SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.09.1.05
Vol.9 (21) (2013), 57–70

CERTAIN SUBCLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS DEFINED BY USING A CERTAIN FRACTIONAL DERIVATIVE OPERATOR

M. K. AOUF AND B. A. FRASIN

ABSTRACT. Making use of a certain operator of fractional derivative, a new subclass $F_\lambda(n,p,\alpha,\mu)$ of analytic and p-valent functions with negative coefficients is introduced and studied here rather systematically. Coefficient estimates, a distortion theorem and radii of p-valently close-to-convexity, starlikeness and convexity are given. Finally several applications involving an integral operator and a certain fractional calculus operator are also considered.

1. INTRODUCTION

Let $T_p(n)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \; (a_k \geq 0; \;\; p,n \in \mathbb{N} = \{1,2,\ldots.\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$. Various operators of fractional calculus (that is, fractional integral and fractional derivative) have been studied in the literature rather extensively (cf., e.g., [9], [11], [12] and [13]; see also the various references cited therein). For our present investigations, we recall the following definitions.

Definition 1 (Fractional Integral Operator)**.** *The frcational integral operator of order λ is defined, for a function $f(z)$ by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0)\,, \tag{1.2}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Multivalent functions, fractional calculus.

Definition 2 (Fractional Derivative Operator)**.** *The fractional derivative of order λ is defined, for a function $f(z)$ by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1)\,, \tag{1.3}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 2.

Definition 3 (Extended Fractional Derivative Operator)**.** *Under the hypothesis of Definition* 3*, the fractional derivative of order $n+\lambda$ is defined, for a function $f(z)$ by*

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \le \lambda < 1;\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})\,. \tag{1.4}$$

In terms of the fractional derivative operator D_z^{λ} of order λ, defined by (1.3), with

$$D_z^0 f(z) = f(z) \quad \text{and} \quad D_z^1 f(z) = f^{'}(z)\,, \tag{1.5}$$

Srivastava and Aouf [11] defined and studied the operator:

$$\Omega_z^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda-p} D_z^{\lambda} f(z) \quad (0 \le \lambda \le 1;\ p \in \mathbb{N})\,. \tag{1.6}$$

In this paper we shall study some properties of the class $F_{\lambda}(n,p,\alpha,\mu)$, defined as follows:

Definition 4. *Let $F_{\lambda}(n,p,\alpha,\mu)$ be the subclass of $T_p(n)$ consisting of functions of the form* (1.1) *which satisfy the following inequality:*

$$\Re\left\{\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda-p}\left[(1-\mu)D_z^{\lambda} f(z) + \frac{\mu}{p} z(D_z^{\lambda} f(z))^{'}\right]\right\} > \frac{\alpha}{p} \tag{1.7}$$

where $z \in U;\ 0 \le \alpha < p;\ p \in \mathbb{N};\ 0 \le \lambda \le 1;\ \alpha + \lambda < p;\ 0 \le \mu \le 1)$.

By specializing the parameters λ, μ, α and p, we obtain the following subclasses studied by various authors:

(i) $F_{\lambda}(n,1,\alpha,\mu) = F_{\lambda}(n,\alpha,\mu)$ $(0 \le \alpha < 1)$ (Altintas et al. [1]);
(ii) $F_0(1,p,\alpha,\mu) = F_p(\mu,\alpha)$ $(0 \le \alpha < p;\ p \in \mathbb{N};\ \mu \ge 0)$ (Lee et al. [6] and Aouf and Darwish [2]);
(iii) $F_0(1,1,\alpha,\mu) = F_{\mu}(\alpha)$ $(0 \le \alpha < 1;\ 0 \le \mu \le 1)$ (Bhoosnurmath and Swamy [4]);
(iv) $F_{\lambda}(1,p,p\alpha,0) = F_p(\alpha,\lambda)$ $(0 \le \alpha < 1;\ 0 \le \lambda \le 1;\ p \in \mathbb{N})$

$$= \left\{f(z) \in T_p(1) = T_p : \Re\left[\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda-p} D_z^{\lambda} f(z)\right] > \alpha,\ z \in U\right\}; \tag{1.8}$$

(see [10]);
(v) $F_0(1,p,p\alpha,0) = F_p(0,p\alpha) \quad (0 \le \alpha < 1)$

$$= \left\{ f(z) \in T_p(1) = T_p : \Re\left(\frac{f(z)}{z^p}\right) > \alpha\,, \quad z \in U \right\}\,; \tag{1.9}$$

(see [6]);
(vi) $F_0(1,p,\alpha,1) = F_p(1,\alpha) \quad (0 \le \alpha < p)$

$$= \left\{ f(z) \in T_p(1) = T_p : \Re\left(\frac{f'(z)}{z^{p-1}}\right) > \alpha\,,\ z \in U \right\}. \tag{1.10}$$

(see [6]).

Also we note that: $F_\lambda(n,p,\alpha,1) = F_\lambda(n,p,\alpha)$

$$= \left\{ f(z) \in T_p(n) : \Re\left[\frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} \cdot \frac{(D_z^\lambda f(z))'}{z^{p-\lambda-1}}\right] > \alpha \right\}, \tag{1.11}$$

where $0 \le \alpha < p,\ p \in \mathbb{N},\ \ z \in U$.

In our present paper, we shall make use of the familiar operator $J_{c,p}$ defined by (cf. [3], [7] and [8]; see also [12])

$$(J_{c,p}f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)\,dt \ \ (c > -p)\,. \tag{1.12}$$

2. Coefficient estimates

Theorem 1. *Let the function $f(z) \in T_p(n)$ be given by* (1.1)*. Then $f(z) \in F_\lambda(n,p,\alpha,\mu)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{[p+\mu(k-p-\lambda)]\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k \le p - \mu\lambda - \alpha \tag{2.1}$$

where $\alpha + \lambda < p$.

