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FUNCTIONS COMMUTING WITH AN ARBITRARY FIXED BIJECTION

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Abstract. In this note a characterization of all bijective functions commuting with a fixed bijection $\varphi : X \to X$, where X is an arbitrary nonempty set, is given.

Introduction

One can find commuting functions in many papers devoted to functional equations or iteration theory, see for example [1], [2], [3], [5], [6]. Therefore, the problem of describing commuting functions with a given one seems to be interesting. The aim of this paper is a solution of the commutativity problem for all bijective functions. We present the form of commuting functions separately for closed and open orbits (see [4], page 15) of a given bijection $\varphi : X \to X$.

1. Functions commuting with a solution of Babbage functional equation

1.1. **Notations, definitions, lemma.** Let us start with introducing some notations and definitions. Let X be an arbitrary nonempty set. Let $\varphi : X \to X$ satisfy the Babbage equation (described in [4], page 288), this means

$$\varphi^n(x) = x, \qquad n \geq 2, \tag{1.1}$$

(by φ^n we denote n-th iteration).

Let us define

$$D := \{m : \ m \text{ divides } n\}$$

and

$$X^{(m)} := \{x \in X : \ \varphi^m(x) = x \text{ and } \varphi^k(x) \neq x, \text{ for every } 1 \leq k < m\}.$$

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Lemma 1.1 ([5] or [7])**.** *The sets $\{X^{(m)} :\ m \in D\}$ are pairwise disjoint and the equality*
$$\bigcup_{m\in D} X^{(m)} = X$$
holds.

For convenience of the reader we quote the proof from the paper [7].

Proof. It is evident that the sets $\{X^{(m)}\}_{m\in D}$ are pairwise disjoint and $\bigcup_{m\in D} X^{(m)} \subset X$. To prove the reverse inclusion, we reason as follows: if $x \in X$ and $p \in \{1, 2, \ldots, n\}$ is the minimal number such that $\varphi^p(x) = x$ then, evidently, for $t \in \mathbf{N} \cup \{0\}$ satisfying the inequalities
$$\frac{n-p}{p} \le t \le \frac{n-1}{p},$$
we have
$$1 \le n - tp \le p \quad \text{and} \quad \varphi^{tp}(x) = x.$$
Therefore
$$\varphi^{n-tp}(x) = \varphi^{n-tp}(\varphi^{tp}(x)) = \varphi^n(x) = x.$$
Hence $n - tp = p$, whence $p(t + 1) = n$. □

For every $m \in D$, by S_m, we denote an arbitrary selection of the family of orbits
$$\Big\{\{x, \varphi(x), \varphi^2(x), \ldots, \varphi^{m-1}(x)\} :\ x \in X^{(m)}\Big\}.$$

1.2. **Main constructions for a solution of the Babbage equation.**

Proposition 1.1. *Let us suppose that $m \in D$ is fixed, $X = X^{(m)}$ and $S = S_m$. Let $g^* : S \to S$ be an arbitrary bijection. Moreover, let $k : X \to \{0, 1, \ldots, m-1\}$ be defined as follows*
$$k(x) := k \quad \Leftrightarrow \quad \varphi^k(x) \in S. \tag{1.2}$$
The function $g : X \to X$ defined by the formula
$$g(x) := \Big[\varphi^{m-k(x)} \circ g^* \circ \varphi^{k(x)}\Big](x) \tag{1.3}$$
commutes with the function φ.

