

VARIATIONS ON A-BROWDER-TYPE THEOREMS

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ABSTRACT. We introduce and we study the new spectral properties (SBw) , $(SBaw)$, $(SBab)$ and (SBb) . Among other results, we show that if T is a bounded linear operator acting on a Banach space X , then T possesses property (SBb) if and only if T possesses property (b) and $\Pi^0(T) = \Pi_a(T)$.

1. INTRODUCTION

This paper is a continuation of our recent investigations in the subject of Weyl type theorems. As in [12], we investigate other new variants of a-Browder's and a-Weyl's theorem, and we define the properties (SBw) , (SBb) , $(SBab)$ and $(SBaw)$ (see Definitions 2.1 and 2.10) for bounded linear operators, in connection with Weyl-type theorems. The essential results obtained are summarized in the diagram presented in the last part of this paper (see Table 1). For further definitions and symbols we refer the reader to [12], and we also refer to [4, 5, 6, 9, 11, 13] for more details. The inclusion of the meaning of various known theorems and properties (see Table 2) is motivated by having a global overview of the subject. In addition, we have the following usual notations that will be needed later:

- $E(T)$ – eigenvalues of T isolated in the spectrum $\sigma(T)$
- $E^0(T)$ – eigenvalues of T of finite multiplicity isolated in $\sigma(T)$
- $E_a(T)$ – eigenvalues of T isolated in the approximate point spectrum $\sigma_a(T)$
- $E_a^0(T)$ – eigenvalues of T of finite multiplicity isolated in $\sigma_a(T)$
- $\Pi(T)$ – poles of T
- $\Pi^0(T)$ – poles of T of finite rank
- $\Pi_a(T)$ – left poles of T
- $\Pi_a^0(T)$ – left poles of T of finite rank

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- $\sigma_{BW}(T)$ – B-Weyl spectrum of T
- $\sigma_W(T)$ – Weyl spectrum of T
- $\sigma_{SBF_+^-}(T)$ – essential semi-B-Fredholm spectrum of T
- $\sigma_{SF_+^-}(T)$ – Weyl essential approximate point spectrum of T

2. MAIN RESULTS

We denote by $L(X)$ the Banach algebra of all operators acting on a Banach space X . For $T \in L(X)$, let $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, $\Delta_a^g(T) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$ and let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$.

An operator $T \in L(X)$ is said to have the *single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open neighborhood \mathcal{U} of λ_0 , the only analytic function $f : \mathcal{U} \rightarrow X$ which satisfies the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have the SVEP if T has this property at every $\lambda \in \mathbb{C}$. (See [16] for more details about this concept).

Definition 2.1. A bounded linear operator $T \in L(X)$ is said to possess property *(SBw)* if $\Delta_a^g(T) = E^0(T)$ and is said to possess property *(SBb)* if $\Delta_a^g(T) = \Pi^0(T)$.

Recall that $T \in L(X)$ satisfies property *(w)* if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$.

Theorem 2.2. *Let $T \in L(X)$. Then T possesses property (SBw) if and only if T possesses property (w) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

Proof. Suppose that T possesses property *(SBw)*, that is $\Delta_a^g(T) = E^0(T)$. Since $\Delta_a(T) \subset \Delta_a^g(T)$ then $\Delta_a(T) \subset E^0(T)$. Now if $\lambda \in E^0(T)$, then $T - \lambda I$ is an upper semi-B-Fredholm operator. As λ is an eigenvalue of T of finite multiplicity, then from [11, Lemma 2.2], $T - \lambda I$ is an upper semi-Fredholm operator. Thus $\Delta_a(T) \supset E^0(T)$. Hence T possesses property *(w)*. We then have $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^0(T)$ and $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E^0(T)$. So $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.

