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# ANALYTIC CONTINUATION OF THE EXTENDED HURWITZ–LERCH ZETA FUNCTION

RAM K. SAXENA AND TIBOR K. POGÁNY

Abstract. The object of this paper is to investigate the analytic continuation and asymptotic expansions for families of the generalized Hurwich–Lerch Zeta functions defined by Srivastava *et al.* [9]. The result obtained is of general character and includes, as special cases, in the same fashion results about the Gauss hypergeometric function, the generalized hypergeometric function and for Fox–Wright function given earlier by Kilbas *et al.* [4] and others.

## 1. Introduction and preliminaries

Analytic continuation formulae and asymptotic estimates of the higher transcendental functions play a very important role in the development of the theory of many areas of science, engineering and technology. Without their help it would not be possible to analyze the solution of various problems for large values of the argument. In recent years interest was developed by many authors in the study of the analytic and statistical properties of the generalized Hurwitz–Lerch Zeta function, see e.g. the recent publications [8, 9] and the references therein. This has motivated the authors to investigate the analytic continuation and asymptotic expansion of the generalized Hurwitz–Lerch Zeta function, defined and studied by Srivastava *et al.* [8] for large values of the argument. The results obtained are of a most general nature and contain numerous results for Hurwitz–Lerch Zeta functions, Gauss ${}_2F_1$ and generalized hypergeometric ${}_pF_q$ functions, Fox–Wright $\Psi$ functions, and other potentially useful functions like Bessel and Whittaker functions and their generalizations can be deduced as corollaries of our main findings.

Throughout, as usual $\mathbb{N}_p = \{p, p+1, \dots\}$, $\mathbb{N}_1 \equiv \mathbb{N}, p$ nonnegative integer, $\mathbb{Z}_0^- = \mathbb{Z} \setminus \mathbb{N}$, while $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$ stand for the sets of real, *positive* real and complex numbers, respectively.

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We begin our study by defining the Fox–Wright generalized hypergeometric function ${}_p\Psi_q^*[\cdot]$, $p, q \in \mathbb{N}_0$ defined by [3, p. 56]

$$ {}_p\Psi_q^*\left[\begin{array}{ll} (a_1,\rho_1),\dots,(a_p,\rho_p); & \\ (b_1,\sigma_1),\dots,(b_q,\sigma_q); & z\end{array}\right] = \sum_{n=0}^\infty \frac{\prod\limits_{j=1}^p (a_j)_{\rho_j n}}{\prod\limits_{j=1}^q (b_j)_{\sigma_j n}} \frac{z^n}{n!}, \tag{1.1} $$

where $a_j,\ b_k \in \mathbb{C}$, $\rho_j,\ \sigma_k \in \mathbb{R}^+$, $j = 1,\dots,p;\ k = 1,\dots,q$. Here $(\lambda)_\mu$ denotes the Pochhammer symbol (or the *shifted factorial*) defined, in terms of Euler's Gamma function, by

$$ (\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & \big(\mu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}\big) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \big(\mu = n \in \mathbb{N};\ \lambda \in \mathbb{C}\big), \end{cases} $$

it being understood *conventionally* that $(0)_0 := 1$. The defining series in (1.1) converges in the whole complex $z$-plane when

$$ \Delta := \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j > -1 \tag{1.2} $$

while when $\Delta = 0$, then the series in (1.1) converges for $|z| < \nabla$, where

$$ \nabla := \prod_{j=1}^p \rho_j^{-\rho_j} \cdot \prod_{j=1}^q \sigma_j^{\sigma_j}\,. \tag{1.3} $$

If, in the definition (1.1), we set $\rho_1 = \cdots = \rho_p = 1$ and $\sigma_1 = \cdots = \sigma_q = 1$ we get the relatively more familiar generalized hypergeometric function ${}_pF_q[z]$, [3].