Proof. Firstly, let us remark that (1.2) implies $k(\varphi(x)) = k(x) - 1$ if $k(x) \neq 0$ and $k(\varphi(x)) = m - 1$ if $k(x) = 0$. We have for $k(x) \neq 0$
$$g\Big(\varphi(x)\Big) = \Big[\varphi^{m-k(\varphi(x))} \circ g^* \circ \varphi^{k(\varphi(x))}\Big]\Big(\varphi(x)\Big) = \Big[\varphi^{m-k(\varphi(x))} \circ g^* \circ \varphi^{k(\varphi(x))+1}\Big](x)$$
$$= \Big[\varphi^{m-k(x)+1} \circ g^* \circ \varphi^{k(x)-1+1}\Big](x) = \varphi\Big(\Big[\varphi^{m-k(x)} \circ g^* \circ \varphi^{k(x)}\Big](x)\Big) = \varphi\Big(g(x)\Big).$$

Similarly, for $k(x)=0$ we have

$$g\Big(\varphi(x)\Big)=\Big[\varphi^{m-k(\varphi(x))}\circ g^*\circ\varphi^{k(\varphi(x))}\Big]\Big(\varphi(x)\Big)=\Big[\varphi^{m-k(\varphi(x))}\circ g^*\circ\varphi^{k(\varphi(x))+1}\Big](x)$$

$$=\Big[\varphi^{m-m+1}\circ g^*\circ\varphi^{m-1+1}\Big](x)=\varphi\Big(\Big[\varphi^{m-k(x)}\circ g^*\circ\varphi^{k(x)}\Big](x)\Big)=\varphi\Big(g(x)\Big).$$

□

Remark 1.2. Let us remark that in the case $m=1$ we have $\varphi=id_X$, $S=X$ and by the formula (1.3) we get $g=g^*:X\to X$, i.e. every bijection commutes with the identity function.

Proposition 1.2. *The function g given by the equality $g=\bigcup_{m\in D} g_m$, where $g_m: X^{(m)}\to X^{(m)}$ has the form* (1.3) *commutes with the function $\varphi: X\to X$ such that eq.* (1.1) *holds.*

The proof results immediately from $\varphi(X^{(m)})=X^{(m)}$, Lemma 1.1 and Proposition 1.1.

Example 1.3. Using Propositions 1.1 and 1.2 one can easily obtain the following: if $g^*:]\frac{1}{2},\infty[\to]\frac{1}{2},\infty[$ is an arbitrary bijection then the function

$$g(x)=\begin{cases} g^*(x) & \text{for} \quad x>\frac{1}{2},\\ \frac{1}{2} & \text{for} \quad x=\frac{1}{2},\\ 1-g^*(1-x) & \text{for} \quad x<\frac{1}{2},\end{cases} \tag{1.4}$$

commutes with the function $\varphi(x)=1-x$, for $x\in\mathbb{R}$. Particularly, the functions

$$g(x)=\begin{cases} \frac{1}{2}+\frac{2}{2x-1} & \text{for} \quad x>\frac{1}{2},\\ \frac{1}{2} & \text{for} \quad x=\frac{1}{2},\\ \frac{1}{2}-\frac{2}{1-2x} & \text{for} \quad x<\frac{1}{2},\end{cases} \qquad g(x)=\begin{cases} \frac{1}{2}+\ln(x+\frac{1}{2}) & \text{for} \quad x>\frac{1}{2},\\ \frac{1}{2} & \text{for} \quad x=\frac{1}{2},\\ \frac{1}{2}-\ln(\frac{3}{2}-x) & \text{for} \quad x<\frac{1}{2},\end{cases}$$

commute with the function $\varphi(x)=1-x$, for $x\in\mathbb{R}$.

Example 1.4. Let $X=\mathbb{R}\setminus\{0\}$ and $\varphi(x):=\frac{1}{x}, x\in X$. Let $g_1^*:\{-1,1\}\to\{-1,1\}$ be an arbitrary bijection. Let us consider the following cases:

a) $S:=]-1,0[\,\cup\,]0,1[$, or
b) $S:=]-1,0[\,\cup\,]1,+\infty[$, or
c) $S:=]-\infty,-1[\,\cup\,]0,1[$, or
d) $S:=]-\infty,-1[\,\cup\,]1,+\infty[$.