Conversely, the condition $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$ entails that $\Delta_a^g(T) = \Delta_a(T)$. The property *(w)* for T implies that $\Delta_a^g(T) = E^0(T)$ and T possesses property *(SBw)*. \square

Generally, property *(w)* does not imply property *(SBw)*. Indeed, let $U \in L(\ell^2(\mathbb{N}))$ be defined by $U(x_1, x_2, x_3, \dots) = (0, x_2, x_3, \dots)$. Then $\sigma_a(U) = \{0, 1\}$, $\sigma_{SF_+^-}(U) = \{1\}$ and $E^0(U) = \{0\}$. Thus $\Delta_a(U) = E^0(U)$, i.e. U possesses property *(w)*. Moreover, $\sigma_{SBF_+^-}(U) = \emptyset$. This implies that $\Delta_a^g(U) \neq E^0(U)$ and U does not possess property *(SBw)*.

Remark 2.3. 1) The property (SBw) is not intermediate between property (w) and property (gw) . Indeed, let $T = 0 \oplus S$ be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$, where S is defined on $\ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, \dots) = (0, \frac{1}{2}x_1, \frac{1}{3}x_2, \dots)$. Then $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \{0\}$ and $E(T) = \{0\}$. So $\Delta_a^g(T) \neq E(T)$, i.e. T does not possess property (gw) . On the other hand, since $E^0(T) = \emptyset$ we have $\Delta_a^g(T) = E^0(T)$, i.e. T possess property (SBw) . Now if we consider the operator U defined above, then $E(U) = \{0, 1\}$. So $\Delta_a^g(U) = E(U)$, i.e. U possesses property (gw) . But it does not possess property (SBw) .

2) Generally, the property (SBw) is not transmitted from an operator T to its dual T^* . To see this, consider the operator S be defined as in part 1) of this remark, then S possesses property (SBw) because $\sigma_a(S) = \sigma_{SBF_+^-}(S) = \{0\}$ and $E^0(S) = \emptyset$. But its adjoint which is defined by $S^*(x_1, x_2, x_3, \dots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \dots)$ does not possess this property because $\sigma_a(S^*) = \sigma_{SBF_+^-}(S^*) = \{0\}$ and $E^0(S^*) = \{0\}$.

From Theorem 2.2, we deduce immediately that an operator possessing property (SBw) possesses property (SBb) , but the converse does not hold in general as shown by the following example: let T be the operator be defined on the Hilbert space $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, \dots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \frac{1}{4}x_4, \dots)$. Then $\sigma_a(T) = \sigma_{SBF_+^-}(T) = \{0\}$, $E^0(T) = \{0\}$ and $\Pi^0(T) = \emptyset$. This implies that $\Delta_a^g(T) = \Pi^0(T)$ and $\Delta_a^g(T) \neq E^0(T)$, so that T possesses property (SBb) , but it does not possess property (SBw) . However, we have the following corollary in which we give a condition for the equivalence of these properties.

Corollary 2.4. *Let $T \in L(X)$. Then T possesses property (SBw) if and only if T possesses property (SBb) and $E^0(T) = \Pi^0(T)$.*

The next theorem gives characterizations of operators possessing property (SBb) . Recall that $T \in L(X)$ satisfies property (b) (respectively property (gb)) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$ (respectively: $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$).

Theorem 2.5. *Let $T \in L(X)$. Then the following statements are equivalent.*

- (i) T possesses property (SBb) ;
- (ii) T possesses property (b) and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$;
- (iii) T possesses property (b) and $\Pi^0(T) = \Pi_a(T)$;
- (iv) T possesses property (gb) and $\Pi^0(T) = \Pi(T)$.

Proof.

(i) \iff (ii). Suppose that T possesses property (SBb) , that is $\Delta_a^g(T) = \Pi^0(T)$. Since $\Delta_a(T) \subset \Delta_a^g(T)$, $\Delta_a(T) \subset \Pi^0(T)$. As $\Delta_a(T) \supset \Pi^0(T)$ is always true, then $\Delta_a(T) = \Pi^0(T)$ and T possesses property (b) . We then have

$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi^0(T)$ and $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi^0(T)$. So $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$. Conversely, the condition $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$ and property (b) for T imply that $\Delta_a^g(T) = \Pi^0(T)$ and T possesses property (SBb).

