (X, Γ) is said to be $\mathcal{I}^{\mathcal{K}}$-convergent to $x_0 \in X$ if there exists a $M \in \mathcal{F}(\mathcal{I})$ and the net $\{t_\alpha : \alpha \in D\}$ defined by $t_\alpha = s_\alpha$ if $\alpha \in M$ and $t_\alpha = x_0$ if $\alpha \in D \setminus M$ is $\mathcal{K}$-convergent to x_0.

Definition 2.2. A net $\{s_\alpha : \alpha \in D\}$ in a uniform space (X, Γ) is said to be $\mathcal{I}^{\mathcal{K}}$-Cauchy if there exists a $M \in \mathcal{F}(\mathcal{I})$ and the net $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$-Cauchy.

If one takes $\mathcal{I} = \mathcal{I}_d$ and $\mathcal{K} = \mathcal{I}_{fin}$ then $\mathcal{I}^{\mathcal{K}}$-convergence becomes s^*-convergence and $\mathcal{I}^{\mathcal{K}}$-Cauchy condition becomes s^*-Cauchy condition.

Theorem 2.1. *In a uniform space (X, Γ) every $\mathcal{I}^{\mathcal{K}}$-convergent net satisfies $\mathcal{I}^{\mathcal{K}}$-Cauchy condition.*

Proof. Let $\{s_\alpha : \alpha \in D\}$ be $\mathcal{I}^{\mathcal{K}}$-convergent to x_0. Then there exists a $M \in \mathcal{F}(\mathcal{I})$ and the net $\{t_\alpha : \alpha \in D\}$ defined by $t_\alpha = s_\alpha$ if $\alpha \in M$ and $t_\alpha = x_0$ if $\alpha \in D \setminus M$ is $\mathcal{K}$-convergent to x_0. i.e. the net $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$ -convergent to x_0. Hence $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$ -Cauchy (by Theorem 2 of [4]). i.e. $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}^{\mathcal{K}}$-Cauchy. □

Theorem 2.2. *Let $\mathcal{I}$, $\mathcal{K}$ be two D-admissible ideals with $\mathcal{K} \subset \mathcal{I}$. Then an $\mathcal{I}^{\mathcal{K}}$-Cauchy net in X is also $\mathcal{I}$-Cauchy.*

Proof. Let $U \in \Gamma$ and $\{s_\alpha : \alpha \in D\}$ be $\mathcal{I}^{\mathcal{K}}$-Cauchy. Then there exists a $M \in \mathcal{F}(\mathcal{I})$ and the net $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$-Cauchy. Therefore there exists a $A \in \mathcal{K}\restriction_M$ such that $\alpha,\ \beta \notin A$ $(\alpha, \beta \in M)$ implies $(s_\beta, s_\alpha) \in U$ (by Theorem 1 of [4]). Choose $B = D \setminus (M \cap (D \setminus A))$. Then $B \in \mathcal{I}$ (as $D \setminus A \in \mathcal{F}(\mathcal{K})$ and $\mathcal{F}(\mathcal{K}) \subset \mathcal{F}(\mathcal{I})$ and $M \in \mathcal{F}(\mathcal{I})$). Now $\gamma,\ \delta \notin B$ implies $\gamma,\ \delta \in M \cap (D \setminus A)$ i.e. $\gamma,\ \delta \in M$ and $\gamma,\ \delta \notin A$ which implies that $(s_\gamma, s_\delta) \in U$. Hence $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}$-Cauchy (by Theorem 1 of [4]). □

The following example shows that in general, the above result is not true if $\mathcal{K} \nsubseteq \mathcal{I}$.

