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HYPERCYCLIC AND TOPOLOGICALLY MIXING PROPERTIES OF ABSTRACT TIME-FRACTIONAL EQUATIONS WITH DISCRETE SHIFTS

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Abstract. The most valuable theoretical results about hypercyclic and topologically mixing properties of some special subclasses of the abstract time-fractional equations of the following form:

$$\mathbf{D}_t^{\alpha_n}u(t)+A_{n-1}\mathbf{D}_t^{\alpha_{n-1}}u(t)+\cdots+A_1\mathbf{D}_t^{\alpha_1}u(t)=A_0\mathbf{D}_t^{\alpha}u(t),\ t>0,$$
$$u^{(k)}(0)=u_k,\ k=0,\cdots,\lceil\alpha_n\rceil-1. \quad (1)$$

where $n\in\mathbb{N}\setminus\{1\}$, $A_0,A_1,\cdots,A_{n-1}$ are closed linear operators acting on a separable infinite-dimensional complex Banach space E, $0\leq\alpha_1<\cdots<\alpha_n$, $0\leq\alpha<\alpha_n$, and $\mathbf{D}_t^{\alpha}$ denotes the Caputo fractional derivative of order α ([1]), have been recently clarified in [12]-[13]. In this paper, we continue the analysis contained in [12]-[13] by assuming that, for every $j\in\mathbb{N}_{n-1}$, the operator A_j is a certain function of unilateral backward shifts acting on weighted $l^1(\mathbb{C})$-spaces.

1. Introduction and preliminaries

The basic hypercyclic and topologically mixing properties of some special subclasses of the abstract time-fractional equations of the form (1) have been recently analyzed by the author in [12]-[13]. The blank hypothesis in [13] was that there exist complex constants $c_1,\cdots,c_{n-1}$ such that, for every $j\in\mathbb{N}_{n-1}$, the operator A_j satisfies the equality $A_j=c_jI$, where we denote by I the identity operator on E. In this paper, we shall consider topologically mixing solutions of the equation (1) with $A_0,A_1,\cdots,A_{n-1}$ being functions of unilateral backward shift operators. Here we would like to observe that various types of hypercyclic and topologically mixing properties of backward shift operators on Banach or Fréchet sequence spaces have been widely studied (cf. [2, 4, 6-8, 19-21] for further information in this direction). On the other hand, fractional differential equations have gained importance

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and popularity during the past three decades or so, mainly due to their applications in various fields of physics, chemistry, mathematical biology and engineering. Concerning the theory of abstract fractional equations and abstract Volterra integro-differential equations in Banach spaces, the references [1, 5, 10, 18, 22] are of crucial importance.

We use the standard notation throughout the paper. By $(E, ||\cdot||)$ we denote a separable infinite-dimensional complex Banach space; $A_0, \cdots, A_{n-1}$ denote closed linear operators acting on E, $n \in \mathbb{N} \setminus \{1\}$, $0 \leq \alpha_1 < \cdots < \alpha_n$ and $0 \leq \alpha < \alpha_n$. The space of continuous linear mappings from E into E is denoted by $L(E)$. For a closed linear operator B on E, we denote by $D(B)$ and $D_\infty(B)$ its domain and the set $\bigcap_{n\in\mathbb{N}} D(B^n)$, respectively. Given $s \in \mathbb{R}$ in advance, put $\lceil s \rceil := \inf\{k \in \mathbb{Z} : s \leq k\}$. The Gamma function is denoted by $\Gamma(\cdot)$ and the principal branch is always used to take the powers. Set $\mathbb{N}_l := \{1, \cdots, l\}$, $\mathbb{N}_l^0 := \{0, 1, \cdots, l\}$, $0^\zeta := 0$ and $g_\zeta(t) := t^{\zeta-1}/\Gamma(\zeta)$ ($\zeta > 0$, $t > 0$). Define $m_j := \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m := m_0 := \lceil \alpha \rceil$ and $\alpha_0 := \alpha$. Put, for every $i \in \mathbb{N}^0_{m_n-1}$, $D_i := \{j \in \mathbb{N}_{n-1} : m_j - 1 \geq i\}$.

We refer the reader to [15]-[16] for the most important facts concerning the well-posedness of problem (1), in particular, for the notions of mild (strong) solutions of (1). As a special case of (1), we quote the abstract Cauchy problem

$$(ACP_n) : \begin{cases} u^{(n)}(t) + A_{n-1}u^{(n-1)}(t) + \cdots + A_1 u'(t) + A_0 u(t) = 0, \ t \geq 0, \\ u^{(k)}(0) = u_k, \ k = 0, \cdots, n-1. \end{cases}$$

Fairly complete information on the general theory of abstract higher-order differential equations can be obtained by consulting the monograph [22] by T.-J. Xiao and J. Liang.

The following definition is generally enough for our purposes (cf. also [15, Definition 2.2] for the notion of various types of (C_1, C_2)-existence and uniqueness propagation families for (1)).