Lin-Srivastava generalized Hurwitz-Lerch Zeta function in the form

$$ \Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}\cdot n!} \frac{z^n}{(a+n)^s}\,, \tag{1.4} $$

where $\lambda,\mu \in \mathbb{C}; \nu \in \mathbb{C}\setminus\mathbb{Z}_0^-; \min\{\rho,\sigma,\kappa\} > 0; \kappa-\rho-\sigma > -1$ when $s, z \in \mathbb{C}$; $\kappa-\rho-\sigma = -1$ and $s \in \mathbb{C}$ when $|z| < \delta = \kappa^\kappa\rho^{-\rho}\sigma^{-\sigma}$; while $\kappa-\rho-\sigma = -1$ and $\Re\{s+\nu-\lambda-\mu\} > 1$ when $|z| = \delta$. In a recent article by Srivastava *et al.* [9, p. 491, Eq.(1.20)] (1.4) has been extended along the lines of the Fox-Wright hypergeometric function ${}_p\Psi_q^*$ defined above. We recall here the definition of the extended Hurwitz-Lerch Zeta functions as follows.

**Definition.** [9, p. 503, Eq. (6.2)] The family of the extended Hurwitz-Lerch Zeta functions:

$$ \Phi_{\lambda_1,\dots,\lambda_p;\mu_1,\dots,\mu_q}^{(\rho_1,\dots,\rho_p,\sigma_1,\dots,\sigma_q)}(z,s,a), $$

with $p+q$ upper parameters and $p+q+2$ lower parameters, is given by

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a) = \Phi^{(\rho_1,\ldots,\rho_p,\sigma_1,\ldots,\sigma_q)}_{\lambda_1,\ldots,\lambda_p;\mu_1,\ldots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p}(\lambda_j)_{n\rho_j}}{n!\,\prod\limits_{j=1}^{q}(\mu_j)_{n\sigma_j}}\,\frac{z^n}{(n+a)^s}\,. \tag{1.5}$$

The parameters $p,q\in\mathbb{N}_0$; $\lambda_j\in\mathbb{C}$, $j=1,\ldots,p$; $a,\mu_j\in\mathbb{C}\setminus Z_0^-$, $j=1,\ldots,q$; $\rho_j,\sigma_k\in\mathbb{R}^+$, $j=1,\ldots,p; k=1,\ldots,q$. Further $\Delta>-1$ when $s,z\in\mathbb{C}$; while $\Delta=-1$, and $s\in\mathbb{C}$ when $|z|<\nabla$; $\Delta=-1$ and $\Re(\Xi)>\frac{1}{2}$ when $|z|=\nabla$. Here $\Delta$ and $\nabla$ are given by (1.2) and (1.3), respectively, and

$$\Xi := s+\sum_{j=1}^{q}\mu_j-\sum_{j=1}^{p}\lambda_j+\frac{p-q}{2}\,.$$

Here empty products are interpreted by convention as unity.

The extended Hurwitz-Lerch Zeta function $\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a)$ in (1.5), contains a set of special functions such as the Riemann Zeta function $\zeta(s)$, the Hurwitz Zeta function $\zeta(s,a)$, the Lerch Zeta function $\ell_s(\xi)$, the Polylogarithm (or *Jonquère's function*) $\mathrm{Li}_s(z)$, the Hurwitz-Lerch Zeta $\Phi(z,s,a)$ and its various generalizations (see, for details, [7, 9] and the references therein). However, a comprehensive and detailed account of the integral and computational representations for the extended Hurwitz-Lerch Zeta function can be found in [8].