Example 2.1. Let $\mathcal{I} = \mathcal{I}_{fin} = \{A \subset \mathbb{N} : A \text{ is finite}\}$ and $\mathcal{K} = \mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then clearly $\mathcal{F}(\mathcal{K}) \nsubseteq \mathcal{F}(\mathcal{I})$. Consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined as $x_n = k$ if $n = k^2$ and $x_n = \frac{1}{n}$ if $n \neq k^2$. Then $\{x_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-convergent and also not $\mathcal{I}$-Cauchy (since $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy condition coincides for complete metric spaces [8]). Put $M = \mathbb{N}$. Then $M \in \mathcal{F}(\mathcal{I})$. Define $y_n = x_n$ if $n \neq 1$ and $x_n = 0$ if $n = 1$. Then $\{x_n\}_{n\in M}$ is $\mathcal{K}\restriction_M$-convergent to 0 i.e. $\{x_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^{\mathcal{K}}$-convergent to 0 and so is $\mathcal{I}^{\mathcal{K}}$-Cauchy.

Theorem 2.3. *If (X, Γ) is a uniform space with $\Delta \in \Gamma$ then $\mathcal{I}$-Cauchy condition implies $\mathcal{I}^{\mathcal{K}}$-Cauchy condition for D-admissible ideals $\mathcal{I}$ and $\mathcal{K}$.*

Proof. Let $\{s_\alpha : \alpha \in D\}$ be $\mathcal{I}$-Cauchy. We have to show that $\{s_\alpha : \alpha \in D\}$ is also $\mathcal{I}^{\mathcal{K}}$-Cauchy. Since $\Delta \in \Gamma$, there is a $A \in \mathcal{I}$ such that $\gamma, \delta \notin A$ implies $(s_\gamma, s_\delta) \in \Delta$. Put $M = D \setminus A$. Then $M \in \mathcal{F}(\mathcal{I})$ and we can show that M is directed with respect to the binary relation induced from $(D, \geq)$. Take $\beta \in M$ (as $M \neq \emptyset$). Let $s_\beta = x_0$. Now for all $\alpha \in M$, $s_\alpha = s_\beta = x_0$. Thus $\{s_\alpha : \alpha \in M\}$ is a constant net and so is Cauchy i.e. $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}^{\mathcal{K}}$-Cauchy. □

The following example shows that in general $\mathcal{I}$-Cauchy condition does not imply $\mathcal{I}^{\mathcal{K}}$-Cauchy condition even if $\mathcal{I}$ and $\mathcal{K}$ are D-admissible ideals such that $\mathcal{F}(\mathcal{K}) \subset \mathcal{F}(\mathcal{I})$. However, the following example is modeled after a similar example from [5].

Example 2.2. If a uniform space (X, Γ) with the property that $\bigcap_{U\in\Gamma} U = \Delta$ has a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct points then there exists an admissible non-trivial ideal $\mathcal{I}$ of $\mathbb{N}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ in X such that $\{y_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy but not $\mathcal{I}^{\mathcal{K}}$-Cauchy where $\mathcal{K}$ is an admissible ideal of $\mathbb{N}$ with condition (AP) and $\mathcal{K} \subset \mathcal{I}$.

Proof. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} A_j$ be a decomposition of $\mathbb{N}$ such that each A_j is infinite and $A_i \cap A_j = \emptyset$ for $i \neq j$. Let $\mathcal{I}$ denote the class of all $A \subset \mathbb{N}$ which intersects at most a finite number of A_j's. Then $\mathcal{I}$ is a non-trivial admissible ideal. Note that any A_j is a member of $\mathcal{I}$.

Let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence defined by $y_n = x_j$ if $n \in A_j$. Let $U \in \Gamma$. Since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy, there exists a $m_0 \in \mathbb{N}$ such that $(x_m, x_n) \in U$ for all $m, n \in \mathbb{N}$ with $m, n \geq m_0$. Clearly $A_1 \cup A_2 \cup \cdots \cup A_{m_0} = C \in \mathcal{I}$ and $m, n \notin C$ implies $(y_m, y_n) = (x_p, x_q)$ for some $p, q \geq m_0$, and so belongs to U. Hence $\{y_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-Cauchy.