(i) \iff (iii). Assume that T possesses property (SBb), then T possesses property (b). From [11, Theorem 2.5], T satisfies a-Browder's theorem, and since a-Browder's theorem is equivalent to generalized a-Browder's theorem then $\Delta_a^g(T) = \Pi_a(T)$. Hence $\Pi^0(T) = \Pi_a(T)$. Conversely, assume that T possesses property (b) and $\Pi^0(T) = \Pi_a(T)$. Then $\Delta_a^g(T) = \Pi_a(T)$ and $\Pi^0(T) = \Pi_a(T)$. Thus $\Delta_a^g(T) = \Pi^0(T)$ and T possesses property (SBb).

(i) \iff (iv). If T possesses property (SBb), that is $\Delta_a^g(T) = \Pi^0(T)$, then $\Delta_a^g(T) \subset \Pi(T)$. Since we have always that $\Delta_a^g(T) \supset \Pi(T)$, then $\Delta_a^g(T) = \Pi(T) = \Pi^0(T)$. So T possesses property (gb). The reverse implication is trivial. \square

Remark 2.6. 1) From Theorem 2.5, we have if $T \in L(X)$ possesses property (SBb) then it possesses property (gb). However, the converse is not true in general: for example, let T be defined on $\ell^2(\mathbb{N})$ by $T(x_1, x_2, x_3, \dots) = (0, \frac{1}{2}x_1, 0, 0, \dots)$. Then $\sigma_a(T) = \{0\}$, $\sigma_{SBF_+^-}(T) = \emptyset$ and $\Pi(T) = \Pi_a(T) = \{0\}$. So $\Delta_a^g(T) = \Pi(T)$, i.e. T possesses property (gb). But it does not possess property (SBb), since $\Pi^0(T) = \emptyset$. Here $\sigma_{SF_+^-}(T) = \{0\}$.

2) Property (SBb) as well as property (SBw), does not pass from an operator to its dual. For this, we consider on the Hilbert space $\ell^2(\mathbb{N})$ the operator T defined by $T(x_1, x_2, x_3, \dots) = (\epsilon x_1, 0, x_2, x_3, \dots)$ for fixed $0 < \epsilon < 1$. Then $\sigma(T) = D(0, 1)$ the closed unit disc in \mathbb{C} and $\sigma_a(T^*) = C(0, 1) \cup \{\epsilon\}$ where $C(0, 1)$ is the unit circle of \mathbb{C} . This implies that T^* has the SVEP, which implies from [10, Theorem 2.5] that T possesses property (gb). As $\Pi(T) = \Pi^0(T) = \emptyset$, then T possesses property (SBb). On the other hand, $\sigma_{SF_+^-}(T^*) = C(0, 1)$. Thus $\Delta_a(T^*) \neq \Pi^0(T^*)$ and T^* does not possess property (b). Consequently T^* does not possess property (SBb).

Recall that an operator $T \in L(X)$ is said to possess property (Bw) if $\Delta^g(T) = E^0(T)$ and is said to possess property (Bb) if $\Delta^g(T) = \Pi^0(T)$, where $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$. The properties (Bw) and (Bb) have been introduced very recently in [15] and [19], respectively as variants of Weyl's and Browder's theorem. In the next theorem, we investigate the relationship between property (SBw) and property (Bw).

Theorem 2.7. *Let $T \in L(X)$. Then T possesses property (SBw) if and only if:*

- (i) T possesses property (Bw);
- (ii) $\text{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a^g(T)$; where $\text{ind}(T - \lambda I)$ is the index of $T - \lambda I$.