Finally, we recall that it has been shown in [9, p. 504, Theorem 8] that

$$\Phi^{(\boldsymbol{\rho},\boldsymbol{\sigma})}_{\boldsymbol{\lambda};\boldsymbol{\mu}}(z,s,a) = \frac{\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}}}{2\pi\mathrm{i}}\int_{\mathfrak{L}}\frac{\Gamma(-\xi)\Gamma^s(\xi+a)\prod_{j=1}^{p}\Gamma(\lambda_j+\xi\rho_j)}{\Gamma^s(\xi+a+1)\prod_{j=1}^{q}\Gamma(\mu_j+\sigma\rho_j)}\,(-z)^{\xi}\,\mathrm{d}\xi \tag{1.6}$$

whenever $\big(\min\{\Re(a),\Re(s)\}>0;\ |z|<1, |\arg(-z)|<\pi\big)$, provided that each member of the assertion (1.6) exists. Here and in what follows

$$\boldsymbol{\Gamma}^{\boldsymbol{\mu}}_{\boldsymbol{\lambda}} := \prod_{j=1}^{q}\Gamma(\mu_j)\left\{\prod_{j=1}^{p}\Gamma(\lambda_j)\right\}^{-1}.$$

Further, the poles of the gamma functions occurring in the integrand of (1.6) are all simple, the path of integration $\mathfrak{L}$ is a contour starting at the point $w-\mathrm{i}\infty$, and terminating at the point $w+\mathrm{i}\infty$ with $w\in\mathbb{R}$, with indentations, if necessary, in such a manner so as to separate the poles of $\Gamma(-\xi)$ at the points $\xi_{1,n}=n, n\in\mathbb{N}_0$, from the poles of $\Gamma(\lambda_h+\xi\rho_h)$ at the points $\xi_{2,h}(r)=-(\lambda_h+r)/\rho_h$, $h=\overline{1,p}$, $r\in\mathbb{N}_0$.

Next we present our main analytic continuation results related to the previously introduced generalized multi–parameter Hurwitz–Lerch Zeta function $\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z,s,a)$.

## 2. Analytic continuation formulae

This section is devoted for investigating the analytic continuation formulae for the generalized Hurwich–Lerch Zeta function (1.5).

**Theorem 1.** *Let* $p,q\in\mathbb{N}_0$*;* $\lambda_j\in\mathbb{C}, j=\overline{1,p}$*;* $a,\mu_j\in\mathbb{C}\setminus\mathbb{Z}_0^-, j=\overline{1,q}$*;* $\rho_j,\sigma_k>0, j=\overline{1,p}, k=\overline{1,q}$ *and* $\Delta>-1$ *for* $s,z\in\mathbb{C}$*; while for* $|z|>\nabla, s\in\mathbb{C}$ *when* $\Delta=-1$*. Then*

$$\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z,s,a)=\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\cdot\sum_{h=1}^{p}\sum_{r=0}^{\infty}\frac{\Gamma\big(\frac{\lambda_h+r}{\rho_h}\big)\prod\limits_{j\neq h,j=1}^{p}\Gamma\big(\lambda_j-\rho_j\frac{\lambda_h+r}{\rho_h}\big)}{\prod\limits_{j=1}^{q}\Gamma\big(\mu_j-\sigma_j\frac{\lambda_h+r}{\rho_h}\big)\big(\frac{\lambda_h+r}{\rho_h}+a\big)^s}\frac{(-1)^r}{\rho_h\cdot r!}(-z)^{-\frac{\lambda_h+r}{\rho_h}}, \tag{2.7}$$

*when* $|\arg(-z)|<\pi$ *and provided the series* (2.7) *converges.*

*Proof.* Consider the Mellin–Barnes type contour integral expression (1.6) of $\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z,s,a)$. If we calculate the residues at the simple poles of the Gamma functions $\Gamma(\lambda_h+\rho_h\xi)$ which are

$$\xi_{2,h}(r)=-\frac{\lambda_h+r}{\rho_h},\qquad h=\overline{1,p},\quad r\in\mathbb{N}_0\,,$$

then by the application of calculus of the residues we arrive at the desired result (2.7). The convergence of the above series can be proved in a manner similar to that adopted by Kilbas *et al.* [3, p. 129] in the case of Fox–Wright function. □

Letting $s\to 0$ in (2.7) we arrive at the following analytic continuation formula for the Fox-Wright ${}_q\Psi_p$–function given by Kilbas *et al.* [3].