Now if possible suppose that $\{y_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}^{\mathcal{K}}$-Cauchy. Then there exists a $H \in \mathcal{I}$ such that $\{y_n\}_{n\in M}$, where $M = \mathbb{N} \setminus H$, is $\mathcal{K}\restriction_M$-Cauchy. From the construction of $\mathcal{I}$ it follows that there exists a $l \in \mathbb{N}$ such that $H \subset A_1 \cup A_2 \cup \cdots \cup A_l$ and then $A_i \subset M = \mathbb{N} \setminus H$ for $i \geq l+1$. We first observe that $\mathcal{K}\restriction_M$ also satisfies the condition (AP). For this let $\{A_1, A_2, \dots\}$ be a countable family of mutually disjoint sets from $\mathcal{K}\restriction_M$. As $\mathcal{K}\restriction_M$ is a sub-ideal of $\mathcal{K}$ and $\mathcal{K}$ satisfies the condition (AP), there exists a countable family of subsets $\{B_1, B_2, \dots\}$ of $\mathbb{N}$ such that for each j, $A_j \Delta B_j$ is finite and $B = \bigcup_j B_j \in \mathcal{K}$. Put $C_j = B_j \cap M$. Then for each j, $A_j \Delta C_j$ is finite and $C = \bigcup_j C_j \in \mathcal{K}\restriction_M$. Now $\{y_n\}_{n \in M}$ is $\mathcal{K}\restriction_M{}^*$-Cauchy (see [3, 17]). Hence there exists a set $M' \in \mathcal{F}(\mathcal{K}\restriction_{\mathcal{M}})$ such that $\{y_n\}_{n \in M \cap M'}$ is Cauchy. Observe that $M \setminus M' \in \mathcal{K}\restriction_M$ i.e. $M \setminus M' = A \cap M$ where $A \in \mathcal{K}$. Hence $M' = (D \setminus (A \cap M)) \cup (D \setminus M)$. Since $D \setminus (A \cap M) \in \mathcal{F}(\mathcal{K})$, so $M' \in \mathcal{F}(\mathcal{K})$. Again since $\mathcal{F}(\mathcal{K}) \subset \mathcal{F}(\mathcal{I})$, so $M' \in \mathcal{F}(\mathcal{I})$. Hence from the construction of $\mathcal{I}$, there exists a $l_1 \in \mathbb{N}$ such that $A_i \subset M'$ for $i \geq l+1$. Thus $A_i \subset M \cap M'$ for $i \geq l_0 + 1$ where $l_0 = \max\{l, l_1\}$. Choose $i, j \in \mathbb{N}$ with $i, j \geq l_0 + 1$ so that $A_i, A_j \subset M \cap M'$. So $\{y_n\}_{n \in M \cap M'}$ contains an infinite number of terms which are equal to x_i and x_j respectively. Now by our assumption $\bigcap_{U \in \Gamma} U = \Delta$. Since $x_i \neq x_j$, there is a $V \in \Gamma$ such that $(x_i, x_j) \notin V$. Then there does not exist any $m_0 \in M \cap M'$ with the property that $m, n \geq m_0$ implies $(y_m, y_n) \in V$, which contradicts the fact that $\{y_n\}_{n \in M \cap M'}$ is Cauchy. Therefore $\{y_n\}_{n \in \mathbb{N}}$ is not $\mathcal{I}^{\mathcal{K}}$-Cauchy. □

We shall now study the equivalence of $\mathcal{I}$ and $\mathcal{I}^{\mathcal{K}}$-Cauchy conditions under certain assumptions (namely conditions $AP(\mathcal{I}, \mathcal{K})$) which becomes necessary as well as sufficient on certain restrictions on the space.