Proof. Suppose that T possesses property (SBw) and let $\lambda \in \Delta^g(T)$. Since $\sigma_{SBF_+^-}(T) \subset \sigma_{BW}(T)$, then $\lambda \notin \sigma_{SBF_+^-}(T)$. We also have that $\lambda \in \sigma_a(T)$. Indeed, if $\lambda \notin \sigma_a(T)$, as $\lambda \notin \sigma_{BW}(T)$, then $T - \lambda I$ will be invertible and this is impossible since $\lambda \in \sigma(T)$. As T possesses property (SBw) , then $\lambda \in E^0(T)$. This implies that $\Delta^g(T) \subset E^0(T)$. To show the opposite inclusion, let $\lambda \in E^0(T)$ be arbitrary. Since T possesses property (SBw) , then $\lambda \notin \sigma_{SBF_+^-}(T)$ and so $\text{ind}(T - \lambda I) \leq 0$. As $\lambda \in \text{iso}\sigma(T)$, then T^* has the SVEP at λ . By [7, Corollary 2.8], we deduce that $\text{ind}(T - \lambda I) \geq 0$. So $\text{ind}(T - \lambda I) = 0$ and $\lambda \notin \sigma_{BW}(T)$. Hence $\Delta^g(T) = E^0(T)$ and $\text{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a^g(T)$. Conversely, assume that T possesses property (Bw) and $\text{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a^g(T)$. If $\lambda \in \Delta_a^g(T)$, then $T - \lambda I$ is a semi-B-Fredholm operator such that $\text{ind}(T - \lambda I) = 0$. Thus $T - \lambda I$ is a B-Weyl operator. Since T possesses property (Bw) , then $\lambda \in E^0(T)$ and hence $\Delta_a^g(T) \subset E^0(T)$. Now if $\lambda \in E^0(T)$, then $T - \lambda I$ is a B-Weyl operator and $\lambda \in \sigma(T)$. Hence $\lambda \in \Delta_a^g(T)$ and T possesses property (SBw) . \square

Similarly to Theorem 2.7, we have the following relationship between property (SBb) and property (Bb) which we give without proof.

Theorem 2.8. *Let $T \in L(X)$. Then T possesses property (SBb) if and only if:*

- (i) T possesses property (Bb) ;
- (ii) $\text{ind}(T - \lambda I) = 0$ for all $\lambda \in \Delta_a^g(T)$.

From Theorem 2.7 and Theorem 2.8, we have: the properties (SBw) and (SBb) imply the properties (Bw) and (Bb) , respectively. But the following example shows that the converses do not hold in general.

Example 2.9. On the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ we consider the operator $T = 0 \oplus R$ where R is the unilateral right shift operator. Then $\sigma(T) = \sigma_{BW}(T) = D(0, 1)$ and $E^0(T) = \Pi^0(T) = \emptyset$. This implies that $\Delta^g(T) = E^0(T)$ and $\Delta^g(T) = \Pi^0(T)$, i.e. T possesses property (Bw) and property (Bb) . On the other hand, $\sigma_a(T) = C(0, 1) \cup \{0\}$ and $\sigma_{SBF_+^-}(T) = C(0, 1)$. Thus $\Delta_a^g(T) \neq \Pi^0(T)$ and $\Delta_a^g(T) \neq E^0(T)$, i.e. T does not possess either property (SBw) nor property (SBb) . Here $\Delta_a^g(T) = \{0\}$ but $\text{ind}(T) \neq 0$.

Definition 2.10. A bounded linear operator $T \in L(X)$ is said to possess property $(SBaw)$ if $\Delta_a^g(T) = E_a^0(T)$ and is said to possess property $(SBab)$ if $\Delta_a^g(T) = \Pi_a^0(T)$.

From Theorem 2.5 we deduce that property (SBb) implies property $(SBab)$, but the converse is not true in general as shown by the following example: let $T = R \oplus U$ be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ where R is the unilateral right shift operator defined on $\ell^2(\mathbb{N})$, and U is defined as

above. Then $\sigma_a(T) = C(0, 1) \cup \{0\}$, $\sigma_{SBF_+^-}(T) = C(0, 1)$ and $\Pi_a^0(T) = \{0\}$. So T possesses property $(SBab)$. But it does not possess property (SBb) , since $\Pi^0(T) = \emptyset$. Observe that the two properties (SBw) and $(SBaw)$ for $T \in L(X)$ are independent. Indeed, the example $R \oplus U$ defined above shows that property $(SBaw)$ does not imply property (SBw) because $E_a^0(R \oplus U) = \{0\}$, $E^0(R \oplus U) = \emptyset$. But the example $R \oplus S$ defined below (see Remark 2.16) shows that property (SBw) does not imply property $(SBaw)$ because $E^0(R \oplus S) = \emptyset$.

However, we give in the next corollary conditions for the equivalence of two properties (SBb) and $(SBab)$.