Let $g_2^*:S\to S$ be an arbitrary bijection. Using Propositions 1.1 and 1.2 the function

$$g(x)=\begin{cases} g_1^*(x) & \text{for} \quad x\in\{-1,1\},\\ g_2^*(x) & \text{for} \quad x\in S,\\ \frac{1}{g_2^*(\frac{1}{x})} & \text{for} \quad x\in X\setminus S,\end{cases} \tag{1.5}$$

commutes with the function $\varphi(x)$. Particularly, the functions $g(x) = x^n$, $g(x) = \frac{1}{x^n}$,

$$g(x) = \begin{cases} 1 & \text{for } x = -1, \\ -1 & \text{for } x = 1, \\ \sin\frac{\pi}{2}x & \text{for } x \in]-1,0[\cup]0,1[, \\ \frac{1}{\sin\frac{\pi}{2x}} & \text{for } x \in]-\infty,-1[\cup]1,+\infty[, \end{cases}$$

$$g(x) = \begin{cases} id_{\{-1,1\}} & \text{for } x \in \{-1,1\}, \\ e^{x+1} & \text{for } x \in]-\infty,-1[, \\ \ln x - 1 & \text{for } x \in]0,1[, \\ \frac{1}{e^{\frac{1}{x}+1}} & \text{for } x \in]-1,0[, \\ \frac{1}{\ln\frac{1}{x}-1} & \text{for } x \in]1,+\infty[, \end{cases}$$

commute with the function $\varphi(x) = \frac{1}{x}$, for $x \in \mathbb{R} \setminus \{0\}$.

Example 1.5. Let A and Y be arbitrary nonempty sets and $cardY \geq 12$. Let us write $I_p := \{0,1,\dots,p\}$, for $p = 0,1,\dots$. Let $X := A \cup B$, where $B := \{\sigma_i^j \in Y : i \in I_3, j \in I_4\}$ and $cardB = 12$. Let $\varphi \in X^X$ be defined by

$$\varphi(x) := \begin{cases} x, & \text{if } x \in A, \\ \\ \sigma^j_{(i+1)_4}, & \text{if } x = \sigma_i^j,\ i \in I_3, j \in I_4 \end{cases}$$

where $(i+1)_4$ denotes the remainder of division of $i+1$ by 4. Then $\varphi^4(x) = x, x \in X$. In this case we have $n = 4$, $X^{(1)} = A$, $X^{(2)} = \emptyset$, $X^{(4)} = B$. Let $g_1^* : A \to A$ be an arbitrary bijection. Let $S := \{\sigma_0^j : j \in I_4\}$. Take the bijection $g_2^* : S \to S$ defined as follows: $g_2^*(\sigma_0^j) := \sigma_0^{(j+1)_5}$. According to Propositions 1.1 and 1.2 the function $g : X \to X$ defined by the formula

$$g(x) = \begin{cases} g_1^*(x) & \text{for } x \in A, \\ \sigma^{(j+1)_5}_{(i)_4} & \text{for } x = \sigma_i^j \in B, \end{cases}$$

commutes with φ.

We define the relation in X by

$$x \sim y \quad \Leftrightarrow \quad \varphi^p(x) = y, \text{ for a } p \in \{0,1,\dots,m-1\}.$$

One can easily observe that it is an equivalence relation.

Let S be a selection of the family of orbits $\{\{x,\varphi(x),\varphi^2(x),\dots,\varphi^{m-1}(x)\} : x \in X\}$. Then there exists a unique function $h_S : X \to S$ such that $h_S(x) \sim x$. Note that $h_S(\varphi(x)) = h_S(x)$.

Theorem 1.6. *Let X be an arbitrary nonempty set and $n \geq 2$. Let $\varphi : X \to X$ be a solution of the Babbage equation* (1.1) *and suppose that $X =$*

$X^{(m)}$ for a fixed $m \in D$. The bijection $g : X \to X$ commutes with the solution φ if and only if there exist a selection S of the family of orbits $\{\{x, \varphi(x), \varphi^2(x), \dots, \varphi^{m-1}(x)\} : x \in X\}$, a bijection $g^* : S \to S$ and a function $p : S \to \{0, 1, \dots, m-1\}$ such that g has the form

$$g(x) := \Big[\varphi^{m-k(x)+p(h_S(x))} \circ g^* \circ \varphi^{k(x)}\Big](x) \tag{1.6}$$

where $k : X \to \{0, 1, \dots, m-1\}$ is defined by (1.2).

