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$\omega_{\mathcal{I},\gamma}$-CONTINUOUS FUNCTIONS AND WEAKLY $\omega_{\mathcal{I},\gamma}$-CONTINUOUS FUNCTIONS

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Abstract. Using the $\omega_{\mathcal{I},\gamma}$-closed sets defined in [3], we introduce the notions of $\omega_{\mathcal{I},\gamma}$-continuous functions and weakly $\omega_{\mathcal{I},\gamma}$-continuous functions. Characterizations and properties of this new class of functions are obtained and studied.

1. Introduction

In 2012, Carpintero et al. [3], introduced the concept of $\omega_{\mathcal{I},\gamma}$-closed sets in terms of operators on topological spaces as a generalization of the concept given by Arhangel'skiĭ [1] and Ekici et al. in [6]. The purpose of this article is to introduce and to study a new class of functions called $\omega_{\mathcal{I},\gamma}$-continuous and weakly $\omega_{\mathcal{I},\gamma}$-continuous in terms of the $\omega_{\mathcal{I},\gamma}$-closed sets. These new functions form a more general class than the class of functions given in [4], [5], [6], [7], [12] and [13]. We give some characterizations and properties of the $\omega_{\mathcal{I},\gamma}$-continuous functions and weakly $\omega_{\mathcal{I},\gamma}$-continuous functions. Finally, we study some notions of compactness and connectedness on this new class of functions.

2. Preliminaries

An ideal $\mathcal{I}$ on a nonempty set X [11] is a collection of subsets of X that satisfies the following properties:

(1) $A_1 \in \mathcal{I}$ and $A_2 \in \mathcal{I}$ imply that $A_1 \cup A_2 \in \mathcal{I}$;
(2) $A_1 \in \mathcal{I}$ and $A_2 \subset A_1$ imply that $A_2 \in \mathcal{I}$.

An operator associated to a topology τ on X [10] is an application $\gamma : 2^X \to 2^X$ such that $U \subset \gamma(U)$ for all $U \in \tau$. For a subset $A \subset X$, the γ-closure of A and the γ-interior of A are defined as follows:

(1) $\gamma\text{-}\operatorname{Cl}(A) = \{x \in X, \gamma(U) \cap A \neq \emptyset,\ U \in \tau,\ x \in U\}$.

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(2) γ-$int(A) = \{x \in U, U \in \tau, \gamma(U) \subset A\}$.

A subset A of X is said to be γ-open if $A = \gamma$-$int(A)$, and the complement of a γ-open set is a γ-closed set. Thus when γ is the closure operator, we obtain the concept of θ-closure [17].

The properties of γ-closure and γ-interior have been studied in [10] and [15].

Throughout the present paper $(X,\tau,\mathcal{I},\gamma)$ denote a topological space (X,τ) together with an ideal $\mathcal{I}$ on X and an operator γ associated to a topology τ.

Definition 2.1. [3]. Let $(X,\tau,\mathcal{I},\gamma)$ be given. A subset A of X is said to be $\omega_{\mathcal{I},\gamma}$-closed set if

$$\gamma\text{-}cl(B) \subset A \text{ for all } B \subset A,\ B \in \mathcal{I}.$$

The complement of an $\omega_{\mathcal{I},\gamma}$-closed set is called $\omega_{\mathcal{I},\gamma}$-open. Each γ-open set is an $\omega_{\mathcal{I},\gamma}$-open set [3].

In a similar form as for the closure and the interior of a subset A in a topological space, we can define [3]:

(1) The $\omega_{\mathcal{I},\gamma}$-closure of A, denoted by $\omega_{\mathcal{I},\gamma}$-$cl(A)$, is the intersection of all $\omega_{\mathcal{I},\gamma}$-closed sets containing A.
(2) The $\omega_{\mathcal{I},\gamma}$-interior of A, denoted by $\omega_{\mathcal{I},\gamma}$-$int(A)$, is the union of all $\omega_{\mathcal{I},\gamma}$-open sets contained in A.

The properties of $\omega_{\mathcal{I},\gamma}$-closed sets and $\omega_{\mathcal{I},\gamma}$-open sets have been studied in detail in [3].

3. Weakly forms of continuity using $\omega_{\mathcal{I},\gamma}$-open sets

Using the $\omega_{\mathcal{I},\gamma}$-open sets, we define a new weak form of continuity between topological spaces in order to generalize some forms of continuity studied in [4], [6], [5], [12], [13], [7], and we give some characterizations of those.