Corollary 2.11. *Let $T \in L(X)$. Then the following statements are equivalent.*

- (i) T possesses property (SBb) ;
- (ii) T possesses property $(SBab)$ and $\Pi^0(T) = \Pi_a^0(T)$;
- (iii) T possesses property $(SBab)$ and $\Pi^0(T) = \Pi_a(T)$.

Proof.

(i) \iff (ii) Suppose that T possesses property (SBb) . By Theorem 2.5, T possesses property (b) . This implies from [11, Corollary 2.7] that $\Pi^0(T) = \Pi_a^0(T)$. Thus $\Delta_a^g(T) = \Pi_a^0(T)$ and T possesses property $(SBab)$. The reverse implication is trivial.

(ii) \iff (iii) Follows directly by Theorem 2.5. □

Theorem 2.12. *Let $T \in L(X)$. Then T possesses property $(SBaw)$ if and only if T satisfies a-Weyl's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.*

Proof. Suppose that T possesses property $(SBaw)$, that is $\Delta_a^g(T) = E_a^0(T)$. Since $\Delta_a(T) \subset \Delta_a^g(T)$ then $\Delta_a(T) \subset E_a^0(T)$. Now if $\lambda \in E_a^0(T)$, then $T - \lambda I$ is an upper semi-B-Fredholm operator. Since λ is an eigenvalues of T of finite multiplicity, by [11, Lemma 2.2] we have $T - \lambda I$ is an upper semi-Fredholm operator. Thus $\Delta_a(T) \supset E_a^0(T)$ and $\Delta_a(T) = E_a^0(T)$. So T satisfies a-Weyl's theorem. We then have $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E_a^0(T)$ and $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_a^0(T)$. So $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$.

Conversely, the condition $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$ entails that $\Delta_a^g(T) = \Delta_a(T)$ and a-Weyl's theorem for T implies that $\Delta_a^g(T) = E_a^0(T)$. So T possesses property $(SBaw)$. □

Remark 2.13. 1) The operator T defined as in part 1) of Remark 2.6 shows that generally, a-Weyl's theorem does not imply the property $(SBaw)$. Indeed, since $\sigma_{SBF_+^-}(T) = E_a^0(T) = \emptyset$, $\sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\}$ then $\Delta_a(T) = E_a^0(T)$, but $\Delta_a^g(T) \neq E_a^0(T)$.

2) The property $(SBaw)$ is not intermediate between a-Weyl's theorem and generalized a-Weyl's theorem. Indeed, we define T on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ by $T = I \oplus S$ where I is the identity operator on $\ell^2(\mathbb{N})$ and S is an injective quasinilpotent operator on $\ell^2(\mathbb{N})$ which is not nilpotent. Then $\sigma_a(T) = \{0, 1\}$, $\sigma_{SBF_+^-}(T) = \{0\}$, $E_a(T) = \{1\}$ and $E_a^0(T) = \emptyset$. So $\Delta_a^g(T) = E_a(T)$, i.e. T satisfies the generalized a-Weyl's theorem, but it does not possess property $(SBaw)$ because $\Delta_a^g(T) \neq E_a^0(T)$. On the other hand, let Q be defined for each $x = (\xi_i) \in \ell^1$ by $Q(\xi_1, \xi_2, \xi_3, \dots, \xi_k, \dots) = (0, \alpha_1\xi_1, \alpha_2\xi_2, \dots, \alpha_{k-1}\xi_{k-1}, \dots)$, where (α_i) is a sequence of complex numbers such that $0 < |\alpha_i| \leq 1$ and $\sum_{i=1}^\infty |\alpha_i| < \infty$ and define T on $X = \ell^1 \oplus \ell^1$ by $T = Q \oplus 0$. Then $\sigma_a(T) = \{0\}$, $E_a^0(T) = \emptyset$, $E_a(T) = \{0\}$ and $\sigma_{SBF_+^-}(T) = \{0\}$. Thus $\Delta_a^g(T) = E_a^0(T)$, i.e. T possesses property $(SBaw)$, but it does not satisfy the generalized a-Weyl's theorem because $\Delta_a^g(T) \neq E_a(T)$.

3) The operator defined as in part 2) of Remark 2.3 shows that in general, property $(SBaw)$ is not transmitted from an operator to its dual.