**Corollary 1.1.** *Let the parameter space be the same as in* Theorem 1. *Then we have*

$$ {}_p\Psi_q\left[\begin{matrix}(\lambda_1,\rho_1),\ldots,(\lambda_p,\rho_p);\\ (\mu_1,\sigma_1),\ldots,(\mu_q,\sigma_q);\end{matrix}\; z\right]=\boldsymbol{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}\cdot{}_p\Psi_q^*\left[\begin{matrix}(\lambda_1,\rho_1),\ldots,(\lambda_p,\rho_p);\\ (\mu_1,\sigma_1),\ldots,(\mu_q,\sigma_q);\end{matrix}\; z\right]$$

$$=\sum_{h=1}^{p}\sum_{r=0}^{\infty}\frac{\Gamma\big(\frac{\lambda_h+r}{\rho_h}\big)\prod\limits_{j\neq h,j=1}^{p}\Gamma\big(\lambda_j-\rho_j\frac{\lambda_h+r}{\rho_h}\big)}{\prod\limits_{j=1}^{q}\Gamma\big(\mu_j-\sigma_j\frac{\lambda_h+r}{\rho_h}\big)}\frac{(-1)^r}{\rho_h\cdot r!}(-z)^{-\frac{\lambda_h+r}{\rho_h}}, \tag{2.8}$$

*when* $|\arg(-z)| < \pi$ *and the series* (2.8) *is convergent.*

**Corollary 1.2.** *The following analytic continuation formula holds true for all* $|\arg(-z)| < \pi, |z| > 1$*:*

$$
{}_pF_q\left[\begin{array}{c}\lambda_1,\dots,\lambda_p;\\ \mu_1,\dots,\mu_q;\end{array} z\right] = \mathbf{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \cdot \sum_{h=1}^{p} (-z)^{-\lambda_h} \frac{\Gamma(\lambda_h) \prod\limits_{j\neq h, j=1}^{p} \Gamma(\lambda_j-\lambda_h)}{\prod\limits_{j=1}^{q} \Gamma(\mu_j-\lambda_h)}
$$

$$
\times\, {}_{q+1}F_{p-1}\left[\begin{array}{c}\lambda_h, 1-\mu_1+\lambda_h,\dots,1-\mu_q+\lambda_h;\\ 1-\lambda_1+\lambda_h,\dots * \dots,1-\lambda_p+\lambda_h;\end{array} \frac{(-1)^{q-p+1}}{z}\right]. \tag{2.9}
$$

*Here* $\lambda_j - \lambda_h \notin \mathbb{Z}_0^-; j \neq h, j = \overline{1,p}, h \in \mathbb{N}_0$*, and the asterisk indicates the omission of the term* $1 \equiv 1 - \lambda_h + \lambda_h$*.*

*Proof.* Taking $\rho_1 = \cdots = \rho_p = \sigma_1 = \cdots = \sigma_q = 1$ in (2.7) we arrive at

$$
{}_pF_q\left[\begin{array}{c}\lambda_1,\dots,\lambda_p;\\ \mu_1,\dots,\mu_q;\end{array} z\right]
$$

$$
= \mathbf{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \cdot \sum_{h=1}^{p}\sum_{r=0}^{\infty} \frac{\Gamma\big(\lambda_h+r\big) \prod\limits_{j\neq h, j=1}^{p} \Gamma(\lambda_j-\lambda_h-r)}{\prod\limits_{j=1}^{q} \Gamma(\mu_j-\lambda_h-r)} \frac{(-1)^{\lambda_h}}{r!} z^{-\lambda_h-r}. \tag{2.10}
$$

By employing the formula

$$
(g)_r = \frac{(-1)^r}{(1-g)_r}, \qquad r \in \mathbb{N},\, g \notin \mathbb{Z}\,,
$$

we get

$$
\prod_{j\neq h, j=1}^{p} \Gamma(\lambda_j-\lambda_h-r) = \prod_{j\neq h, j=1}^{p} \Gamma(\lambda_j-\lambda_h-r)(\lambda_j-\lambda_h)_r
$$

$$
= (-1)^{r(p-1)} \prod_{j\neq h, j=1}^{p} \frac{\Gamma(\lambda_j-\lambda_h)}{(1-\lambda_j+\lambda_h)_r}. \tag{2.11}
$$