Definition 3.1. Let $(X,\tau,\mathcal{I},\gamma)$ and (Y,σ) be a topological space. A function $f : X \to Y$ is said to be $\omega_{\mathcal{I},\gamma}$-continuous at $x \in X$ if for each open set V in Y such that $f(x) \in V$, there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X containing x such that $f(U) \subset V$. The function f is said to be $\omega_{\mathcal{I},\gamma}$-continuous if it is $\omega_{\mathcal{I},\gamma}$-continuous at each $x \in X$.

Observe that if we consider specific operators and ideals, we recover some well known notions of continuity in the literature as follows:

(1) If γ is the identity operator and $\mathcal{I}$ is the ideal of the countable subsets of X, then the $\omega_{\mathcal{I},\gamma}$-continuous functions coincide with the ω-continuous functions defined in [6].

(2) If γ is the identity operator and the ideal $\mathcal{I}$ is the power sets of X, then the $\omega_{\mathcal{I},\gamma}$-continuous functions coincide with the continuous functions defined in [14].
(3) If γ is the closure operator and the ideal $\mathcal{I}$ is the power sets of X, then the $\omega_{\mathcal{I},\gamma}$-continuous functions coincide with the θ-continuous functions defined in [7].
(4) If γ is the interior closure operator and the ideal $\mathcal{I}$ is the power sets of X, the $\omega_{\mathcal{I},\gamma}$-continuous functions coincide with the super continuous functions defined in [13].
(5) If γ is any operator and the ideal $\mathcal{I}$ is the power sets of X, then the $\omega_{\mathcal{I},\gamma}$-continuous functions coincide with the γ-continuous functions defined in [4].
(6) If γ is the closure operator and $\mathcal{I}$ is the ideal of the countable subsets of X, then the $\omega_{\mathcal{I},\gamma}$-continuous functions coincide with the ω_*-continuous functions defined in [6].

Theorem 3.2. *Let $(X,\tau,\mathcal{I},\gamma)$ and (Y,σ) be a topological space. The function $f : X \to Y$ is $\omega_{\mathcal{I},\gamma}$-continuous if and only if $f^{-1}(V)$ is an $\omega_{\mathcal{I},\gamma}$-open subset in X for every open subset V of Y.*

Proof. Let V be an open subset of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$, by hypothesis, there exists an $\omega_{\mathcal{I},\gamma}$-open subset U of X containing x and $f(U) \subset V$, follows $x \in U \subset f^{-1}(V)$ and then, $x \in \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(V))$. Since $\omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(V)) \subset f^{-1}(V)$ then $f^{-1}(V) = \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(V))$ in consequence, $f^{-1}(V)$ is an $\omega_{\mathcal{I},\gamma}$-open subset in X.

Reciprocally. Let $x \in X$ and V be an open set in Y such that $f(x) \in V$, follows $x \in f^{-1}(V)$. By hypothesis, $f^{-1}(V)$ is an $\omega_{\mathcal{I},\gamma}$-open set in X. Therefore, there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X such that $x \in U \subset f^{-1}(V)$. Follows that, $f(U) \subset V$. In consequence, f is an $\omega_{\mathcal{I},\gamma}$-continuous function. □

Definition 3.3. [4] A function $f : X \to Y$ is said to be γ-continuous if $f^{-1}(V)$ is a γ-open subset in X for each open set V in Y.

The following theorem shows that the $\omega_{\mathcal{I},\gamma}$-continuous functions are a more general class than that of the γ-continuous functions.

Theorem 3.4. *Let $(X,\tau,\mathcal{I},\gamma)$ and (Y,σ) be a topological space. If $f : X \to Y$ is a γ-continuous function then it is an $\omega_{\mathcal{I},\gamma}$-continuous function.*

Proof. Let V be an open subset in Y. Since f is a γ-continuous function then $f^{-1}(V)$ is a γ-open set in X and therefore $f^{-1}(V)$ is a $\omega_{\mathcal{I},\gamma}$-open set in X; because all γ-open set is an $\omega_{\mathcal{I},\gamma}$-open set. In consequence, f is an $\omega_{\mathcal{I},\gamma}$-continuous function. □

The converse of the above theorem is not true in general as we show in the following example:

Example 3.5. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$, γ the interior closure operator and the ideal $\mathcal{I} = \{\emptyset, \{b\}\}$. Let $f : (X,\tau) \to (X,\sigma)$ defined as:

$$f(x) = \begin{cases} a & \text{if} \quad x = c \\ \\ b & \text{if} \quad x \neq c \end{cases}$$

We can see that f is $\omega_{\mathcal{I},\gamma}$-continuous but not γ-continuous.