Theorem 2.4. *Let $\mathcal{I}, \mathcal{K}$ be two D-admissible ideals of a directed set $(\mathcal{D}, \geq)$ and $\mathcal{I}$ satisfies the condition $AP(\mathcal{I}, \mathcal{K})$. Let (X, Γ) be a uniform space having a countable base $\mathcal{R}$. Then for any net $\{s_\alpha : \alpha \in D\}$ in X, $\mathcal{I}$-Cauchy condition implies $\mathcal{I}^{\mathcal{K}}$-Cauchy condition.*

Proof. Let $\mathcal{R} = \{U_i : i = 1, 2, 3, \dots\}$ be a countable base of (X, Γ). Without any loss of generality we can assume $\{U_i : i = 1, 2, 3, \dots\}$ to be monotonically decreasing. Since $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}$-Cauchy, for each $U_i \in \mathcal{R}$ there exists a $K_i \in \mathcal{I}$ such that $\beta, \alpha \notin K_i$ implies $(s_\alpha, s_\beta) \in U_i$.

Let $A_1 = K_1, A_2 = K_2 \setminus K_1, A_3 = K_3 \setminus (K_1 \cup K_2), \dots$. Then $\{A_i : i = 1, 2, 3, \dots\}$ is a countable family of mutually disjoint sets in $\mathcal{I}$. By the condition $AP(\mathcal{I}, \mathcal{K})$ there exists a countable family of sets $\{B_i : i = 1, 2, 3, \dots\}$ in $\mathcal{I}$ such that $A_j \Delta B_j \subset D \setminus C_j$ for some $C_j \in \mathcal{F}(\mathcal{K})$ and $B = \cup B_j \in \mathcal{I}$. Let $M = D \setminus B$. Then $M \in \mathcal{F}(\mathcal{I})$. Let $U \in \Gamma$. Since $\mathcal{R}$ is a basis of Γ, there exists $l \in \mathbb{N}$ such that $U_l \subset U$. Now $K_l \setminus B \subset \bigcup_{i=1}^{l} (A_i \setminus B_i) \subset \bigcup_{i=1}^{l} (D \setminus C_i)$ (by condition $AP(\mathcal{I}, \mathcal{K})$, where $C_i \in \mathcal{F}(\mathcal{K})$).

Put $C = C_1 \cap C_2 \cap \cdots \cap C_l$. Then ${K_l}^c \cap M \supset M \cap C$ (for otherwise there is a $\gamma \in M \cap C$ but $\gamma \notin {K_l}^c$ and so $\gamma \in K_l \cap M \subset D \setminus C$ which is a contradiction). This shows that $\alpha, \beta \in C \cap M$ implies $\alpha, \beta \in {K_l}^c$ which consequently implies that $(s_\alpha, s_\beta) \in U_l \subset U$. Thus $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$-Cauchy and so $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}^{\mathcal{K}}$-Cauchy. □

Theorem 2.5. *Let (X, Γ) be a uniform space having a countable basis $\mathcal{R}$ with the property that $\bigcap_{U \in \Gamma} U = \Delta$ and let X has at least one limit point. If for every net $\{s_\alpha : \alpha \in D\}$ $\mathcal{I}$-Cauchy condition implies $\mathcal{I}^{\mathcal{K}}$-Cauchy condition. Then the condition $AP(\mathcal{I}, \mathcal{K})$ holds.*

Proof. Without any loss of generality we can assume that $\mathcal{R} = \{U_i : i = 1, 2, 3, \dots\}$ be a countable basis of Γ satisfying the condition that each U_i is symmetric and $U_{i+1} \circ U_{i+1} \subset U_i$ for all $i = 1, 2, 3, \dots$. Let x_0 be a limit point of X. Then $\{U_i(x_0) : i = 1, 2, 3, \dots\}$ is a monotonically decreasing open base at x_0. Since $\bigcap_{U \in \Gamma} U = \Delta$, it follows that the uniform topology corresponding to the family Γ is also T_1. We can find a sequence $\{x_i\}_{i \in \mathbb{N}}$ of distinct elements in X such that $x_i \in U_i(x_0) \setminus U_{i+1}(x_0)$, $x_i \neq x_0$ for all i and $x_i \to x_0$.