Definition 1.1. A sequence $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ of strongly continuous operator families in $L(E)$ is called a (global) resolvent propagation family for (1) if $R_i(0) = g_{i+1}(0)I$ for all $i \in \mathbb{N}^0_{m_n-1}$, $R_i(t)A_j \subseteq A_j R_i(t)$ for all $t \geq 0$, $i \in \mathbb{N}^0_{m_n-1}$ and $j \in \mathbb{N}^0_{n-1}$, and if the following functional equation holds:

$$\Big[R_i(\cdot)x - g_{i+1}(\cdot)x\Big] + \sum_{j\in D_i} g_{\alpha_n-\alpha_j} * \Big[R_i(\cdot)A_j x - g_{i+1}(\cdot)A_j x\Big]$$
$$+ \sum_{j\in \mathbb{N}_{n-1}\setminus D_i} \big(g_{\alpha_n-\alpha_j} * R_i(\cdot)A_j x\big)(\cdot)$$

$$= \begin{cases} \big(g_{\alpha_n-\alpha} * R_i(\cdot)A_0x\big)(\cdot), \ m-1 < i, \\ g_{\alpha_n-\alpha} * \Big[R_i(\cdot)A_0x - g_{i+1}(\cdot)A_0x\Big](\cdot), \ m-1 \geq i, \end{cases}$$

for any $i = 0, \cdots, m_n - 1$ and $x \in \bigcap_{0 \leq i \leq n-1} D(A_i)$.

The interested reader may consult the papers [11] and [15]-[16] for further information concerning some other types of resolvent families which can be useful in the analysis of (inhomogeneous) abstract Cauchy problems of the form (1). The notions of exponential boundedness and analyticity of resolvent propagation families will be understood in the sense of [15]; we shall always assume that the operator A_j is densely defined for all $j \in \mathbb{N}^0_{n-1}$ as well as that every single operator family $(R_i(t))_{t\geq 0}$ of the tuple $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ is non-degenerate, i.e., that the supposition $R_i(t)x = 0$, $t \geq 0$ implies $x = 0$. With the exception of Example 2.2(i) below, we shall always consider the case in which $\alpha = 0$; then the problem (1) has at most one mild (strong) solution ([15]).

Although the notion of a global ζ-times resolvent family ($\zeta > 0$) is a very special case of the general notion of a resolvent propagation family, it would be very useful to introduce this notion in a separate definition.

Definition 1.2. Let $\zeta > 0$, and let B be a closed densely defined linear operator on E. A strongly continuous operator family $(R_\zeta(t))_{t\geq 0}$ is called a ζ-times regularized resolvent family having B as the integral generator iff the following holds:

(i) $R_\zeta(t)B \subseteq BR_\zeta(t), \ t \geq 0, \ R_\zeta(0) = I,$ *and*
(ii) $R_\zeta(t)x = x + \int_0^t g_\zeta(t-s)BR_\zeta(s)x\,ds, \ t \geq 0, \ x \in D(B).$

Let $\beta > 0$. Denote by $E_\beta(z)$ the Mittag-Leffler function

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \ z \in \mathbb{C},$$

and by $\Phi_\gamma(t)$ the Wright function

$$\Phi_\gamma(t) := \mathcal{L}^{-1}\big(E_\gamma(-\lambda)\big)(t), \ t \geq 0,$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. It is well known that the function $\Phi_\gamma(t)$ can be extended to an entire function, and that there exists a finite constant $M > 0$ such that $0 \leq \Phi_\gamma(t) \leq M$, $t \geq 0$. For more details on the Mittag-Leffler and Wright functions, we refer the reader to [1] and references cited there.

2. Topological dynamics of certain classes of abstract time-fractional PDEs with unilateral backward shifts

Recall that E denotes a separable infinite-dimensional Banach space over the field of complex numbers. In the sequel, we shall assume that $A_0, A_1, \cdots, A_{n-1}$ are bounded linear operators acting on E as well as that $n \in \mathbb{N} \setminus \{1\}$, $0 \leq \alpha_1 < \cdots < \alpha_n$, $0 \leq \alpha < \alpha_n$, $m_j = \lceil \alpha_j \rceil$, $1 \leq j \leq n$, $m = m_0 = \lceil \alpha \rceil$, $\alpha_0 = \alpha$, and that there exists a global resolvent propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ for (1). Then we know (see [15]) that the unique mild solution of (1) is given by $u(t) = \sum_{i=0}^{m_n-1} R_i(t)x_i$, $t \geq 0$.

We need the following definition from [12]-[13].

Definition 2.1. Let $i \in \mathbb{N}^0_{m_n-1}$. Then it is said that $(R_i(t))_{t\geq 0}$ is:

(i) hypercyclic iff there exists $x \in E$ such that $\{R_i(t)x : t \geq 0\}$ is a dense subset of E; such an element is called a hypercyclic vector of $(R_i(t))_{t\geq 0}$,

(ii) topologically transitive iff for every $y,\ z \in E$ and for every $\varepsilon > 0$, there exist $x \in E$ and $t \geq 0$ such that $||y-x|| < \varepsilon$ and $||z - R_i(t)x|| < \varepsilon$,

(iii) topologically mixing iff for every $y,\ z \in E$ and for every $\varepsilon > 0$, there exists $t_0 \geq 0$ such that, for every $t \geq t_0$, there exists $x_t \in E$ such that $||y-x_t|| < \varepsilon$ and $||z - R_i(t)x_t|| < \varepsilon$.

