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NEW SHARP ERROR BOUNDS FOR SOME CORRECTED QUADRATURE FORMULAE

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ABSTRACT. A generalization of the pre-Grüss inequality is used to derive a new sharp L_2 inequality which provides improved versions of some corrected inequalities that appear in the literature. An application to numerical integration is illustrated.

1. INTRODUCTION

In [4], the following sharp bounds for the errors in a unified corrected quadrature formulae are obtained:

Theorem 1. *Let $f : [a,b] \to \mathbf{R}$ be such that f' is absolutely continuous on $[a,b]$ and $f'' \in L_2[a,b]$, then for any $\theta \in [0,1]$,*

$$\left| \int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right) + \theta\frac{f(a)+f(b)}{2}\right] - \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{24\sqrt{5}}(15\theta^2-15\theta+4)^{\frac{1}{2}}\sqrt{\sigma(f'')}, \quad (1)$$

where $\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}(\int_a^b f(t)\,dt)^2$ and $\|f\|_2 := [\int_a^b f^2(t)\,dt]^{\frac{1}{2}}$.

Inequality (1) is sharp in the sense that the constant $\frac{1}{24\sqrt{5}}$ cannot be replaced by a smaller one.

Specifically, if we take $\theta = 1, 0, \frac{1}{3}, \frac{7}{15}$ and $\frac{1}{2}$ in (1), we obtain, respectively, the sharp corrected trapezoid type inequality, the sharp corrected midpoint type inequality, the sharp Simpson type inequality, the sharp corrected Simpson type inequality and the sharp corrected average midpoint-trapezoid type inequality. These are:

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$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}, \tag{2}$$

$$\left|\int_a^b f(t)\,dt - (b-a)f\left(\frac{a+b}{2}\right) - \frac{(b-a)^2}{24}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}}\sqrt{\sigma(f'')}, \tag{3}$$

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{30}}\sqrt{\sigma(f'')}, \tag{4}$$

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] + \frac{(b-a)^2}{60}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{60\sqrt{3}}\sqrt{\sigma(f'')} \tag{5}$$

and

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{(b-a)^2}{48}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}}\sqrt{\sigma(f'')}. \tag{6}$$

Inequalities (2) and (3) have been considered in [1] and [2], the inequality (4) has been considered in [3] without a proof of its sharpness, while the corrected Simpson rule has been considered in [7], [8] and [5].

In [6], there is the following a generalization of the pre-Grüss inequality.

Lemma. *Let $f, g, \Psi \in L_2(a,b)$, then,*

$$S_\Psi(f,g)^2 \le S_\Psi(f,f)S_\Psi(g,g), \tag{7}$$

where

$$S_\Psi(f,g) = \int_a^b f(t)g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\int_a^b g(t)\,dt - \frac{1}{\|\Psi\|_2^2}\int_a^b f(t)\Psi(t)\,dt\int_a^b g(t)\Psi(t)\,dt \tag{8}$$

and Ψ satisfies

$$\int_a^b \Psi(t)\,dt = 0, \tag{9}$$

while, as usual, $\|\cdot\|_2$ *is the norm in* $L_2(a,b)$. *i.e.,*

$$\|\Psi\|_2^2 = \int_a^b \Psi^2(t)\,dt.$$

In this paper, we use this generalization of the pre-Grüss inequality to derive a new sharp L_2 inequality which provides better estimation of error. An application in numerical integration is also considered.

2. Main results

Theorem 2. *Let the assumptions of Theorem* 1 *hold, then for any* $\theta \in [0,1]$,

$$\Big|\int_a^b f(t)\,dt - (b-a)\Big[(1-\theta)f\Big(\frac{a+b}{2}\Big) + \theta\frac{f(a)+f(b)}{2}\Big] - \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\Big| \le \frac{(b-a)^{\frac{5}{2}}}{24\sqrt{5}}\sqrt{4-15\theta+15\theta^2}\,M(f;a,b), \tag{10}$$

where

$$M(f;a,b) = \left\{\|f''\|_2^2 - \frac{[f'(b)-f'(a)]^2}{b-a} - \frac{[2f'(\frac{a+b}{2}) - f'(a) - f'(b)]^2}{b-a}\right\}^{\frac{1}{2}}. \tag{11}$$

The inequality (10) *is sharp.*

Proof. Let

$$p(t) = \begin{cases} 1, & t \in [a, \frac{a+b}{2}], \\ -1, & t \in (\frac{a+b}{2}, b], \end{cases}$$

and

$$\Psi(t) = \begin{cases} \frac{(t-a)^2}{2} - \frac{\theta(b-a)}{2}(t-a) - \frac{1-3\theta}{24}(b-a)^2, & t \in [a, \frac{a+b}{2}], \\ \frac{(t-b)^2}{2} + \frac{\theta(b-a)}{2}(t-b) - \frac{1-3\theta}{24}(b-a)^2, & t \in (\frac{a+b}{2}, b], \end{cases}$$

where $\theta \in [0,1]$.