Theorem 2.14. *Let $T \in L(X)$. Then the following statements are equivalent:*

- (i) T possesses property $(SBab)$;
- (ii) T satisfies a-Browder's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$;
- (iii) T satisfies a-Browder's theorem and $\Pi_a^0(T) = \Pi_a(T)$.

Proof.

(i) \iff (ii) If T possesses property $(SBab)$, that is $\Delta_a^g(T) = \Pi_a^0(T)$, then $\Delta_a(T) \subset \Pi_a^0(T)$. As $\Delta_a(T) \supset \Pi_a^0(T)$ is always true, then $\Delta_a(T) = \Pi_a^0(T)$, i.e. T satisfies a-Browder's theorem. We then have $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \Pi_a^0(T)$ and $\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus \Pi_a^0(T)$. So $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$. The reverse implication is trivial.

(i) \iff (iii) If T possesses property $(SBab)$, then T satisfies a-Browder's theorem. Hence $\Delta_a^g(T) = \Pi_a(T) = \Pi_a^0(T)$. Conversely, if T satisfies a-Browder's theorem, then T satisfies generalized a-Browder's theorem $\Delta_a^g(T) = \Pi_a(T)$. As $\Pi_a(T) = \Pi_a^0(T)$ then $\Delta_a^g(T) = \Pi_a^0(T)$ and T possesses property $(SBab)$. \square

Corollary 2.15. *Let $T \in L(X)$. Then the following statements are equivalent:*

- (i) T possesses property $(SBaw)$;
- (ii) T possesses property $(SBab)$ and $E_a^0(T) = \Pi_a^0(T)$;
- (iii) T possesses property $(SBab)$ and $E_a^0(T) = \Pi_a(T)$.

Proof.

(i) \implies (ii) If T satisfies property $(SBaw)$, then from Theorem 2.12 and [8, Corollary 3.5] we conclude that T satisfies a-Browder's theorem and $\sigma_{SBF_+^-}(T) = \sigma_{SF_+^-}(T)$. So T possesses property $(SBab)$ and $E_a^0(T) = \Pi_a^0(T)$.

(ii) \implies (iii) Obvious, see Theorem 2.14.

(iii) \implies (i) The property $(SBab)$ entails that $\Delta_a^g(T) = \Pi_a(T)$. As $E_a^0(T) = \Pi_a(T)$, then $\Delta_a^g(T) = E_a^0(T)$, a desired. \square

Remark 2.16. 1) From Theorem 2.14, we have property $(SBab)$ for $T \in L(X)$ implies a-Browder's theorem for T . But the operator $T = 0$ shows that the converse is not true in general, since $\sigma_a(T) = \sigma_{SF_+^-}(T) = \{0\}$, $\Pi_a^0(T) = \emptyset$ and $\sigma_{SBF_+^-}(T) = \emptyset$. We also observe that $\Pi_a(T) = \{0\}$.

2) Similarly, we have by Corollary 2.15 that if $T \in L(X)$ possesses property $(SBaw)$ then it possesses property $(SBab)$, but generally, the converse is not true as the following example shows: let $R \oplus S$ be defined on the Banach space $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ where R is the unilateral right shift operator and S be defined on $\ell^2(\mathbb{N})$ by $S(x_1, x_2, x_3, \dots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$, then $\sigma_a(R \oplus S) = \sigma_{SBF_+^-}(R \oplus S) = C(0, 1) \cup \{0\}$, $\Pi_a^0(R \oplus S) = \emptyset$ and $E_a^0(R \oplus S) = \{0\}$. This proves that $R \oplus S$ possesses property $(SBab)$, but it does not possess property $(SBaw)$. Here $\Pi_a(R \oplus S) = \emptyset$.