Let $\{A_i : i = 1, 2, 3, \dots\}$ be a mutually disjoint countable family of nonempty sets from $\mathcal{I}$. Define a net $\{s_\alpha : \alpha \in D\}$ by $s_\alpha = x_j$ if $\alpha \in A_j$ and $s_\alpha = x_0$ if $\alpha \notin A_j$ for any $j \in \mathbb{N}$. As in Theorem 6 [5] we can show that $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}$-Cauchy. By our assumption $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}^{\mathcal{K}}$-Cauchy. Hence there exists a set $H \in \mathcal{I}$ such that $M = D \setminus H \in \mathcal{F}(\mathcal{I})$ and $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$-Cauchy. Now let $B_j = A_j \cap H$ for all $j \in \mathbb{N}$. We consider following cases.

If for each j there exists $C_j \in \mathcal{F}(\mathcal{K})$ such that A_j is disjoint from $M \cap C_j$ then clearly we have $A_j \subset B_j \cup (M \setminus C_j)$ and so $A_j \Delta B_j = A_j \setminus B_j \subset M \setminus C_j \subset D \setminus C_j$. Also since $B_j \in \mathcal{I}$ for all $j \in \mathbb{N}$ and $B = \cup B_j \subset H \in \mathcal{I}$ so the condition $AP(\mathcal{I}, \mathcal{K})$ holds.

Next suppose that the condition of the previous case does not hold. First suppose that there is only one $j \in \mathbb{N}$ for which the condition specified in the above case does not hold good. Then re-defining $B_i = A_i \cap H$ for $i \neq j$ and $B_j = A_j$ we can see that the sequence of sets $\{B_i : i = 1, 2, 3, \dots\}$ has the property that $B_i \in \mathcal{I}$ for all i and $\cup B_i \subset A_j \cup H \in \mathcal{I}$. Clearly $A_i \Delta B_i \subset D \setminus C_i$ for all $i \neq j$ for some $C_i \in \mathcal{F}(\mathcal{K})$ and $A_i \Delta B_i = \emptyset$. Hence the condition $AP(\mathcal{I}, \mathcal{K})$ holds.

Finally suppose that there are more than one $j \in \mathbb{N}$ for which the condition specified above does not hold. Take any two of these members of $\mathbb{N}$ say i and j $(i \neq j)$. We have $C_1 = A_i \cap M \notin \mathcal{K}$ and $C_2 = A_j \cap M \notin \mathcal{K}$. Now any $E \in \mathcal{F}(\mathcal{K}\restriction_M)$ is of the form $E = C \cap M$ where $C \in \mathcal{F}(\mathcal{K})$. Again $C \cap C_1 \neq \emptyset$ for otherwise we will have $C_1 \subset D \setminus C \in \mathcal{K}$ which is not the case. By similar reasoning $C \cap C_2 \neq \emptyset$. This implies that there exists a $\gamma \in A_i$ and a $\delta \in A_j$. Clearly $s_\gamma = x_i$ and $s_\delta = x_j$. Since $x_i \neq x_j$ and by our assumption $\bigcap_{U \in \Gamma} U = \Delta$, so there exists a $U \in \mathcal{R}$ such that $(x_i, x_j) \notin U$. Since any

$E \in \mathcal{F}(\mathcal{K}\restriction_M)$ is of the form $C \cap M$ where $C \in \mathcal{F}(\mathcal{K})$ so we get that for any $E \in \mathcal{F}(\mathcal{K}\restriction_M)$ there are $\gamma, \delta \in E$ such that $(s_\gamma, s_\delta) = (x_i, x_j) \notin U$. This contradicts the fact that $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$-Cauchy. Hence there cannot be more than one $j \in \mathbb{N}$ for which the above condition does not hold. This completes the proof of the theorem. □

In the rest of the paper we primarily investigate when $\mathcal{I}^{\mathcal{K}}$-Cauchy nets are $\mathcal{I}^{\mathcal{K}}$-convergent and its relation with completeness.