It is not difficult to verify that

$$\int_a^b p(t)\,dt = 0, \tag{12}$$

$$\int_a^b \Psi(t)\,dt = 0, \tag{13}$$

$$\int_a^b p(t)\Psi(t)\,dt = 0. \tag{14}$$

Also,

$$\|p\|_2^2 = \int_a^b p^2(t)\,dt = b - a, \tag{15}$$

$$\|\Psi\|_2^2 = \int_a^b \Psi^2(t)\,dt = \frac{(b-a)^5}{2880}(4 - 15\theta + 15\theta^2), \tag{16}$$

and

$$\int_a^b f''(t)p(t)\,dt = 2f'\Big(\frac{a+b}{2}\Big) - f'(a) - f'(b). \tag{17}$$

Integrating by parts,

$$\begin{aligned}\int_a^b f''(t)\Psi(t)\,dt = \int_a^b f(t)\,dt - (b-a)\Big[(1-\theta)f\Big(\frac{a+b}{2}\Big) + \theta\frac{f(a)+f(b)}{2}\Big]&\\ - \frac{1-3\theta}{24}(b-a)^2[f'(b) - f'(a)].&\end{aligned} \tag{18}$$

From (12), (14), (17) and (8),

$$\begin{aligned}S_\Psi(f'', p) &= \int_a^b f''(t)p(t)\,dt - \frac{1}{b-a}\int_a^b f''(t)\,dt\int_a^b p(t)\,dt\\ &\qquad - \frac{1}{\|\Psi\|_2^2}\int_a^b f''(t)\Psi(t)\,dt\int_a^b p(t)\Psi(t)\,dt\\ &= 2f'\Big(\frac{a+b}{2}\Big) - f'(a) - f'(b).\end{aligned} \tag{19}$$

From (12), (14) and (8),

$$\begin{aligned}S_\Psi(p,p) &= \|p\|_2^2 - \frac{1}{b-a}\Big(\int_a^b p(t)\,dt\Big)^2 - \frac{1}{\|\Psi\|_2^2}\Big(\int_a^b p(t)\Psi(t)\,dt\Big)^2\\ &= b - a.\end{aligned} \tag{20}$$

From (16), (18) and (8),

$$\begin{aligned}S_\Psi(f'', f'') &= \|f''\|_2^2 - \frac{1}{b-a}\Big(\int_a^b f''(t)\,dt\Big)^2 - \frac{1}{\|\Psi\|_2^2}\Big(\int_a^b f''(t)\Psi(t)\,dt\Big)^2\\ &= \|f''\|_2^2 - \frac{[f'(b) - f'(a)]^2}{b-a} - \frac{2880}{(4 - 15\theta + 15\theta^2)(b-a)^5}\\ &\quad\times\Big\{\int_a^b f(t)\,dt - (b-a)\Big[(1-\theta)f\Big(\frac{a+b}{2}\Big) + \theta\frac{f(a)+f(b)}{2}\Big]\\ &\qquad - \frac{1-3\theta}{24}(b-a)^2[f'(b) - f'(a)]\Big\}^2.\end{aligned} \tag{21}$$

Thus from (19)–(21) and (7) we can obtain

$$\frac{[2f'(\frac{a+b}{2})-f'(a)-f'(b)]^2}{b-a}\le \|f''\|_2^2-\frac{[f'(b)-f'(a)]^2}{b-a}$$
$$-\frac{2880}{(4-15\theta+15\theta^2)(b-a)^5}\left\{\int_a^b f(t)\,dt-(b-a)\left[(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\right\}^2, \quad (22)$$

which is equivalent to

$$\frac{2880}{(4-15\theta+15\theta^2)(b-a)^5}\left\{\int_a^b f(t)\,dt-(b-a)\left[(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\right\}^2$$
$$\le \|f'\|_2^2-\frac{[f(b)-f(a)]^2}{b-a}-\frac{[2f'(\frac{a+b}{2})-f'(a)-f'(b)]^2}{b-a}. \quad (23)$$

Inequality (10) follows from (23).