In particular, the above definition specifies the basic hypercyclic properties of global ζ-times regularized resolvent families; observe also that the notions of hypercyclicity and topologically transitivity (mixing) can be introduced for an arbitrary strongly continuous operator family $(R(t))_{t\geq 0} \subseteq L(E)$. Recall that the topological transitivity of $(R_i(t))_{t\geq 0}$ for some $i \in \mathbb{N}^0_{m_n-1}$ implies that $(R_i(t))_{t\geq 0}$ is hypercyclic and that the set of all hypercyclic vectors of $(R_i(t))_{t\geq 0}$ is a dense G_δ-subset of E ([13]).

Before proceeding further, we would like to present two illustrate examples of abstract time-fractional equations (1) with topologically mixing solutions.

Example 2.2.

(i) ([9], [13]) Let X be a symmetric space of non-compact type and rank one, let $p > 2$, let the parabolic domain P_p, the operator $\Delta^\natural_{X,p}$ and the positive real number c_p possess the same meaning as in [9], and let $P(z) = \sum_{j=0}^n a_j z^j$, $z \in \mathbb{C}$ be a non-constant complex polynomial with $a_n > 0$. Suppose first $\zeta \in (1,2)$, $\pi - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} > 0$ and $\theta \in \left(n \arctan \frac{|p-2|}{2\sqrt{p-1}} + \zeta\frac{\pi}{2} - \pi, \pi - n \arctan \frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2}\right)$. Then $-e^{i\theta} P(\Delta^\natural_{X,p})$ is the integral generator of an exponentially bounded, analytic ζ-times regularized resolvent family $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ of angle

$\frac{1}{\zeta}\big(\pi - n\arctan\frac{|p-2|}{2\sqrt{p-1}} - \zeta\frac{\pi}{2} - |\theta|\big)$. Furthermore, the condition

$$-e^{i\theta}P\big(\operatorname{int}(P_p)\big) \ \cap \ \Big\{te^{\pm i\zeta\frac{\pi}{2}} : t \geq 0\Big\} \ \neq \emptyset$$

implies that $(R_{\zeta,\theta,P}(t))_{t\geq 0}$ is topologically mixing. Suppose now $n = 2$, $0 < a < 2$, $\alpha_2 = 2a$, $\alpha_1 = 0$, $\alpha = a$, $c_1 > 0$, $i = 0$ and $|\theta| < \min\big(\frac{\pi}{2} - n\arctan\frac{|p-2|}{2\sqrt{p-1}}, \frac{\pi}{2} - n\arctan\frac{|p-2|}{2\sqrt{p-1}} - \frac{\pi}{2}a\big)$. Then $D_0 = \emptyset$ and the operator $-e^{i\theta}P(\Delta^{\natural}_{X,p})$ is the integral generator of an exponentially bounded, analytic resolvent propagation family $((R_{\theta,P,0}(t))_{t\geq 0}, \cdots, (R_{\theta,P,\lceil 2a\rceil -1}(t))_{t\geq 0})$ of angle $\min\Big(\frac{\pi - n\arctan\frac{|p-2|}{2\sqrt{p-1}} - |\theta|}{a} - \frac{\pi}{2}, \frac{\pi}{2}\Big)$. The condition

$$-e^{i\theta}P\big(\operatorname{int}(P_p)\big) \ \cap \ \Big\{\big(it\big)^{a} + c_1\big(it\big)^{-a} : t \in \mathbb{R}\setminus\{0\}\Big\} \ \neq \emptyset$$

implies that $(R_{\theta,P,0}(t))_{t\geq 0}$ is topologically mixing.

(ii) ([3], [13]) Suppose $\zeta \in (0,1)$, $E = L^2(\mathbb{R})$ and $c > b/2 > 0$. Set $\Omega := \{\lambda \in \mathbb{C} : \Re\lambda < c - b/2\}$, $\phi \in E^* = E$ and $\mathcal{A}_c u := u'' + 2bxu' + cu$ acts on E with domain $D(\mathcal{A}_c) := \{u \in L^2(\mathbb{R}) \cap W^{2,2}_{loc}(\mathbb{R}) : \mathcal{A}_c u \in L^2(\mathbb{R})\}$. Then $\mathcal{A}_c$ is the integral generator of a topologically mixing ζ-times regularized resolvent family $(R_\zeta(t))_{t\geq 0}$, which cannot be hypercyclic provided $b < 0$ or $c \leq b/2$.