Theorem 2.6. *Let (X, Γ) be a uniform space. If for every directed set D there exists two arbitrary D-admissible ideals $\mathcal{I}$ and $\mathcal{K}$ such that every $\mathcal{I}^{\mathcal{K}}$-Cauchy net $\{s_\alpha : \alpha \in D\}$ in X is $\mathcal{I}^{\mathcal{K}}$-convergent in X, then X is complete.*

Proof. Let $\{s_\alpha : \alpha \in D\}$ be a Cauchy net in X and $\mathcal{I}, \mathcal{K}$ be two D-admissible ideals. Now $D \in \mathcal{F}(\mathcal{I})$, $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}_0\restriction_D$-Cauchy and so is $\mathcal{K}\restriction_D$-Cauchy as $\mathcal{K}$ is D-admissible. Thus $\{s_\alpha : \alpha \in D\}$ is $\mathcal{I}^{\mathcal{K}}$-Cauchy and so by our assumption $\{s_\alpha : \alpha \in D\}$ is also $\mathcal{I}^{\mathcal{K}}$-convergent. Let $\{s_\alpha : \alpha \in D\}$ be $\mathcal{I}^{\mathcal{K}}$-convergent to $x_0 \in X$. Then there exists $M \in \mathcal{F}(\mathcal{I})$ and the net $\{s_\alpha : \alpha \in M\}$ is $\mathcal{K}\restriction_M$ convergent to x_0.

Let $E = \{(U(x_0), \beta) : U \in \Gamma,\ \beta \in M\}$. For $(U(x_0), \beta)$ and $(V(x_0), \alpha)$ define $(U(x_0), \beta) \geq (V(x_0), \alpha)$ if and only if $U(x_0) \subset V(x_0)$ and $\beta \geq \alpha$. Then $(E, \geq)$ is a directed set.

Let $(U(x_0), \beta) \in E$. Now for each $U \in \Gamma$, $\{\alpha \in M : s_\alpha \notin U(x_0)\} \in \mathcal{K}\restriction_M$. Then $\{\alpha \in M : s_\alpha \in U(x_0)\} \in \mathcal{F}(\mathcal{K}\restriction_{\mathcal{M}})$ and so belongs to $\mathcal{F}(\mathcal{K})$ as was shown earlier. Therefore $M' = \{\alpha \in M : s_\alpha \in U(x_0)\} \cap \{\alpha \in M : \alpha \geq \beta\} \in \mathcal{F}(\mathcal{K})$. Choose $\gamma \in M'$. Now define $i : E \to M$ by $i(U(x_0), \beta) = \gamma$ and $t : E \to X$ by $t(U(x_0), \beta) = s_\gamma$. Then it is easy to check that $\{t_\delta : \delta \in E\}$ is a subnet of $\{s_\alpha : \alpha \in M\}$ and hence of $\{s_\alpha : \alpha \in D\}$ which converges to x_0. This shows that (X, Γ) is complete. □

Theorem 2.7. *In a complete uniform space (X, Γ) having a countable base $\mathfrak{R}$, every $\mathcal{I}^{\mathcal{K}}$-Cauchy net $\{s_\alpha : \alpha \in D\}$ has a subnet $\{t_\beta : \beta \in E\}$ which is convergent.*

The proof is analogous to the proof of Theorem 6 [4] with minor modifications and so is omitted.

Theorem 2.8. *In a complete uniform space (X, Γ) having a countable base $\mathfrak{R}$, every maximal $\mathcal{I}^{\mathcal{K}}$-Cauchy net is $\mathcal{I}^{\mathcal{K}}$-convergent for every $D-$admissible ideals $\mathcal{I}$ and $\mathcal{K}$.*

Proof. Follows from Theorem 2.7. □

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