In order to prove that the inequality (10) is sharp, for any $\theta\in[0,1]$, we define the function,

$$f(t)=\begin{cases}\frac{1}{24}t^4-\frac{\theta}{12}t^3, & t\in[0,\frac12],\\ \frac{1}{24}(t-1)^4+\frac{\theta}{12}(t-1)^3+\frac{1-3\theta}{24}(t-\frac12), & t\in(\frac12,1]\end{cases} \quad (24)$$

from which it follows that

$$f'(t)=\begin{cases}\frac{1}{6}t^3-\frac{\theta}{4}t^2, & t\in[0,\frac12],\\ \frac{1}{6}(t-1)^3+\frac{\theta}{4}(t-1)^2+\frac{1-3\theta}{24}, & t\in(\frac12,1]\end{cases} \quad (25)$$

and

$$f''(t)=\begin{cases}\frac{1}{2}t^2-\frac{\theta}{2}t, & t\in[0,\frac12],\\ \frac{1}{2}(t-1)^2+\frac{\theta}{2}(t-1), & t\in(\frac12,1].\end{cases} \quad (26)$$

The function given in (24) is absolutely continuous since it is a continuous piecewise polynomial function.

We now suppose that (10) holds with a constant $K>0$ as

$$\left|\int_a^b f(t)\,dt-(b-a)\left[(1-\theta)f\Big(\frac{a+b}{2}\Big)+\theta\frac{f(a)+f(b)}{2}\right]\right.$$
$$\left.-\frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\right|\le K(b-a)^{\frac52}\sqrt{4-15\theta+15\theta^2}M(f;a,b), \quad (27)$$

where $M(f;a,b)$ is as defined in (11).

Choosing $a = 0$, $b = 1$, and f as defined in (24), we get

$$\int_0^1 f(t)\,dt = \frac{11-35\theta}{1920},$$

$$f(0) = 0,\ \ f(1) = \frac{1-3\theta}{48},\ \ f(\frac{1}{2}) = \frac{1-4\theta}{384},$$

$$f'(0) = 0,\ \ f'(1) = \frac{1-3\theta}{24},\ \ f'(\frac{1}{2}) = \frac{1-3\theta}{48},$$

$$\int_0^1 (f''(t))^2\,dt = \frac{3-15\theta+20\theta^2}{960}$$

such that the LHS of (28) becomes $\frac{4-15\theta+15\theta^2}{2880}$, and the RHS becomes $\frac{K(4-15\theta+15\theta^2)}{24\sqrt{5}}$.

Thus from (27), we find that $K \geq \frac{1}{24\sqrt{5}}$, proving that the constant $\frac{1}{24\sqrt{5}}$ is the best possible in (10). □

Remark 1. It is obvious that the error estimation in (10) is better than that in (1).

Remark 2. If we take $\theta = 1$ and $\theta = 0$ in (10), we obtain the following sharp, corrected trapezoid type and corrected midpoint type inequalities, respectively, as

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} M(f;a,b) \tag{28}$$

and

$$\left| \int_a^b f(t)\,dt - (b-a)f\Big(\frac{a+b}{2}\Big) - \frac{(b-a)^2}{24}[f'(b)-f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{5}} M(f;a,b). \tag{29}$$

Remark 3. If $\theta = \frac{1}{3}$ in (10), we obtain a sharp Simpson type inequality of the form

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b) \right] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{30}} M(f;a,b), \tag{30}$$

and if $\theta = \frac{7}{15}$ in (10), we obtain a sharp corrected Simpson type inequality of the form

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{30}\left[7f(a) + 16f\Big(\frac{a+b}{2}\Big) + 7f(b) \right] + \frac{(b-a)^2}{60}[f'(b) - f'(a)] \right| \leq \frac{(b-a)^{\frac{5}{2}}}{60\sqrt{3}} M(f;a,b). \tag{31}$$

From (30) and (31), we see that the corrected Simpson rule provides better results than the Simpson rule.

Remark 4. If $\theta = \frac{1}{2}$ in (10), we obtain the following sharp, corrected average midpoint-trapezoid type inequality,

$$\left| \int_a^b f(t)\,dt - \frac{b-a}{4}\left[f(a) + 2f\Big(\frac{a+b}{2}\Big) + f(b)\right] + \frac{(b-a)^2}{48}[f'(b) - f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}} M(f;a,b). \quad (32)$$

It is interesting to note that the smallest bound for (10) is obtained at $\theta = \frac{1}{2}$. Thus the corrected averaged midpoint-trapezoid rule is optimal in the current situation.