Proof of the "if" part. Evidently, we have if $x \sim h_S(x)$, $h_S(\varphi(x)) = h_S(x)$ and further reasoning is similar as in Proposition 1.1. We have for $k(x) \neq 0$

$$\begin{aligned} g\Big(\varphi(x)\Big) &= \Big[\varphi^{m-k(\varphi(x))+p(h_S(\varphi(x)))} \circ g^* \circ \varphi^{k(\varphi(x))}\Big]\Big(\varphi(x)\Big) \\ &= \Big[\varphi^{m-k(\varphi(x))+p(h_S(\varphi(x)))} \circ g^* \circ \varphi^{k(\varphi(x))+1}\Big](x) \\ &= \Big[\varphi^{m-k(x)+1+p(h_S(x))} \circ g^* \circ \varphi^{k(x)-1+1}\Big](x) \\ &= \varphi\Big(\Big[\varphi^{m-k(x)+p(h_S(x))} \circ g^* \circ \varphi^{k(x)}\Big](x)\Big) = \varphi\Big(g(x)\Big). \end{aligned}$$

Similarly, for $k(x) = 0$ we have

$$\begin{aligned} g\Big(\varphi(x)\Big) &= \Big[\varphi^{m-k(\varphi(x))+p(h_S(\varphi(x)))} \circ g^* \circ \varphi^{k(\varphi(x))}\Big]\Big(\varphi(x)\Big) \\ &= \Big[\varphi^{m-k(\varphi(x))+p(h_S(\varphi(x)))} \circ g^* \circ \varphi^{k(\varphi(x))+1}\Big](x) \\ &= \Big[\varphi^{m-m+1+p(h_S(x))} \circ g^* \circ \varphi^{m-1+1}\Big](x) \\ &= \varphi\Big(\Big[\varphi^{m-k(x)+p(h_S(x))} \circ g^* \circ \varphi^{k(x)}\Big](x)\Big) = \varphi\Big(g(x)\Big). \end{aligned}$$

□

Proof of the "only if" part. Let $g : X \to X$ be a bijection commuting with φ. Take an arbitrary selection S of the family of orbits $\{\{x, \varphi(x), \varphi^2(x), \dots, \varphi^{m-1}(x)\} : x \in X\}$. One can observe that every function $g^* : S \to S$ defined in such a way that $g^*(s) = t$ for a t if and only if $g(s) \sim t$, is a bijection. Let $x \in X$ and put $s := h_S(x)$. There exists $p(s) \in \{0, 1, \dots, m-1\}$ such that $\varphi^{p(s)}(g^*(s)) = g(s)$. Moreover we have

$$\varphi^{k(x)}(x) = s, \; \varphi^{m-k(x)}(s) = x.$$

From the above

$$\begin{aligned} \Big[\varphi^{m-k(x)+p(s)} \circ g^* \circ \varphi^{k(x)}\Big](x) &= \varphi^{m-k(x)+p(s)}\Big[g^*\Big(s\Big)\Big] = \varphi^{m-k(x)+p(s)}\Big(t\Big) \\ &= \varphi^{m-k(x)}\Big(g(s)\Big) = g\Big(\varphi^{m-k(x)}(s)\Big) = g(x). \end{aligned}$$

□

Theorem 1.7. *Let X be an arbitrary nonempty set and $n \geq 2$. Let $\varphi : X \to X$ satisfy the Babbage equation* (1.1)*. The bijection $g : X \to X$ commutes with the solution φ if and only if $g = \bigcup_{m \in D} g_m$, where $g_m : X^{(m)} \to X^{(m)}$ have the form* (1.6)*.*

The proof results immediately from $\varphi(X^{(m)}) = X^{(m)}$, Lemma 1.1 and Theorem 1.6.