Let $\zeta > 0$, and let $(r_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers satisfying that there exists $M > 0$ such that $r_k r_{k+1}^{-1} \leq M$ for all $k \in \mathbb{N}$. Consider the weighted l^1-space

$$l^1_r := \Big\{\big(x_k\big)_{k\in\mathbb{N}} : x_k \in \mathbb{C}, \ \sum_{k=1}^{\infty} r_k|x_k| < \infty\Big\},$$

normed by

$$\Big\|\big(x_k\big)_{k\in\mathbb{N}}\Big\| := \sum_{k=1}^{\infty} r_k\big|x_k\big|, \quad \big(x_k\big)_{k\in\mathbb{N}} \in l^1_r.$$

Define now the unilateral backward shift $A : l^1_r \to l^1_r$ by $A(x_k)_{k\in\mathbb{N}} := (x_{k+1})_{k\in\mathbb{N}}$, $(x_k)_{k\in\mathbb{N}} \in l^1_r$. Clearly, $A \in L(l^1_r)$ and the norm of A can be majorized by the constant M mentioned above. Recall that H. R. Salas [19] has proved that the operator $I + A$ is hypercyclic. Details of his proof have been essentially used by W. Desch, W. Schappacher and G. F. Webb [4], where it has been shown that the strongly continuous semigroup $(T(t))_{t\geq 0}$, generated by A, is hypercyclic. Observe further that [1, Theorem 2.5] and its proof imply that the operator A is the integral generator of a global exponentially bounded ζ-times regularized resolvent family

$$\left(R^{\zeta}(t) \equiv \sum_{k=0}^{\infty} \frac{t^{\zeta k}}{\Gamma(\zeta k+1)} A^k\right)_{t\geq 0}.$$

A slight modification of the arguments given in the proof of [4, Theorem 5.2] implies that the following theorem holds good:

Theorem 2.3. *Let $\zeta > 0$, and let A be defined as above. Denote by $(R^{\zeta}(t))_{t\geq 0}$ the ζ-times regularized resolvent family generated by A. Then $(R^{\zeta}(t))_{t\geq 0}$ is topologically mixing.*

The importance of Theorem 2.3 lies in the fact that, for any arbitrarily large finite number $\zeta > 0$, we have the existence of a topologically mixing ζ-times regularized resolvent family on a Banach space, here concretely on l_r^1.

Now we would like to mention the following problem connected with the existence of topologically mixing solutions of the abstract Cauchy problem (ACP_n) : if there exist at least two indices $i,\ j \in \mathbb{N}_{n-1}^0$ such that the operators A_i and A_j are not scalar multiples of the identity operator, then we could not find in the existing literature an example of the abstract Cauchy problem (ACP_n) with topologically mixing solution $u(t)$. The main goal of following theorem is to show that, for every $n \geq 2$, there exists an example of the abstract Cauchy problem (ACP_n) with such properties. In order to help one to better understand the proof, we will consider separately the cases $n = 2$ and $n > 2$.

Theorem 2.4. *Let $A \in L(l_r^1)$ be as in the formulation of Theorem 2.3, and let $(T(t))_{t\geq 0}$ be the strongly continuous semigroup generated by A.*

(i) *Consider the abstract Cauchy problem*

$$(P_2) : \begin{cases} u''(t) - (2A - I)u'(t) + A(A - I)u(t) = 0,\ t \geq 0, \\ u(0) = x,\ u'(0) = y. \end{cases}$$

Then there exists a resolvent propagation family $((R_0(t))_{t\geq 0}$, $(R_1(t))_{t\geq 0})$ for (P_2), given by $R_0(t)x = T(t)(x-Ax)+e^{-t}T(t)Ax$ and $R_1(t)x = T(t)(1-e^{-t})x$ $(t \geq 0,\ x \in l_r^1)$. Furthermore, $(R_0(t))_{t\geq 0}$ and $(R_1(t))_{t\geq 0}$ are topologically mixing, and the operator family $(R_0(t)+R_1(t))_{t\geq 0}$ is also topologically mixing.

(ii) *Suppose $n > 2$ and $0 < c_1 < \cdots < c_{n-1}$. Consider the abstract Cauchy problem*

$$(P_n) : \begin{cases} \prod_{i=0}^{n-1}\Big(\frac{d}{dt} - \big(-c_i + A\big)\Big)u(t) = 0,\ t \geq 0, \\ u^{(k)}(0) = x_k,\ k = 0, \cdots, n-1, \end{cases}$$

with $c_0 = 0$. Then there exists a global exponentially bounded resolvent propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{n-1}(t))_{t\geq 0})$ for (P_n). Furthermore, $(R_i(t))_{t\geq 0}$ is topologically mixing for any $i \in \mathbb{N}^0_{n-1}$, and the operator family $(R_0(t) + \cdots + R_{n-1}(t))_{t\geq 0}$ is also topologically mixing.