Similarly, we have

$$
\prod_{j=1}^{q} \Gamma(\mu_j-\lambda_h-r) = (-1)^{rq} \prod_{j=1}^{q} \frac{\Gamma(\mu_j-\lambda_h)}{(1-\mu_j+\lambda_h)_r}. \tag{2.12}
$$

By using (2.11) and (2.12) the right–hand side of (2.10) becomes

$$\sum_{h=1}^{p}(-z)^{-\lambda_h}\,\frac{\Gamma(\lambda_h)\prod\limits_{j\neq h,j=1}^{p}\Gamma(\lambda_j-\lambda_h)}{\prod\limits_{j=1}^{q}\Gamma(\mu_j-\lambda_h)}$$
$$\times\; {}_{q+1}F_{p-1}\left[\begin{matrix}\lambda_h, 1-\mu_1+\lambda_h,\dots,1-\mu_q+\lambda_h; \\ 1-\lambda_1+\lambda_h,\cdots*\dots,1-\lambda_p+\lambda_h;\end{matrix}\;\frac{(-1)^{q-p+1}}{z}\right],$$

where $\lambda_j-\lambda_h\notin\mathbb{Z}_0^-; j\neq h, j=\overline{1,p}, h\in\mathbb{N}_0$, $|\arg(-z)|<\pi, |z|>1$ and the asterisk indicates the omission of the term $1\equiv 1-\lambda_h+\lambda_h$. □

**Remark 1.** It is interesting to observe that for $p=q+1$, (2.9) reduces to a known result in [6, p. 445, 7.23(77)]. Let us emphasize the case of Gaussian hypergeometric function, see [1, p. 63, Equation 2.10 (2)]:

$$\begin{aligned} {}_2F_1[\lambda,\nu\,;\mu\,;z] &= \frac{\Gamma(\nu-\lambda)}{\Gamma(\nu)\Gamma(\mu-\lambda)}\,(-z)^{-\lambda}\,{}_2F_1[\lambda,1-\mu+\lambda;1-\nu+\lambda;-z^{-1}] \\ &\quad+\frac{\Gamma(\lambda-\nu)}{\Gamma(\lambda)\Gamma(\mu-\nu)}\,(-z)^{-\nu}\,{}_2F_1[\nu,1-\mu+\nu;1-\mu+\nu\,;-z^{-1}], \end{aligned} \tag{2.13}$$

where $|\arg(-z)|<\pi$; $-\mu\notin\mathbb{N}_0$ and $\lambda-\nu$ is not an integer; $|z|>1$.

For $p=q=3$ in (2.7) with some changes in the parameters, the following representation holds true.

**Corollary 1.3.** [9, p. 500, Theorem 6] *We have*

$$\begin{aligned} &\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu;\nu}(z,s,a) \\ &\quad=\frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\,(-z)^{-\lambda/\rho}\sum_{n=0}^{\infty}\frac{\Gamma\big(\frac{\lambda+n}{\rho}\big)\Gamma\big(\mu-\frac{(\lambda+n)\sigma}{\rho}\big)}{\rho n!\Gamma(\nu-\frac{(\lambda+n)\kappa}{\rho})}\,\frac{[-(-z)^{-1/\rho}]^n}{(a-\frac{\lambda+n}{\rho})^s} \\ &\quad+\frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\mu)}\,(-z)^{-\mu/\sigma}\sum_{n=0}^{\infty}\frac{\Gamma\big(\frac{\mu+n}{\sigma}\big)\Gamma\big(\lambda-\frac{(\mu+n)\rho}{\sigma}\big)}{\sigma\, n!\,\Gamma(\nu-\frac{(\mu+n)\kappa}{\sigma})}\,\frac{[-(-z)^{-1/\sigma}]^n}{(a-\frac{\mu+n}{\sigma})^s}, \end{aligned} \tag{2.14}$$