Example 1.8. Let $X = B$, where B is defined as in Example 1.5. Let φ be defined by $\varphi(\sigma_i^j) = \sigma_{(i+1)_4}^j$, for $i \in I_3, j \in I_4$. Let S be the same as in Example 1.5, so $S := \{\sigma_0^j : j \in I_4\}$. Take the bijection $g^* : S \to S$ defined as follows: $g^*(\sigma_0^j) := \sigma_0^{(j+2)_5}$. Define the function $p : S \to \{0, 1, 2, 3\}$ by $p(\sigma_0^j) = (j)_4$. By the formula (1.6) we get the function $g(\sigma_i^j) = \sigma_{(j+i)_4}^{(j+2)_5}$. One can easily verify that g commutes with φ.

2. The case of open orbits

Let X be an arbitrary nonempty set. Let $\varphi : X \to X$ be a bijection such that

$$\varphi^n(x) \neq x, \quad \forall n \geq 1. \tag{2.1}$$

In this case all orbits are as follows:

$$\{\dots, \varphi^{-3}(x), \varphi^{-2}(x), \varphi^{-1}(x), x, \varphi(x), \varphi^2(x), \varphi^3(x), \varphi^4(x), \dots\}.$$

Now, we define the equivalence relation in X by

$$x \sim y \quad \Leftrightarrow \quad \varphi^p(x) = y, \text{ for a } p \in \mathbb{Z}.$$

Let S be a selection of the family of orbits

$$\Big\{\{\dots, \varphi^{-3}(x), \varphi^{-2}(x), \varphi^{-1}(x), x, \varphi(x), \varphi^2(x), \varphi^3(x), \varphi^4(x), \dots\} : x \in X\Big\}.$$

Then there exists a unique function $h_S : X \to S$ such that $h_S(x) \sim x$. Note that $h_S(\varphi(x)) = h_S(x)$.

Theorem 2.1. *Let X be an arbitrary nonempty set. Let $\varphi : X \to X$ be a bijective function satisfying* (2.1)*. The bijection $g : X \to X$ commutes with the function φ if and only if there exist a selection S of the family of orbits*

$$\Big\{\{\dots, \varphi^{-3}(x), \varphi^{-2}(x), \varphi^{-1}(x), x, \varphi(x), \varphi^2(x), \varphi^3(x), \varphi^4(x), \dots\} : \ x \in X\Big\},$$

a bijection $g^* : S \to S$ and a function $p : S \to \mathbb{Z}$ such that g has the form

$$g(x) := \Big[\varphi^{-k(x)+p(h_S(x))} \circ g^* \circ \varphi^{k(x)}\Big](x) \tag{2.2}$$

where $k : X \to \mathbb{Z}$ is defined by (1.2)*.*

Proof of the "if" part. Evidently, we have if $x \sim h_S(x)$, $h_S(\varphi(x)) = h_S(x)$ and further reasoning is similar as in Proposition 1.1 and in Theorem 1.6. Indeed, we have $k(\varphi(x)) = k(x) - 1$ for all $x \in X$, so

$$\begin{aligned} g\Big(\varphi(x)\Big) &= \Big[\varphi^{-k(\varphi(x))+p(h_S(\varphi(x)))} \circ g^* \circ \varphi^{k(\varphi(x))}\Big]\Big(\varphi(x)\Big) \\ &= \Big[\varphi^{-k(\varphi(x))+p(h_S(\varphi(x)))} \circ g^* \circ \varphi^{k(\varphi(x))+1}\Big](x) \\ &= \Big[\varphi^{-k(x)+1+p(h_S(x))} \circ g^* \circ \varphi^{k(x)-1+1}\Big](x) \\ &= \varphi\Big(\Big[\varphi^{-k(x)+p(h_S(x))} \circ g^* \circ \varphi^{k(x)}\Big](x)\Big) = \varphi\Big(g(x)\Big). \end{aligned}$$