Proof. Notice that the problem (P_2) is a special case of the problem (P_n) with $n = 2$ and $c_1 = 1$, so that the first statement in (i) is an almost immediate consequence of [5, Theorem 25.6]. Suppose now that $y = (y_k)_{k\in\mathbb{N}}$ and $z = (z_k)_{k\in\mathbb{N}}$ belong to the dense subset

$$D := \Big\{ \big(x_k\big)_{k\in\mathbb{N}} : \exists L \in \mathbb{N} \ \ \forall k > L \ \ x_k = 0 \Big\}$$

of l^1_r. Let $y_k = z_k = 0$ for $k > L$. For any sufficiently large number $t > 0$, we will construct the vector $v(t) = (v_k(t))_{k\in\mathbb{N}} \in l^1_r$ such that

$$\big\|y - v(t)\big\| = O\big(t^{-1}\big) \text{ as } t \to +\infty, \text{ and } \big\|z - R_0(t)v(t)\big\| = O\big(t^{-1}\big) \text{ as } t \to +\infty. \tag{2}$$

Towards this end, observe that

$$T(t)\big(x_k\big)_{k\in\mathbb{N}} = \left(\sum_{j=k}^{\infty} \frac{t^{j-k}}{(j-k)!} x_j\right)_{k\in\mathbb{N}}, \ t \geq 0, \ \big(x_k\big)_{k\in\mathbb{N}} \in l^1_r,$$

and

$$R_0(t)\big(x_k\big)_{k\in\mathbb{N}} = \left(\sum_{j=k}^{\infty} \frac{t^{j-k}}{(j-k)!}\Big(x_j + \big(e^{-t} - 1\big)x_{j+1}\Big)\right)_{k\in\mathbb{N}}, \tag{3}$$

where $t \geq 0$, $\big(x_k\big)_{k\in\mathbb{N}} \in l^1_r$.

Define $v_k(t) := y_k$ for $1 \leq k \leq L$, and $v_k(t) := 0$ for $L + 1 \leq k \leq 2L$ and $k \geq 3L + 1$. The numbers $v_{2L+1}(t), \cdots, v_{3L}(t)$ are defined as the unique solutions of system (cf. the first L elements of sequence appearing on the right hand side of (3), with x_j replaced by $v_j(t)$)

$$(S): \begin{cases} \sum\limits_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!}\Big(v_j(t) + \big(e^{-t} - 1\big)v_{j+1}(t)\Big) = z_1 \\ \sum\limits_{j=2}^{\infty} \frac{t^{j-2}}{(j-2)!}\Big(v_j(t) + \big(e^{-t} - 1\big)v_{j+1}(t)\Big) = z_2 \\ \qquad\qquad \cdots \\ \sum\limits_{j=L}^{\infty} \frac{t^{j-L}}{(j-L)!}\Big(v_j(t) + \big(e^{-t} - 1\big)v_{j+1}(t)\Big) = z_L \end{cases}$$

i.e., $v_{2L+1}(t), \cdots, v_{3L}(t)$ satisfy the following matrix equality:

$$A(t)\big[v_{2L+1}(t) \cdots v_{3L}(t)\big]^T = \big[z_1 \cdots z_L\big]^T - B(t)\big[y_1 \cdots y_L\big]^T,$$

where

$$A(t) = \Big[a_{ij}(t)\Big]_{L\times L} = \Bigg[\Big(e^{-t}-1\Big)\frac{t^{2L-i+j-1}}{(2L-i+j-1)!} + \frac{t^{2L-i+j}}{(2L-i+j)!}\Bigg]_{L\times L}$$

and $B(t) = [b_{ij}(t)]_{L\times L}$ with

$$b_{ij}(t) = \begin{cases} \big(e^{-t}-1\big)\frac{t^{j-i-1}}{(j-i-1)!} + \frac{t^{j-i}}{(j-i)!}, \text{ for } j > i, \\ 1, \text{ for } i = j, \\ 0, \text{ for } i > j. \end{cases}$$

Notice that any element $a_{ij}(t)$ $[b_{ij}(t)]$ of the matrix $A(t)$ $[B(t)]$ asymptotically behave as $t \to +\infty$ like the corresponding element of the matrix $\tilde{A}(t)$ $[\tilde{B}(t)]$, where

$$\tilde{A}(t) = \Big[\widetilde{a_{ij}}(t)\Big]_{L\times L} = \Bigg[\frac{t^{2L-i+j}}{(2L-i+j)!}\Bigg]_{L\times L}$$

and $\tilde{B}(t) = [\widetilde{b_{ij}}(t)]_{L\times L}$ with

$$\widetilde{b_{ij}}(t) = \begin{cases} \frac{t^{j-i}}{(j-i)!}, \text{ for } j > i, \\ 1, \text{ for } i = j, \\ 0, \text{ for } i > j. \end{cases}$$

The matrices $\tilde{A}(t)$ and $\tilde{B}(t)$ play an important role in the proof of [4, Theorem 5.2], which in combination with the above given arguments also shows that there exists an absolute constant $C_2 > 0$ such that, for every $k \in \{2L+1, \cdots, 3L\}$,

$$\big|v_k(t)\big| \le C_2 t^{L-k}. \tag{4}$$

Now it is not difficult to prove that (2) holds as well as that, for every $y_1,\ z_1 \in l_r^1$ and for every $\varepsilon > 0$, there exists $t_0 \ge 0$ such that, for every $t \ge t_0$, there exists $v_1(t) \in l_r^1$ such that $||y_1 - v_1(t)|| < \varepsilon$ and $||z_1 - R_0(t)v_1(t)|| < \varepsilon$. Hence, $(R_0(t))_{t\ge 0}$ is topologically mixing. The proof of the topologically mixing property for $(R_1(t))_{t\ge 0}$ and $(R_0(t) + R_1(t))_{t\ge 0}$ is quite similar and as such will not be given. Consider now the assertion (ii). Denote by X the operator Van der Monde matrix $X = [x_{kl}]_{n\times n} = \big[(-c_{l-1}+A)^{k-1}\big]_{n\times n}$. Let $i \in \mathbb{N}_{n-1}^0$, let $x \in l_r^1$, and let