Theorem 3.6. *Let $(X,\tau,\mathcal{I},\gamma)$, (Y,σ) be a topological space and $f : X \to Y$ a function. The following statements are equivalent:*

(1) *f is an $\omega_{\mathcal{I},\gamma}$-continuous function.*
(2) *$f(\omega_{\mathcal{I},\gamma}\text{-}cl(A)) \subset cl(f(A))$ for all subsets A of X.*
(3) *$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(B)) \subset f^{-1}(cl(B))$ for all subsets B of Y.*
(4) *$f^{-1}(int(B)) \subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(B))$ for all subsets B of Y.*
(5) *$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(K)) = f^{-1}(K)$ for all closed subsets K of Y.*
(6) *$\omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(V)) = f^{-1}(V)$ for all open subsets V of Y.*

Proof.
(1) → (2). Let $y \in f(\omega_{\mathcal{I},\gamma}\text{-}cl(A))$, then $y = f(x)$ with $x \in \omega_{\mathcal{I},\gamma}\text{-}cl(A)$. Let V be an open set in Y such that $y \in V$, then $x \in f^{-1}(V)$. Since f is an $\omega_{\mathcal{I},\gamma}$-continuous function, $f^{-1}(V)$ is an $\omega_{\mathcal{I},\gamma}$-open set in X containing x. But $x \in \omega_{\mathcal{I},\gamma}\text{-}cl(A)$, so that $f^{-1}(V) \cap A \neq \emptyset$. It follows that $V \cap f(A) \neq \emptyset$; and consequence $y \in cl(f(A))$.
(2) → (3). Let B be any subset of Y. By hypothesis,

$$f(\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(B))) \quad \subset \quad cl(f(f^{-1}(B))).$$

Then,

$$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(B)) \quad \subset \quad f^{-1}(cl(B)).$$

(3) → (4). Let B be any subset of Y, then

$$\begin{aligned} X \setminus \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(B)) &= \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(Y \setminus B)) \\ &\subset f^{-1}(cl(Y \setminus B)) \\ &= X \setminus f^{-1}(int(B)). \end{aligned}$$

Taking complements, we obtain $f^{-1}(int(B)) \subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(B))$.

(4) → (5). Let K be any closed subset in Y, then

$$\begin{aligned} X \setminus f^{-1}(K) = f^{-1}(int(Y \setminus K)) &\subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(Y \setminus K)) \\ &= X \setminus \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(K)). \end{aligned}$$

Taking complements, we obtain $\omega_{\mathcal{I},\gamma}$-$cl(f^{-1}(K)) \subset f^{-1}(K)$, and therefore

$$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(K)) = f^{-1}(K).$$

(5) → (6). Let V be any open subset in Y, then

$$\begin{aligned} X \setminus f^{-1}(V) &= f^{-1}(Y \setminus V) \\ &= \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(Y \setminus V)) \\ &= \omega_{\mathcal{I},\gamma}\text{-}cl(X \setminus f^{-1}(V)) \\ &= X \setminus \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(V)). \end{aligned}$$

Taking complements, we obtain

$$\omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(V)) = f^{-1}(V).$$

(6) → (1). Given $x \in X$, let V be any open subset in Y such that $f(x) \in V$; then $x \in f^{-1}(V) = \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(V))$, and there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X, such that $x \in U$ and $U \subset f^{-1}(V)$. In consequence, $f(U) \subset V$ and f is an $\omega_{\mathcal{I},\gamma}$-continuous function. □

Definition 3.7. Let $(X, \tau, \mathcal{I}, \gamma)$ and (Y, σ) be a topological space. A function $f : X \to Y$ is said to be weakly $\omega_{\mathcal{I},\gamma}$-continuous at the point $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X containing x and $f(U) \subset cl(V)$. The function f is said to be weakly $\omega_{\mathcal{I},\gamma}$-continuous if it is weakly $\omega_{\mathcal{I},\gamma}$-continuous at each point $x \in X$.

Observe that if we consider specific operators and ideals, we recover some well known notions of weak continuity in the literature as we show:

(1) If γ is the identity operator and $\mathcal{I}$ is the ideal of the countable subset of X, then the weakly $\omega_{\mathcal{I},\gamma}$-continuous function coincides with the weakly ω-continuous function defined in [6].
(2) If γ is the identity operator and the ideal $\mathcal{I}$ is the power set of X, then the weakly $\omega_{\mathcal{I},\gamma}$-continuous function coincides with the weakly continuous function defined in [12].
(3) If γ is the closure operator and $\mathcal{I}$ is the ideal of the countable subset of X, then the weakly $\omega_{\mathcal{I},\gamma}$-continuous function coincides with the weakly ω_*-continuous functions defined in [6].
(4) If γ is any operator and the ideal $\mathcal{I}$ is the power set of X, then the weakly $\omega_{\mathcal{I},\gamma}$-continuous function coincides with the weakly γ-continuous function defined in [5].