□

Proof of the "only if" part. Let $g : X \to X$ be a bijection commuting with φ. Take an arbitrary selection S of the family of orbits

$$\Big\{\{\dots, \varphi^{-3}(x), \varphi^{-2}(x), \varphi^{-1}(x), x, \varphi(x), \varphi^2(x), \varphi^3(x), \varphi^4(x), \dots\} : \; x \in X\Big\}.$$

One can observe that every function $g^* : S \to S$ defined in such a way that $g^*(s) = t$ for a t if and only if $g(s) \sim t$, is a bijection. Let $x \in X$ and put $s := h_S(x)$. There exists $p(s) \in \mathbb{Z}$ such that $\varphi^{p(s)}(g^*(s)) = g(s)$. Moreover we have

$$\varphi^{k(x)}(x) = s, \; \varphi^{-k(x)}(s) = x.$$

From the above

$$\Big[\varphi^{-k(x)+p(s)} \circ g^* \circ \varphi^{k(x)}\Big](x) = \varphi^{-k(x)+p(s)}\Big[g^*\Big(s\Big)\Big] = \varphi^{-k(x)+p(s)}\Big(t\Big)$$

$$= \varphi^{-k(x)}\Big(g(s)\Big) = g\Big(\varphi^{-k(x)}(s)\Big) = g(x).$$

□

3. Final remarks, examples and problems

Remark 3.1. The results of this paper can be useful in finding all functions commuting with a given one. Namely, by Theorem 1.7, we have a ready to use description of commuting functions for all closed orbits of the given bijective function. Similarly, in Theorem 2.1, we have a ready to use construction of commuting functions for all open orbits of the given bijective function. Since the union of the closed orbits X_I and the union of open orbits X_{II} are disjoint and $\varphi(X_I) = X_I$, $\varphi(X_{II}) = X_{II}$, then the presented results are complete for the characterization of all commutative functions with a given bijection φ. More precisely, the domain X of the bijection φ can be decomposed onto disjoint sets as follows

$$X = \bigcup_{n=1}^{\infty} X^{(n)} \quad \cup \quad X^{(\infty)},$$

where $X^{(n)} = \{x \in X : \ \varphi^n(x) = x \ \text{ and } \ \varphi^k(x) \neq x \text{ for } 0 < k < n\}$ and $X^{(\infty)} = \{x \in X : \ \varphi^n(x) \neq x \text{ for } n \geq 1\}$. Theorems 1.6, 1.7 and Remark 1.2 describe the form of commuting function g on the sets $X^{(n)}$, $n \geq 1$ and Theorem 2.1 describes the form of g on the set $X^{(\infty)}$.

Example 3.2. Let us consider the bijection $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula $\varphi(x) = 3x$. Using the presented theorems we can construct all functions g commuting with φ. We have $\mathbb{R}_+ = \mathbb{R}_+^{(1)} \cup \mathbb{R}_+^{(\infty)}$, where $\mathbb{R}_+^{(1)} = \{0\}$ and $\mathbb{R}_+^{(\infty)} = \mathbb{R}_+ \setminus \{0\}$. Therefore $g(0) = 0$ and to define g on the set $\mathbb{R}_+^{(\infty)}$ we use Theorem 2.1. The interval $[1, 3[$ forms a selection of the family of orbits

$$\Big\{\{\ldots, \varphi^{-3}(x), \varphi^{-2}(x), \varphi^{-1}(x), x, \varphi(x), \varphi^2(x), \varphi^3(x), \varphi^4(x), \ldots\} : \ x \in \mathbb{R}_+^{(\infty)}\Big\}.$$

Moreover

$$\mathbb{R}_+^{(\infty)} = \bigcup_{l \in \mathbb{Z}} [3^l, 3^{l+1}[$$

and for $x \in [3^l, 3^{l+1}[$, $l \in \mathbb{Z}$ - according with (1.2) - we have $k(x) = -l$. The formula below gives a family of functions which commute with φ.