$$\big[y_{0,i}(x)\ y_{1,i}(x)\ \cdots\ y_{n-1,i}(x)\big]^T = X^{-1}\big[0\ \cdots\ x\ \cdots\ 0\big]^T,$$

where x appears in the ith place of the last vector column, starting from 0. The existence of a global exponentially bounded resolvent propagation family $((R_0(t))_{t\ge 0}, \cdots, (R_{n-1}(t))_{t\ge 0})$ for (P_n) follows again from an application of [5, Theorem 25.6]. This theorem yields that, for every $i \in \mathbb{N}^0_{n-1}$, one has:

$$R_i(t)x = \sum_{l=0}^{n-1} e^{-c_l t} T(t) y_{l,i}(x),\ t \geq 0,\ x \in l^1_r.$$

Using the analysis given on page 15 of [17] (cf. the problems 245-246), one can simply prove that there exist $m \in \mathbb{N}$ and complex polynomials $P_{l,i}(z) \equiv a^m_{l,i} z^m + \cdots + a^0_{l,i}$ $(0 \leq l \leq n-1,\ 0 \leq i \leq n-1)$ such that the following holds:

(a) $y_{l,i}(x) = P_{l,i}(A)x$ $(0 \leq l \leq n-1,\ 0 \leq i \leq n-1)$, where the operator $P_{l,i}(A)$ is defined in the obvious way,
(b) $a^0_{0,i} \neq 0$ $(0 \leq i \leq n-1)$,
(c) $\sum_{i=0}^{n-1} a^0_{0,i} \neq 0$.

This implies that, for every $x = (x_k)_{k \in \mathbb{N}} \in l^1_r$,

$$R_i(t)\bigl(x_k\bigr)_{k\in\mathbb{N}} = \Biggl(\sum_{j=k} \frac{t^{j-k}}{(j-k)!}\Biggl\{\Bigl[a^m_{0,i}x_{j+m} + \cdots + a^0_{0,i}x_j\Bigr] + \sum_{l=1}^{n-1} e^{-c_l t}\Bigl[a^m_{l,i}x_{j+m} + \cdots + a^0_{l,i}x_j\Bigr]\Biggr\}\Biggr)_{k\in\mathbb{N}}. \tag{5}$$

Suppose now that $y = (y_k)_{k\in\mathbb{N}}$ and $z = (z_k)_{k\in\mathbb{N}}$ belong to D, and that $y_k = z_k = 0$ for $k > L$. Now we will construct the vector $w(t) = (w_k(t))_{k\in\mathbb{N}} \in l^1_r$ such that (2) holds with $R_0(\cdot)$ and $v(\cdot)$ replaced respectively with $R_i(\cdot)$ and $w(\cdot)$. The sequence $w(t)$ is defined by $w_k(t) := y_k$ for $1 \leq k \leq L$, and $w_k(t) := 0$ for $L+1 \leq k \leq 2L$ and $k \geq 3L+1$; similarly as in the first part of proof, the numbers $w_{2L+1}(t), \cdots, w_{3L}(t)$ satisfy the following system of equations (cf. (5) and the first part of proof):

$$(S'):\ \begin{cases} \sum\limits_{j=1} \frac{t^{j-1}}{(j-1)!}\Bigl\{\Bigl[a^m_{0,i}w_{j+m}(t) + \cdots + a^0_{0,i}w_j(t)\Bigr] \\ \quad + \sum\limits_{l=1}^{n-1} e^{-c_l t}\Bigl[a^m_{l,i}w_{j+m}(t) + \cdots + a^0_{l,i}w_j(t)\Bigr]\Bigr\} = z_1 \\ \sum\limits_{j=2} \frac{t^{j-2}}{(j-2)!}\Bigl\{\Bigl[a^m_{0,i}w_{j+m}(t) + \cdots + a^0_{0,i}w_j(t)\Bigr] \\ \quad + \sum\limits_{l=1}^{n-1} e^{-c_l t}\Bigl[a^m_{l,i}w_{j+m}(t) + \cdots + a^0_{l,i}w_j(t)\Bigr]\Bigr\} = z_2 \\ \qquad \cdots \\ \sum\limits_{j=L} \frac{t^{j-L}}{(j-L)!}\Bigl\{\Bigl[a^m_{0,i}w_{j+m}(t) + \cdots + a^0_{0,i}w_j(t)\Bigr] \\ \quad + \sum\limits_{l=1}^{n-1} e^{-c_l t}\Bigl[a^m_{l,i}w_{j+m}(t) + \cdots + a^0_{l,i}w_j(t)\Bigr]\Bigr\} = z_L. \end{cases}$$