Theorem 3.8. *Let $(X,\tau,\mathcal{I},\gamma)$, (Y,σ) be a topological space and $f : X \to Y$ a function. If $f^{-1}(cl(V))$ is an $\omega_{\mathcal{I},\gamma}$-open set in X for all open subsets V in Y then f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function.*

Proof. Let $x \in X$ and V be an open subset in Y containing $f(x)$. Then $f(x) \in cl(V)$ and therefore $x \in f^{-1}(cl(V))$. Take $U = f^{-1}(cl(V))$ and obtain by hypothesis that U is an $\omega_{\mathcal{I},\gamma}$-open set in X containing x. Since, $f(U) = f(f^{-1}(cl(V))) \subset cl(V)$, it follows that f is an $\omega_{\mathcal{I},\gamma}$-continuous function. □

The converse of the above theorem is not true in general as we show in the following example:

Example 3.9. Let γ be the identity operator and $\mathcal{I}$ be the ideal of the countable real sets. The identity function $i : (\mathbb{R},\tau_u) \to (\mathbb{R},\tau_u)$ where τ_u is the usual topology, is weakly $\omega_{\mathcal{I},\gamma}$-continuous. In effect, for $x \in \mathbb{R}$, take V an open set in $\mathbb{R}$ such that $i(x) \in V$, then V is $\omega_{\mathcal{I},\gamma}$-open set in $\mathbb{R}$ such that $i(V) \subset cl(V)$. It follows that, $i : (\mathbb{R},\tau_u) \to (\mathbb{R},\tau_u)$ is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function. But it is not an $\omega_{\mathcal{I},\gamma}$-continuous function. Because, if we take $V = (0,1)$ open set in $\mathbb{R}$, then $i^{-1}(cl((0,1))) = [0,1]$ is not an $\omega_{\mathcal{I},\gamma}$-open set.

The following theorem shows that the class of the weakly $\omega_{\mathcal{I},\gamma}$-continuous functions contain the class of the $\omega_{\mathcal{I},\gamma}$-continuous functions.

Theorem 3.10. *Let $(X,\tau,\mathcal{I},\gamma)$, (Y,σ) be a topological space and $f : X \to Y$ a function. If f is an $\omega_{\mathcal{I},\gamma}$-continuous function then it is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function.*

Proof. Let $x \in X$ and V be an open subset in Y containing $f(x)$. Since f is an $\omega_{\mathcal{I},\gamma}$-continuous function, there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X such that $f(U) \subset V \subset cl(V)$. Therefore, f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function. □

The converse of the above theorem is not true in general as we show in the following example:

Example 3.11. Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}\}$, γ the interior closure operator and $\mathcal{I}$ the ideal of the power set of X. The function $f : (X,\tau) \to (X,\tau)$ defined as follows:

$$f(x) = \begin{cases} a & \text{if} \quad x = a \\ b & \text{if} \quad x \neq a. \end{cases}$$

We can see that f is weakly $\omega_{\mathcal{I},\gamma}$-continuous but not $\omega_{\mathcal{I},\gamma}$-continuous.

Theorem 3.12. *Let $(X,\tau,\mathcal{I},\gamma)$, (Y,σ) be a topological space and $f : X \to Y$ a function. The following statements are equivalent:*

(1) *f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function.*
(2) *$\omega_{\mathcal{I},\gamma}$-$cl(f^{-1}(int(cl(K)))) \subset f^{-1}(cl(K))$ for all $K \subset Y$.*
(3) *$\omega_{\mathcal{I},\gamma}$-$cl(f^{-1}(int(U))) \subset f^{-1}(U)$ for all regular closed sets $U \subset Y$.*
(4) *$\omega_{\mathcal{I},\gamma}$-$cl(f^{-1}(U)) \subset f^{-1}(cl(U))$ for all open sets $U \subset Y$.*
(5) *$f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(U)))$ for all open sets $U \subset Y$.*
(6) *$\omega_{\mathcal{I},\gamma}$-$cl(f^{-1}(U)) \subset f^{-1}(cl(U))$ for all preopen sets $U \subset Y$.*
(7) *$f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(U)))$ for all preopen sets $U \subset Y$.*