*where* $|\arg(-z)|<\pi; \Re\{s\},\Re\{a\}>0; \lambda,\mu\in\mathbb{C}; \nu\in\mathbb{C}\backslash\mathbb{Z}_0^-$*;* $\min\{\rho,\sigma,\kappa\}>0$*; either* $|z|>\delta=\kappa^\kappa\rho^{-\rho}\sigma^{-\sigma}$*, or* $|z|=\delta$ *and* $\Re\{s+\nu-\lambda-\mu\}>1$ *and the series* (2.14) *converge.*

**Remark 2.** Two special cases of (2.14) are worth mentioning: **(i)** when $\rho=\sigma=\kappa=1$, then (1.4) (and *a fortiori* (2.14)) reduces to the generalized Hurwitz–Lerch Zeta function $\Phi_{\lambda,\mu;nu}(a,z,s)$ due to Garg *et al.* [2]; **(ii)** If we set $\rho=\lambda=1$ in (1.4), then (2.14) reduces to the generalized Hurwitz–Lerch Zeta function $\Phi^{(\sigma,\kappa)}_{\mu,\nu}(a,z,s)$ by Lin and Srivastava [5].

## 3. Asymptotic expansions

Under the hypetheses of Corollary 1.3 (that is originally [9, p. 500, Theorem 6]), we have the asymptotic behavior formula

$$\Phi_{\lambda,\mu;\nu}^{(\rho,\sigma,\kappa)}(z,s,a) \sim \mathcal{A}(-z)^{-\lambda/\rho} + \mathcal{B}(-z)^{-\mu/\sigma}, \qquad |z| \to \infty,\ |\arg(-z)| < \pi\,,$$

where the exact values of the constants are [9, p. 500, Eqs. (4.3), (4.4)]

$$\mathcal{A} = \frac{\rho^{s-1}\Gamma(\nu)\Gamma\left(\frac{\lambda}{\rho}\right)\Gamma(\frac{\mu\rho-\lambda\sigma}{\rho})}{(a\rho-\lambda)^s\Gamma(\lambda)\Gamma(\mu)\Gamma(\frac{\nu\rho-\lambda\kappa}{\rho})}\,, \quad \mathcal{B} = \frac{\sigma^{s-1}\Gamma(\nu)\Gamma\left(\frac{\mu}{\sigma}\right)\Gamma(\frac{\lambda\sigma-\mu\rho}{\sigma})}{(a\sigma-\mu)^s\Gamma(\lambda)\Gamma(\mu)\Gamma(\frac{\nu\sigma-\mu\kappa}{\sigma})}\,.$$

In this section, asymptotic expansion of the generalized Hurwitz–Lerch Zeta functions are investigated together with some its special cases associated with the Fox–Wright function and the generalized hypergeometric function.

**Theorem 2.** *Under the conditions of Theorem* 1*, the generalied Hurwich–Lerch Zeta function has the following asymptotic expansion as* $|z| \to \infty$ *and* $|\arg(-z)| < \pi$*:*

$$\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z,s,a) = \boldsymbol{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \cdot \sum_{h=1}^{p} \frac{\rho_h^{s-1}\Gamma\big(\frac{\lambda_h}{\rho_h}\big) \prod\limits_{j\neq h, j=1}^{p} \Gamma\big(\lambda_j - \rho_j \frac{\lambda_h}{\rho_h}\big)}{\prod\limits_{j=1}^{q} \Gamma\big(\mu_j - \sigma_j \frac{\lambda_h}{\rho_h}\big) \cdot (\lambda_h + a\rho_h)^s} (-z)^{-\frac{\lambda_h}{\rho_h}} \big(1 + \mathscr{O}\big(z^{-\frac{1}{\rho_h}}\big)\big)\,;$$

*in particular*

$$\Phi_{\boldsymbol{\lambda};\boldsymbol{\mu}}^{(\boldsymbol{\rho},\boldsymbol{\sigma})}(z,s,a) = \mathscr{O}(z^{-a^*}), \qquad |z| \to \infty,\ |\arg(-z)| < \pi\,,$$

*where*

$$a^* = \min_{1\le h\le p} \frac{\Re\{\lambda_h\}}{\rho_h}\,.$$

The proof of this asymptotic expansion result is the straightforward consequence of Theorem 1, so it is omitted. Further letting $s \to 0$ in Theorem 2, we conclude the following results.