It is clear that the matricial form of system (S') looks like:

$$A_1(t)\big[v_{2L+1}(t)\ \cdots\ v_{3L}(t)\big]^T = \big[z_1\ \cdots\ z_L\big]^T - B_1(t)\big[y_1\ \cdots y_L\big]^T,$$

where any element $a^1_{kl}(t)$ $[b^1_{kl}(t)]$ of the matrix $A_1(t)$ $[B_1(t)]$ asymptotically behave as $t \to +\infty$ like $a^0_{0,i}\tilde{a_{kl}}(t)$ $[a^0_{0,i}\tilde{b_{kl}}(t)]$; cf. also (b). Arguing as in the proof of (i), we get that there exists $C_2 > 0$ such that, for every $k \in \{2L+1, \cdots, 3L\}$, the estimate (4) holds. The proof of the topologically mixing property for $(R_i(t))_{t\geq 0}$ is now completed through a routine argument. Using (c) instead of (b), we obtain similarly that the operator family $(R_0(t) + \cdots + R_{n-1}(t))_{t\geq 0}$ is topologically mixing. □

Let A, l^1_r, $(T(t))_{t\geq 0}$ and $(R^{\zeta}(t))_{t\geq 0}$ be defined as before ($\zeta > 0$), let $n \geq 2$, and let $0 < c_1 < \cdots < c_{n-1}$. Then it is clear that the problem (P_n) is a special case of the abstract Cauchy problem (ACP_n), with the operators $A_0, A_1, \cdots, A_{n-1}$ being certain functions of A; for example, $A_{n-1} = \sum_{j=1}^{n-1} c_j I - nA$ and $A_0 = (-1)^n \prod_{j=0}^{n-1}(-c_j + A)$. Consider now the problem

$$(P_n^{\gamma}) : \begin{cases} \mathbf{D}_t^{n\gamma}u(t) + A_{n-1}\mathbf{D}_t^{(n-1)\gamma}u(t) + \cdots + A_1\mathbf{D}_t^{\gamma}u(t) + A_0u(t) = 0,\ t > 0, \\ u^{(k)}(0) = u_k,\ k = 0, \cdots, \lceil n\gamma\rceil - 1, \end{cases}$$

where $\gamma \in (0,1)$. Then the use of [15, Theorem 2.9-Theorem 2.10] implies that there exists a global exponentially bounded resolvent propagation family $((R_0^{\gamma}(t))_{t\geq 0}, \cdots, (R^{\gamma}_{\lceil n\gamma\rceil-1}(t))_{t\geq 0})$ for (P_n^{γ}), given by

$$R_i^{\gamma}(t)x = \mathcal{L}^{-1}\Bigg(\Big(\lambda^{n\gamma} + \sum_{j=0}^{n-1}\lambda^{j\gamma}A_j\Big)^{-1}\Big(\lambda^{n\gamma-i-1}x + \sum_{j\in D_i'}\lambda^{j\gamma-i-1}A_jx\Big)\Bigg)(t), \tag{6}$$

for any $t \geq 0$, $x \in l^1_r$ and $i \in \mathbb{N}^0_{\lceil n\gamma\rceil-1}$, where $D_i' = \{j \in \mathbb{N}^0_{n-1} : \lceil j\gamma\rceil - 1 \geq i\}$; speaking matter-of-factly, we have that, for every $t \geq 0$, $x \in l^1_r$ and $i \in \mathbb{N}^0_{\lceil n\gamma\rceil-1}$,

$$\begin{aligned} R_i^{\gamma}(t)x &= g_i * \Big[P_0(A)R_{c_0,\gamma}(\cdot)x + \cdots + P_{n-1}(A)R_{c_{n-1},\gamma}(\cdot)x\Big](t) \\ &+ \sum_{j\in D_i'} A_j\Big\{g_{i+(n-j)\gamma} * \Big[P_0(A)R_{c_0,\gamma}(\cdot)x + \cdots + P_{n-1}(A)R_{c_{n-1},\gamma}(\cdot)x\Big](t)\Big\}, \end{aligned}$$

where

$$P_j(A) = \prod_{\substack{0\leq l\leq n-1 \\ j\neq l}} \big(-c_j + c_l\big)\big(-c_j + A\big)^{n-1},\ j \in \mathbb{N}^0_{n-1},$$

and $(R_{c_j,\gamma}(t))_{t\geq 0}$ denotes the γ-times regularized resolvent family generated by $-c_j + A$ ($j \in \mathbb{N}^0_{n-1}$). Making use of [15, Theorem 2.9-Theorem 2.10] again, as well as the representation formula (6) and the proof of subordination principle [1, Theorem 3.1], we obtain that:

$$R_0^\gamma(t)x = t^{-\gamma}\int_0^\infty \Phi_\gamma\big(st^{-\gamma}\big)R_0(s)x\,ds,\ t>0,\ x\in l_r^1, \tag{7}$$

where $((R_0(t))_{t\geq 0},\cdots,(R_{n-1}(t))_{t\geq 0})$ is the resolvent propagation family for (P_n), defined already in the formulation of Theorem 2.4. With the same notation as in the proof of afore-mentioned theorem, we obtain from (5)-(7) that, for every $x=(x_k)_{k\in\mathbb{N}}\in l_r^1$ and $t>0$,