Proof.
(1) $\to$ (2). Let $K \subset Y$ and $x \in X \setminus f^{-1}(cl(K))$. Then $f(x) \notin cl(K)$. It follows that there exists an open set U in Y such that $f(x) \in U$ and $U \cap K = \emptyset$. We obtain $cl(U) \cap int(cl(K)) = \emptyset$. Since f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function, then there exists an $\omega_{\mathcal{I},\gamma}$-open set V such that $x \in V$ and $f(V) \subset cl(U)$. It follows that $V \cap f^{-1}(int(cl(K))) = \emptyset$. In consequence, $x \in X \setminus \omega_{\mathcal{I},\gamma}$-$cl(f^{-1}(int(cl(K))))$.
(2) $\to$ (3). Let U be any regular closed set in Y. Then

$$\begin{aligned}\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(int(U))) &= \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(int(cl(int(U)))))\\ &\subset f^{-1}(cl(int(U))) = f^{-1}(U).\end{aligned}$$

(3) $\to$ (4). Let U be any open set in Y. Since $cl(U)$ is a regular closed set in Y,

$$\begin{aligned}\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(U)) &\subset \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(int(cl(U))))\\ &\subset f^{-1}(cl(U)).\end{aligned}$$

(4) $\to$ (5). Let U be any open set in Y. Since $Y \setminus cl(U)$ is an open set in Y, then

$$\begin{aligned}X \setminus \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U))) &= \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(Y \setminus cl(U)))\\ &\subset f^{-1}(cl(Y \setminus cl(U)))\\ &\subset X \setminus f^{-1}(U).\end{aligned}$$

In consequence, $f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(U)))$.
(5) $\to$ (1). Let $x \in X$ and U be any open subset in Y containing $f(x)$. It follows that,

$$x \in f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U))).$$

Take $V = \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(U)))$ and obtain that $f(V) \subset cl(U)$ and f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function.
(1) $\to$ (6). Let U be any preopen subset in Y and $x \in X \setminus f^{-1}(cl(U))$. It follows that $x \notin f^{-1}(cl(U))$, $f(x) \notin cl(U)$. Then there exists an open set

S such that $f(x) \in S$ and $S \cap U = \emptyset$. We obtain that $cl(U \cap S) = \emptyset$ and therefore

$$U \cap cl(S) \subset cl(U \cap S) = \emptyset.$$

Since f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function and $f(x) \in S$, there exists an $\omega_{\mathcal{I},\gamma}$-open set V in X, with $x \in V$ and $f(V) \subset cl(S)$. It follows that $f(V) \cap U = \emptyset$, $V \cap f^{-1}(U) = \emptyset$. Therefore, $x \in X \setminus \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(U))$ and $\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(U)) \subset f^{-1}(cl(U))$.

<u>(6) → (7).</u> Let U be any preopen subset in Y. Then

$$\begin{aligned} X \setminus \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U))) &= \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(Y \setminus cl(U))) \\ &\subset f^{-1}(cl(Y \setminus cl(U))) \\ &\subset X \setminus f^{-1}(U). \end{aligned}$$

In consequence, $f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U)))$.

<u>(7) → (1).</u> Let $x \in X$ and U be any open set in Y such that $f(x) \in U$. It follows that

$$x \in f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U))).$$

Take $V = \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U)))$, obtain that $f(U) \subset cl(V)$ and f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function. □

Theorem 3.13. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space and $f : X \to Y$ a function. The following statements are equivalent:*

1. *f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function.*
2. *$f(\omega_{\mathcal{I},\gamma}\text{-}cl(G)) \subset \theta\text{-}cl(f(G))$ for any $G \subset X$.*
3. *$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}cl(A))$ for any $A \subset Y$.*
4. *$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(int(\theta\text{-}cl(A)))) \subset f^{-1}(\theta\text{-}cl(A)))$ for any $A \subset Y$.*

Proof.

<u>(1) → (2).</u> Let $G \subset X$ and $x \in \omega_{\mathcal{I},\gamma}\text{-}cl(G)$. Let V be any open set in Y such that $f(x) \in V$. Since f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function, then there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X containing x such that $f(U) \subset cl(V)$. Now $x \in \omega_{\mathcal{I},\gamma}\text{-}cl(G)$ and U is an $\omega_{\mathcal{I},\gamma}$-open set with $x \in U$, so that $U \cap G \neq \emptyset$. It follows that

$$\emptyset \neq f(U) \cap f(G) \subset cl(V) \cap f(G).$$

In consequence, $f(x) \in \theta\text{-}cl(f(G))$ and $f(\omega_{\mathcal{I},\gamma}\text{-}cl(G)) \subset \theta\text{-}cl(f(G))$.