$$g(x) = \begin{cases} 0 & \text{for} \quad x = 0, \\ 3^{p(s)+l} \cdot g^*(3^{-l}x) & \text{for} \quad x \in [3^l, 3^{l+1}[\text{ and } l \in \mathbb{Z}, \end{cases}$$

where $g^* : [1, 3[\to [1, 3[$ is an arbitrary bijection and $p : [1, 3[\to \mathbb{Z}$ is an arbitrary function. Taking the bijection $g^* : [1, 3[\to [1, 3[$ defined by the formula $g^*(x) = 1 + 2 \sin \frac{\pi}{4}(x - 1)$ and taking the function $p(s) = 7$, for $s \in [1, 3[$, we obtain from the above the form of the commuting function g as follows

$$g(x) = \begin{cases} 0 & \text{for} \quad x = 0, \\ 3^{7+l} \cdot [1 + 2 \sin \frac{\pi}{4}(3^{-l}x - 1)] & \text{for} \quad x \in [3^l, 3^{l+1}[\text{ and } l \in \mathbb{Z}. \end{cases}$$

Example 3.3. For the bijection $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula $\varphi(x) = x^3$ we have $\mathbb{R}_+ = \mathbb{R}_+^{(1)} \cup \mathbb{R}_+^{(\infty)}$, where $\mathbb{R}_+^{(1)} = \{0, 1\}$ and $\mathbb{R}_+^{(\infty)} = \mathbb{R}_+ \setminus \{0, 1\}$. Let $g_1^* : \{0, 1\} \to \{0, 1\}$ be an arbitrary bijection. Let $g_2^* : [\frac{1}{8}, \frac{1}{2}[\cup [2, 8[\to [\frac{1}{8}, \frac{1}{2}[\cup [2, 8[$ be an arbitrary bijection and $p : [\frac{1}{8}, \frac{1}{2}[\cup [2, 8[\to \mathbb{Z}$ be an arbitrary function. We have

$$\mathbb{R}_+^{(\infty)} = \bigcup_{l \in \mathbb{Z}} [2^{-3^{l+1}}, 2^{-3^l}[\ \cup \ \bigcup_{l \in \mathbb{Z}} [2^{3^l}, 2^{3^{l+1}}[$$

and $k(x) = -l$, for $x \in [2^{-3^{l+1}}, 2^{-3^l}[\cup[2^{3^l}, 2^{3^{l+1}}[$, $l \in \mathbb{Z}$. The formula below gives a family of functions which commute with φ.

$$g(x) = \begin{cases} g_1^*(x) & \text{for} \quad x \in \{0,1\}, \\ [g_2^*(x^{3^{-l}})]^{3^{p(s)+l}} & \text{for} \quad x \in [2^{-3^{l+1}}, 2^{-3^l}[\cup[2^{3^l}, 2^{3^{l+1}}[\text{and } l \in \mathbb{Z}. \end{cases}$$

Example 3.4. For the bijection $\varphi : \mathbb{R} \to \mathbb{R}$ given by the formula $\varphi(x) = -x^3$ we have $\mathbb{R} = \mathbb{R}^{(1)} \cup \mathbb{R}^{(2)} \cup \mathbb{R}^{(\infty)}$, where $\mathbb{R}^{(1)} = \{0\}$, $\mathbb{R}^{(2)} = \{-1, 1\}$ and $\mathbb{R}^{(\infty)} = \mathbb{R} \setminus \{-1, 0, 1\}$. Theorems 1.6, 1.7 and Remark 1.2 describe the form of the commuting function g on the sets $\mathbb{R}^{(1)}, \mathbb{R}^{(2)}$ and Theorem 2.1 describes the form of g on the set $\mathbb{R}^{(\infty)}$. Determining the parameters as in the previous examples we can obtain all functions commuting with the given function φ.

Problem 3.5. Characterize all functions commuting with a given finction $f : X \to X$ (not necessarily bijective), where X is an arbitrary nonempty set, particularly for $f : \mathbb{R} \to \mathbb{R}$.

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