3. SUMMARY OF RESULTS

In this last part, we give a summary of known Weyl type theorems as in [8], including the properties introduced in [1, 11, 12, 17, 19] and in this paper. We use the abbreviations W , gW , (Bw) , (w) , (gw) , (SBw) , (aw) , (gaw) , (Baw) , aW , gaW and $(SBaw)$ to signify that an operator $T \in L(X)$ obeys Weyl's theorem, generalized Weyl's theorem, property (Bw) , property (w) , property (gw) , property (SBw) , property (aw) , property (gaw) , property (Baw) , a-Weyl's theorem, generalized a-Weyl's theorem and property $(SBaw)$, respectively. Similarly, the abbreviations B , gB , aB , gaB , (Bb) , (Bab) , (b) , (gb) , (ab) , (gab) , (SBb) and $(SBab)$ have analogous meaning with respect to Browder's theorem or the properties introduced in [11] or the properties introduced in [12] or the properties introduced in [19] or the new properties introduced in this paper.

In the following diagram, which extends the similar diagram given in [12], arrows signify implications between Weyl type theorems, Browder type theorems, property (Bw) , property (Bb) , property (Baw) , property (Bab) , property (SBw) , property (SBb) , property $(SBaw)$ and property $(SBab)$. The numbers near the arrows are references to the results in the present paper (numbers without brackets) or to the bibliography therein (the numbers in square brackets).

$$\begin{array}{ccccccc}
 & & gW & & & & \\
 & & \uparrow [13] & & & & \\
 gW & & gaB & \xrightarrow{[12]} & (aw) & & \\
 \uparrow [1] & \uparrow [11] & \downarrow [12] & & & & \\
 (gw) \xrightarrow{[11]} & (gb) \xrightarrow{[12]} & (gab) \xrightarrow{[12]} & gB \xleftarrow{[8]} & gW \xleftarrow{[8]} & gaW \xrightarrow{[8]} & gaB \xrightarrow{[8]} & gB \\
 \downarrow [1] & \downarrow [11] & \downarrow [12] & \downarrow [8] & \downarrow [8] & \downarrow [8] & \Downarrow [2] & \Downarrow [2] \\
 (w) \xrightarrow{[11]} & (b) \xrightarrow{[12]} & (ab) \xrightarrow{[12]} & B \xleftarrow{[3]} & W \xleftarrow{[18]} & aW \xrightarrow{[8]} & aB \xrightarrow{[14]} & B \\
 \downarrow [17] & \downarrow [11] & \uparrow [12] & \uparrow [19] & \uparrow [19] & \uparrow 2.12 & \uparrow 2.14 & \\
 W & aB & (aw) & (Bb) \xleftarrow{[19]} & (Bw) & (SBaw) \xrightarrow{2.15} & (SBab) & \\
 & & \downarrow [13] & \uparrow [19] & \uparrow [19] & & \uparrow 2.11 & \\
 & & W & (Bab) \xleftarrow{[19]} & (Baw) & (SBw) \xrightarrow{2.4} & (SBb) \xrightarrow{2.5} & (gb) \\
 & & & \downarrow [19] & \downarrow [19] & \downarrow 2.7 & \downarrow 2.8 & \\
 & & & (ab) & (aw) & (Bw) & (Bb) &
 \end{array}$$

Table 1

The following table summarizes the meaning of various theorems and properties.

W	$\sigma(T) \setminus \sigma_W(T) = E^0(T)$	B	$\sigma(T) \setminus \sigma_W(T) = \Pi^0(T)$
gW	$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$	gB	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi(T)$
(Bw)	$\sigma(T) \setminus \sigma_{BW}(T) = E^0(T)$	(Bb)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi^0(T)$
(w)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$	(b)	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi^0(T)$
(gw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$	(gb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi(T)$
(SBw)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^0(T)$	(SBb)	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi^0(T)$
(aw)	$\sigma(T) \setminus \sigma_W(T) = E_a^0(T)$	(ab)	$\sigma(T) \setminus \sigma_W(T) = \Pi_a^0(T)$
(gaw)	$\sigma(T) \setminus \sigma_{BW}(T) = E_a(T)$	(gab)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a(T)$
(Baw)	$\sigma(T) \setminus \sigma_{BW}(T) = E_a^0(T)$	(Bab)	$\sigma(T) \setminus \sigma_{BW}(T) = \Pi_a^0(T)$
aW	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$	aB	$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \Pi_a^0(T)$
gaW	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$	gaB	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a(T)$
$(SBaw)$	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a^0(T)$	$(SBab)$	$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \Pi_a^0(T)$

Table 2

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