<u>(2) → (3).</u> Let A be any subset of Y, then

$$\begin{aligned} f(\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(A))) &\subset \theta\text{-}cl(f(f^{-1}(A))) \\ &\subset \theta\text{-}cl(A). \end{aligned}$$

Therefore,

$$\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(A)) \subset f^{-1}(f(\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(A)))) \subset f^{-1}(\theta\text{-}cl(A)).$$

(3) → (4). Let $A \subset Y$. Since $\theta\text{-}cl(A)$ is a closed set in Y, then

$$\begin{aligned}\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(int(\theta\text{-}cl(A)))) &\subset f^{-1}(\theta\text{-}cl(int(\theta\text{-}cl(A))))\\ &= f^{-1}(cl(int(\theta\text{-}cl(A))))\\ &\subset f^{-1}(\theta\text{-}cl(A)).\end{aligned}$$

(4) → (1). Let U be any open set in Y, then $cl(U) = \theta\text{-}cl(U)$. It follows that $U \subset int(cl(U)) = int(\theta\text{-}cl(U))$ and we obtain

$$\begin{aligned}\omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(U)) &\subset \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(int(\theta\text{-}cl(U))))\\ &\subset f^{-1}(\theta\text{-}cl(U))\\ &= f^{-1}(cl(U)).\end{aligned}$$

By Theorem 3.12-(4), f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function. □

Definition 3.14. Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space. A function $f : X \to Y$ is said to be coweakly $\omega_{\mathcal{I},\gamma}$-continuous if $f^{-1}(Fr(U))$ is an $\omega_{\mathcal{I},\gamma}$-closed set in X for each open set U in Y.

Theorem 3.15. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space and $f : X \to Y$ a function. If f is an $\omega_{\mathcal{I},\gamma}$-continuous function then it is a coweakly $\omega_{\mathcal{I},\gamma}$-continuous function.*

Proof. Let U be any open set in Y; then,

$$\begin{aligned}Fr(U) &= cl(U) \cap cl(Y \setminus U)\\ f^{-1}(Fr(U)) &= f^{-1}(cl(U) \cap cl(Y \setminus U))\\ &= f^{-1}(cl(U)) \cap f^{-1}(cl(Y \setminus U)).\end{aligned}$$

Since f is an $\omega_{\mathcal{I},\gamma}$-continuous function and $cl(U)$, $cl(Y \setminus U)$ are closed subsets in Y, then $f^{-1}(cl(U))$ and $f^{-1}(cl(Y \setminus U))$ are $\omega_{\mathcal{I},\gamma}$-closed sets in X. Using that the intersection of $\omega_{\mathcal{I},\gamma}$-closed sets is an $\omega_{\mathcal{I},\gamma}$-closed set [3], then $f^{-1}(Fr(U))$ is a $\omega_{\mathcal{I},\gamma}$-closed set in X. This shows that f is a coweakly $\omega_{\mathcal{I},\gamma}$-continuous function. □

Example 3.16. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, γ the interior closure operator. $\mathcal{I}$ the ideal of the power sets of X. Define $f : (X, \tau) \to (X, \tau)$ as follows:

$$f(x) = \begin{cases} a & \text{if} \quad x = c \\ b & \text{if} \quad x \neq c. \end{cases}$$

It is easy to see that f is a function coweakly $\omega_{\mathcal{I},\gamma}$-continuous but is not $\omega_{\mathcal{I},\gamma}$-continuous.

Remark 3.17. Observe that the class of the $\omega_{\mathcal{I},\gamma}$-continuous functions is contained in the intersection of the classes of the weakly $\omega_{\mathcal{I},\gamma}$-continuous functions and of the coweakly $\omega_{\mathcal{I},\gamma}$-continuous functions. Equality is obtained when the operator γ is a regular operator.

The following example shows that if the operator γ is not a regular operator, there exists a function f that is both weakly $\omega_{\mathcal{I},\gamma}$-continuous and coweakly $\omega_{\mathcal{I},\gamma}$-continuous but not $\omega_{\mathcal{I},\gamma}$-continuous.

Example 3.18. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$, $\mathcal{I}$ the ideal of the power sets of X and γ the operator defined as:

$$\gamma(A) = \begin{cases} cl(A) & \text{if} \quad b \notin A \\ \\ A & \text{if} \quad x \in A. \end{cases}$$

It is easy to see that the operator γ is not a regular operator and the identity function $i : (X, \tau) \to (X, \tau)$ is coweakly $\omega_{\mathcal{I},\gamma}$-continuous and weakly $\omega_{\mathcal{I},\gamma}$-continuous but not $\omega_{\mathcal{I},\gamma}$-continuous, because $\{a\}$ is an open set and $i^{-1}(\{a\}) = \{a\}$ is not an $\omega_{\mathcal{I},\gamma}$-open set.