$$R_0^\gamma(t)\big(x_k\big)_{k\in\mathbb{N}}=t^{-\gamma}\Bigg(\int_0^\infty \Phi_\gamma\big(st^{-\gamma}\big)\Bigg[\sum_{j=k}\frac{s^{j-k}}{(j-k)!}\Bigg\{\Big[a_{0,0}^m x_{j+m}+\cdots+a_{0,0}^0x_j\Big] + \sum_{l=1}^{n-1}e^{-c_ls}\Big[a_{l,0}^m x_{j+m}+\cdots+a_{l,0}^0x_j\Big]\Bigg\}\Bigg]\,ds\Bigg)_{k\in\mathbb{N}}.$$

Applying [1, Theorem 3.1] to $(T(t))_{t\geq 0}$ and $(R^\gamma(t))_{t\geq 0}$, we get that:

$$t^{-\gamma}\int_0^\infty \Phi_\gamma\big(st^{-\gamma}\big)\frac{s^l}{l!}\,ds=\frac{t^{\gamma l}}{\Gamma(\gamma l+1)},\ t>0,\ l\in\mathbb{N}_0,$$

which further implies that, for every $x=(x_k)_{k\in\mathbb{N}}\in l_r^1$ and $t>0$,

$$R_0^\gamma(t)\big(x_k\big)_{k\in\mathbb{N}}=\Bigg(\sum_{j=k}\frac{t^{\gamma(j-k)}}{\Gamma(\gamma(j-k)+1)}\Big[a_{0,0}^m x_{j+m}+\cdots+a_{0,0}^0x_j\Big] + t^{-\gamma}\sum_{j=k}^\infty\sum_{l=1}^{n-1}\int_0^\infty \Phi_\gamma\big(st^{-\gamma}\big)\frac{s^{j-k}}{(j-k)!}e^{-c_ls}\Big[a_{l,0}^m x_{j+m}+\cdots+a_{l,0}^0x_j\Big]\,ds\Bigg)_{k\in\mathbb{N}}. \tag{8}$$

Observe also that

$$t^{-\gamma}\int_0^\infty \Phi_\gamma\big(st^{-\gamma}\big)\sum_{l=1}^{n-1}\frac{s^l}{l!}e^{-c_ls}\,ds\leq Mt^{-\gamma}\sum_{l=1}^{n-1}\int_0^\infty\frac{s^l}{l!}e^{-c_ls}\,ds\to 0 \text{ as } t\to+\infty,$$

and that, for every $t\geq 0$ and $(x_k)_{k\in\mathbb{N}}\in l_r^1$,

$$R^\gamma(t)\big(x_k\big)_{k\in\mathbb{N}}=\Bigg(\sum_{j=k}^\infty\frac{t^{\gamma(j-k)}}{\Gamma(\gamma(j-k)+1)}x_j\Bigg)_{k\in\mathbb{N}},\ t\geq 0,\ \big(x_k\big)_{k\in\mathbb{N}}\in l_r^1.$$

Proceeding as in the proofs of Theorem 2.3 (in this case, there exists $C_2>0$ such that, for every $k\in\{2L+1,\cdots,3L\}$, the corresponding vector $(v_k^\gamma(t))_{k\in\mathbb{N}}\in l_r^1$ satisfies $\big|v_k^\gamma(t)\big|\leq C_2t^{\gamma(L-k)}$) and Theorem 2.4, we obtain that the following theorem is true.

Theorem 2.5. *The operator family $(R_0^\gamma(t))_{t\geq 0}$ is topologically mixing.*

Remark 2.6.

(i) If $0 < \gamma \leq 1/n$, then it makes no sense to define $(R_i(t))_{t\geq 0}$ for $i \geq 1$; if this is not the case, then it is not clear whether there exists an index $i \in \mathbb{N}_{\lceil n\gamma \rceil -1}$ such that the operator family $(R_i(t))_{t\geq 0}$ is topologically mixing.

(ii) Concerning the invariance of hypercyclic and topologically mixing properties under the action of subordination principles, it should be noted that the unilateral backward shifts have some advantages over other operators used in the theory of hypercyclicity (cf. Theorem 2.3, Theorem 2.5, [1, Theorem 3.1], [11, Theorem 2.11, Theorem 3.9-Theorem 3.1], [15, Theorem 4.1, Theorem 4.4] and [12] for further information in this direction).

Finally, we would like to propose the following problem.

Problem. Suppose $n \in \mathbb{N} \setminus \{1\}$, $0 \leq \alpha_1 < \cdots < \alpha_n$ and $0 \leq \alpha < \alpha_n$. Is it possible to construct a separable infinite-dimensional complex Banach space E and closed linear operators $A_0, A_1, \cdots, A_{n-1}$ on E such that there exists a global resolvent propagation family $((R_0(t))_{t\geq 0}, \cdots, (R_{m_n-1}(t))_{t\geq 0})$ for (1) satisfying that some (every) single operator family $(R_i(t))_{t\geq 0}$ of this tuple is topologically mixing?

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