**Corollary 2.1.** *Under the conditions of Theorem 2, the Fox-Wright hypergeometric function has the following asymptotic expansion:*

$$
\begin{aligned}
{}_p\Psi_q \left[ \begin{matrix} (\lambda_1,\rho_1),\dots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\dots,(\mu_q,\sigma_q); \end{matrix} \quad z \right] &= \boldsymbol{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \cdot {}_p\Psi_q^* \left[ \begin{matrix} (\lambda_1,\rho_1),\dots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\dots,(\mu_q,\sigma_q); \end{matrix} \quad z \right] \\
&= \boldsymbol{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \cdot \sum_{h=1}^{p} \frac{\Gamma\big(\frac{\lambda_h}{\rho_h}\big) \prod\limits_{j\neq h, j=1}^{p} \Gamma\big(\lambda_j - \rho_j \frac{\lambda_h}{\rho_h}\big)}{\rho_h \prod\limits_{j=1}^{q} \Gamma\big(\mu_j - \sigma_j \frac{\lambda_h}{\rho_h}\big)} (-z)^{-\frac{\lambda_h}{\rho_h}} \big(1 + \mathscr{O}\big(z^{-\frac{1}{\rho_h}}\big)\big)\,,
\end{aligned}
$$

*when* $|z| \to \infty$, $|\arg(-z)| < \pi$. *Also, we have*

$$
{}_p\Psi_q \left[ \begin{matrix} (\lambda_1,\rho_1),\dots,(\lambda_p,\rho_p); \\ (\mu_1,\sigma_1),\dots,(\mu_q,\sigma_q); \end{matrix} \quad z \right] = \mathscr{O}(z^{-a^*})\,,
$$

*where* $a^*$ *remains the same as above.*

**Corollary 2.2.** *Under the conditions* $\lambda_j - \lambda_h \notin \mathbb{Z}_0^-; j \neq h, j = \overline{1,p}, h \in \mathbb{N}_0$, *we have*

$$
\begin{aligned}
&{}_pF_q \left[ \begin{matrix} \lambda_1,\dots,\lambda_p; \\ \mu_1,\dots,\mu_q; \end{matrix} \quad z \right] \\
&\qquad = \boldsymbol{\Gamma}_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \cdot \big(1 + \mathscr{O}\big(z^{-1}\big)\big) \sum_{h=1}^{p} \frac{\Gamma(\lambda_h) \prod\limits_{j\neq h, j=1}^{p} \Gamma(\lambda_j - \lambda_h)}{\prod\limits_{j=1}^{q} \Gamma(\mu_j - \lambda_h)} (-z)^{-\lambda_h}\,,
\end{aligned}
$$

*for* $|z| \to \infty$, $|\arg(-z)| < \pi$. *Also, we have*

$$
{}_pF_q \left[ \begin{matrix} \lambda_1,\dots,\lambda_p; \\ \mu_1,\dots,\mu_q; \end{matrix} \quad z \right] = \mathscr{O}(z^{-b^*})\,,
$$

*where* $b^* = \min_{1\le h\le p} \Re\{\lambda_h\}$.

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Ram K. Saxena
Department of Mathematics and Statistics
Jai Narain Vyas University
Jodhpur 342004, Rajasthan
India
ram.saxena@yahoo.com

Tibor K. Pogány
Faculty of Maritime Studies
University of Rijeka
Rijeka 51000
Croatia
poganj@brod.pfri.hr