Theorem 3.19. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space and $f : X \to Y$ a function. The following statements are equivalent:*

(1) *f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function at $x \in X$.*
(2) *$x \in \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(V)))$ for any open set V in Y with $f(x) \in V$.*

Proof.
(1) → (2). Let $x \in X$ and V be an open subset in Y containing $f(x)$. Since f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function, then there exists an $\omega_{\mathcal{I},\gamma}$-open set U in X with $x \in U$ and $f(U) \subset cl(V)$. Since $U \subset f^{-1}(cl(V))$ and U is an $\omega_{\mathcal{I},\gamma}$-open set, then $x \in U \subset \omega_{\mathcal{I},\gamma}$-$int(U) \subset \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(V)))$. (2) → (1). Let $x \in \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(V)))$ for any open subset V in Y containing $f(x)$. Put $U = \omega_{\mathcal{I},\gamma}$-$int(f^{-1}(cl(V)))$. It follows that U is an $\omega_{\mathcal{I},\gamma}$-open set and $f(U) \subset cl(V)$. Therefore, f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function at $x \in X$. □

Definition 3.20. [6] A subset A of X is said to be an N-closed set relative to X if for any covering $\{A_i : i \in I\}$ of A by open sets in X, there exists a finite subcollection $I_0 \subset I$ such that $A \subset \bigcup_{i \in I_0} cl(A_i)$.

Theorem 3.21. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space. If $f : X \to Y$ is a weakly $\omega_{\mathcal{I},\gamma}$-continuous and Y is Hausdorff, the following statements hold:*

(1) *For each $(x, y) \notin G(f)$, there exist an $\omega_{\mathcal{I},\gamma}$-open set $G \subset X$ and an open set $U \subset Y$ such that $x \in G$, $y \in U$ and $f(G) \cap int(cl(U)) = \emptyset$.*

(2) *The inverse image of each N-closed set in Y is an $\omega_{\mathcal{I},\gamma}$-closed set in X if γ is a regular operator.*

Proof.

(1) Suppose that $(x, y) \notin G(f)$, so that $y \neq f(x)$. Since Y is Haussdorff, there exist open sets U and V such that $y \in U$, $f(x) \in V$ and $U \cap V = \emptyset$. It follows that $int(cl(U)) \cap cl(V) = \emptyset$. From the fact that f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function, there exists an $\omega_{\mathcal{I},\gamma}$-open set G such that $x \in G$ with $f(G) \subset cl(V)$. Therefore, $f(G) \cap int(cl(U)) = \emptyset$.

(2) Suppose that there exists a N-closed set $W \subset Y$ such that $f^{-1}(W)$ is not an $\omega_{\mathcal{I},\gamma}$-closed set in X; then there exists a point $x \in \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(W)) \setminus f^{-1}(W)$. Since $x \notin f^{-1}(W)$, then $(x, y) \notin G(f)$ for all $y \in Y$. By (1), there exist an $\omega_{\mathcal{I},\gamma}$-open set $G_y(x) \subset X$ and $B(y)$, an open subset of Y, such that $x \in G_y(x)$, $y \in B(y)$ and $f(G_y(x)) \cap int(cl(B(y))) = \emptyset$. The family $\{B(y) : y \in W\}$ is a covering of W by open sets in Y . Since W is a N-closed set, then there exist a finite number of points y_1, y_2,..., y_n in W such that $W \subset \bigcup_{j=1}^{n} int(cl(B(y_j)))$. Put $G = \bigcap_{j=1}^{n} G_{y_j}(x)$; then, $f(G) \cap W = \emptyset$. Observe that G is $\omega_{\mathcal{I},\gamma}$-open because γ is a regular operator [3]. Since $x \in \omega_{\mathcal{I},\gamma}\text{-}cl(f^{-1}(W))$, then $f(G) \cap W \neq \emptyset$. This is a contradiction. □

Theorem 3.22. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space and $f : X \to Y$ a function. If the graph function of f, say $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$, is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function then $f : X \to Y$ is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function.*

Proof. Suppose that g is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function, $x \in X$ and A is an open set in X such that $f(x) \in A$. Then, $X \times A$ is an open set such that $g(x) \in X \times A$. Since g is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function, there exists an $\omega_{\mathcal{I},\gamma}$-open set B containing x such that $g(B) \subset cl(X \times A) = X \times cl(A)$. It follows that, $f(B) \subset cl(A)$ and f is a weakly $\omega_{\mathcal{I},\gamma}$-continuous function. □

4. $\omega_{\mathcal{I},\gamma}$-CONNECTED SPACES AND $\omega_{\mathcal{I},\gamma}$-COMPACT SPACES

In this section we introduce the notions of connectedness and compactness associated with the $\omega_{\mathcal{I},\gamma}$-open sets. Also, we study the behavior of these notions under the action of weakly $\omega_{\mathcal{I},\gamma}$-continuous functions.