Remark 5. It is also clear that the error estimates in (28)-(32) are better than those in the corresponding results, (2)-(6).

3. Applications in numerical integration

Application here is to the averaged midpoint-trapezoid quadrature rule. Similar analysis can be performed on the other results considered in the previous section.

Theorem 3. *Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem* 1 *hold, then,*

$$\left| \int_a^b f(t)\,dt - \frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i) + 2f\Big(\frac{x_i + x_{i+1}}{2}\Big) + f(x_{i+1})\right] + \frac{(b-a)^2}{48n^2}[f'(b) - f'(a)]\right| \le \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}n^2} M_n(f) \le \frac{(b-a)^{\frac{5}{2}}}{48\sqrt{5}n^2}\sqrt{\sigma(f'')}, \quad (33)$$

where

$$M_n(f) = \left\{ \|f''\|_2^2 - \frac{[f'(b) - f'(a)]^2}{b-a} - \frac{1}{b-a}\left[f'(x_0) + f'(x_n) + 2\sum_{i=1}^{n-1} f'(x_i) - 2\sum_{i=0}^{n-1} f'\Big(\frac{x_i + x_{i+1}}{2}\Big)\right]^2 \right\}^{\frac{1}{2}}. \quad (34)$$

Proof. From (32) in Remark 4,

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{4}\Big[f(x_i)+2f\Big(\frac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]+\frac{h^2}{48}[f'(x_{i+1})-f'(x_i)]\right|$$
$$\le \frac{h^{\frac{5}{2}}}{48\sqrt{5}}\Big\{\int_{x_i}^{x_{i+i}} (f''(t))^2\,dt - \frac{1}{h}[f'(x_{i+1})-f'(x_i)]^2$$
$$-\frac{1}{h}\Big[f'(x_i)-2f'\Big(\frac{x_i+x_{i+1}}{2}\Big)+f'(x_{i+1})\Big]^2\Big\}^{\frac{1}{2}}. \quad (35)$$

By summing (35) over i from 0 to $n-1$ and using the generalized triangle inequality, we obtain,

$$\left|\int_a^b f(t)\,dt - \frac{h}{4}\sum_{i=0}^{n-1}\Big[f(x_i)+2f\Big(\frac{x_i+x_{i+1}}{2}\Big)+f(x_{i+1})\Big]+\frac{h^2}{48}[f'(x_{i+1})-f'(x_i)]\right|$$
$$\le \frac{h^{\frac{5}{2}}}{48\sqrt{5}}\sum_{i=0}^{n-1}\Big\{\int_{x_i}^{x_{i+i}} (f''(t))^2\,dt - \frac{1}{h}[f'(x_{i+1})-f'(x_i)]^2$$
$$-\frac{1}{h}\Big[f'(x_i)-2f'\Big(\frac{x_i+x_{i+1}}{2}\Big)+f'(x_{i+1})\Big]^2\Big\}^{\frac{1}{2}}. \quad (36)$$

By using the Cauchy inequality twice, it can be seen that,

$$\sum_{i=0}^{n-1}\Big\{\int_{x_i}^{x_{i+1}} (f''(t))^2\,dt - \frac{1}{h}[f'(x_{i+1})-f'(x_i)]^2$$
$$-\frac{1}{h}\Big[f'(x_i)-2f'\Big(\frac{x_i+x_{i+1}}{2}+f'(x_{i+1})\Big)\Big]^2\Big\}^{\frac{1}{2}}$$
$$\le \sqrt{n}\Big\{\|f''\|_2^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}[f'(x_{i+1})-f'(x_i)]^2$$
$$-\frac{n}{b-a}\sum_{i=0}^{n-1}\Big[f'(x_i)-2f'\Big(\frac{x_i+x_{i+1}}{2}\Big)+f'(x_{i+1})\Big]^2\Big\}^{\frac{1}{2}}$$
$$\le \sqrt{n}\Big\{\|f''\|_2^2 - \frac{[f'(b)-f'(a)]^2}{b-a} - \frac{1}{b-a}\Big[f'(x_0)+f'(x_n)$$
$$+2\sum_{i=1}^{n-1} f'(x_i) - 2\sum_{i=0}^{n-1} f'\Big(\frac{x_i+x_{i+1}}{2}\Big)\Big]^2\Big\}^{\frac{1}{2}}. \quad (37)$$

Consequently, the inequality (33) with (34) follow from (36) and (37). □

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