Definition 4.1. Let $(X, \tau, \mathcal{I}, \gamma)$ be given. An $\omega_{\mathcal{I},\gamma}$-separation of X is a pair U, V of nonempty disjoint $\omega_{\mathcal{I},\gamma}$-open sets of X whose union is X. The space X is said to be an $\omega_{\mathcal{I},\gamma}$-connected if there exists no $\omega_{\mathcal{I},\gamma}$-separation of X.

Theorem 4.2. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space. If $f : X \to Y$ is a weakly $\omega_{\mathcal{I},\gamma}$-continuous, surjective function and X is an $\omega_{\mathcal{I},\gamma}$-connected space, then Y is a connected space.*

Proof. Suppose that Y is not a connected space, then there exist a nonempty open sets U and V such that $Y = U \cup V$ and $U \cap V = \emptyset$. This implies that U and V are clopen subsets in Y. By Theorem 3.12, $f^{-1}(U) \subset \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(cl(U))) = \omega_{\mathcal{I},\gamma}\text{-}int(f^{-1}(U))$. It follows that, $f^{-1}(U)$ is an $\omega_{\mathcal{I},\gamma}$-open set in X. Similarly, $f^{-1}(U)$ is an $\omega_{\mathcal{I},\gamma}$-open set in X. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, $X = f^{-1}(U) \cup f^{-1}(V)$ and $f^{-1}(U)$, $f^{-1}(V)$ are nonempty. In follows that X is not an $\omega_{\mathcal{I},\gamma}$-connected space. □

Definition 4.3. Let $(X, \tau, \mathcal{I}, \gamma)$ be given. A subset A of X is said to be an $\omega_{\mathcal{I},\gamma}$-compact space relative to X if for any covering $\{U_\alpha : \alpha \in I\}$ of A by $\omega_{\mathcal{I},\gamma}$-open sets in X, there exists a finite subset I_0 of I such that $A \subset \bigcup\{U_\alpha : \alpha \in I_0\}$. The space X is said to be $\omega_{\mathcal{I},\gamma}$-compact if it is $\omega_{\mathcal{I},\gamma}$-compact as a subspace.

Theorem 4.4. *Let $(X, \tau, \mathcal{I}, \gamma)$ be an $\omega_{\mathcal{I},\gamma}$-compact space. Then, every $\omega_{\mathcal{I},\gamma}$-closed set B is an $\omega_{\mathcal{I},\gamma}$-compact space.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be a covering of B by $\omega_{\mathcal{I},\gamma}$-open subsets in X. This implies $B \subset \bigcup_{\alpha\in I} U_\alpha$ and $(X \setminus B) \cup (\bigcup_{\alpha\in I} U_\alpha) = X$. By hypothesis, X is an $\omega_{\mathcal{I},\gamma}$-compact space, so there exists a finite subset I_0 of I such that $B \subset \bigcup_{\alpha\in I_0} U_\alpha$. It follows that B is an $\omega_{\mathcal{I},\gamma}$-compact space. □

Theorem 4.5. *Let $(X, \tau, \mathcal{I}, \gamma)$, (Y, σ) be a topological space. If $f : X \to Y$ is an $\omega_{\mathcal{I},\gamma}$-continuous function and X is an $\omega_{\mathcal{I},\gamma}$-compact space then $f(X)$ is a compact set.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be a covering of $f(X)$ by open subsets in Y. Since f is an $\omega_{\mathcal{I},\gamma}$-continuous function, $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a covering of X by $\omega_{\mathcal{I},\gamma}$-open subsets in X. By hypothesis, X is an $\omega_{\mathcal{I},\gamma}$-compact space, it follows that there exists a finite subset I_0 of I such that $X = \bigcup_{\alpha=1}^{n} f^{-1}(U_\alpha)$. Then,

$$\begin{aligned} f(X) &= f\Big(\bigcup_{\alpha=1}^{n} f^{-1}(U_\alpha)\Big) \\ &= f\Big(f^{-1}\Big(\bigcup_{\alpha=1}^{n} U_\alpha\Big)\Big) \\ &\subset \bigcup_{\alpha=1}^{n} U_\alpha. \end{aligned}$$

This says that $\{U_1, \ldots, U_n\}$ is a finite open subcover of $f(X)$. Therefore, $f(X)$ is a compact set. □

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