Proof. Assume that the inequality (2.1) holds true. Then we find that

$$\left| \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda-p} \left[(1-\mu)D_z^\lambda f(z) + \frac{\mu}{p} z (D_z^\lambda f(z))'\right] - (1 - \frac{\mu}{p}\lambda) \right|$$
$$= \left| - \sum_{k=n+p}^{\infty} \left[\frac{p+\mu(k-p-\lambda)}{p} \, \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)}\right] a_k z^{k-p} \right|$$
$$\le \sum_{k=n+p}^{\infty} \frac{[p+\mu(k-p-\lambda)]\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k \le p - \mu\lambda - \alpha,$$

where $z \in U;\ 0 \le \alpha < p;\ \alpha + \lambda < p;\ p, n \in \mathbb{N};\ 0 \le \lambda \le 1;\ 0 \le \mu \le 1.$

This shows that the values of the function

$$\Phi(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda-p} \left[(1-\mu) D_z^{\lambda} f(z) + \frac{\mu}{p} z (D_z^{\lambda} f(z))' \right] \tag{2.2}$$

lie in a circle which is centered at $w = 1 - \frac{\mu\lambda}{p}$ and whose radius is $1 - \frac{\mu\lambda}{p} - \alpha$. Hence $f(z)$ satisfies the condition (1.7).

Conversely, assume that the function $f(z)$ defined by (1.1) is in the class $F_{\lambda}(n,p,\mu,\alpha)$. Then we have

$$\Re\left\{ 1 - \frac{\mu}{p}\lambda - \sum_{k=n+p}^{\infty} \left[\frac{p+\mu(k-p-\lambda)}{p} \right] \frac{\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k z^{k-p} \right\} > \frac{\alpha}{p} \tag{2.3}$$

for some $\alpha (0 \le \alpha < p)$, $0 \le \lambda \le 1$, $\alpha + \lambda < p$, $0 \le \mu \le 1$, $p, n \in \mathbb{N}$ and $z \in U$. Choose values of z on the real axis so that $\Phi(z)$ given by (2.2) is real.

Letting $z \to 1^-$ through real values, we can see that

$$p - \mu\lambda - \sum_{k=n+p}^{\infty} \frac{[p+\mu(k-p-\lambda)]\Gamma(k+1)\Gamma(p+1-\lambda)}{\Gamma(p+1)\Gamma(k+1-\lambda)} a_k \ge \alpha \tag{2.4}$$

for $0 \le \alpha < p$; $p, n \in \mathbb{N}$; $0 \le \lambda \le 1$; $\alpha + \lambda < p$; $0 \le \mu \le 1$, which is equivalent to the assertion (2.1) of Theorem 1. □

Putting $p = 1$ in Theorem 1, we obtain the following result.

Corollary 1. *Let the function $f(z)$ be defined by* (1.1) *(with $p = 1$). Then $f(z) \in F_{\lambda}(n, \alpha, \mu)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{[1+\mu(k-1-\lambda)]\,\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k \le 1 - \mu\lambda - \alpha \tag{2.5}$$

for $\alpha + \lambda < 1$.

Remark 1. We note that the result obtained by Altintas et al. [1, Theorem 1] is not correct. The correct result is given by Corollary 1.

Corollary 2. *Let the function $f(z)$ be defined by* (1.1) *be in the class $F_{\lambda}(n,p,\alpha,\mu)$. Then*

$$a_k \le \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)}{[p+\mu(k-p-\lambda)]\,\Gamma(k+1)\Gamma(p+1-\lambda)} \tag{2.6}$$

for $k \ge n+p$; $p, n \in \mathbb{N}$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)}{[p+\mu(k-p-\lambda)]\,\Gamma(k+1)\Gamma(p+1-\lambda)} z^k,$$

where $k \geq n+p;\ p,n \in \mathbb{N}$.

3. Distortion theorem

Theorem 2. *If a function $f(z)$ defined by (1.1) is in the class $F_\lambda(n,p,\alpha,\mu)$, then*

$$\left\{\frac{p!}{(p-j)!} - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)(n+p)!}{[p+\mu(n-\lambda)]\,\Gamma(n+p+1)\Gamma(p+1-\lambda)(n+p-j)!}|z|^n\right\}|z|^{p-j}$$

$$\leq \left|f^{(j)}(z)\right| \leq \left\{\frac{p!}{(p-j)!} + \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)(n+p)!}{[p+\mu(n-\lambda)]\,\Gamma(n+p+1)\Gamma(p+1-\lambda)(n+p-j)!}|z|^n\right\}|z|^{p-j}, \quad (3.1)$$

where $z \in U;\ 0 \leq \alpha < p;\ 0 \leq \lambda \leq 1; \alpha+\lambda < p;\ p,n \in \mathbb{N};\ 0 \leq \mu \leq 1;\ j \in \mathbb{N}_0; p > j$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\,\Gamma(n+p+1)\Gamma(p+1-\lambda)}z^{n+p} \quad (p,n \in \mathbb{N}). \quad (3.2)$$

Proof. In view of Theorem 1, we have

$$\frac{[p+\mu(n-\lambda)]\Gamma(n+p+1)\Gamma(p+1-\lambda)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)(n+p)!}\sum_{k=n+p}^{\infty} k!a_k \leq \sum_{k=n+p}^{\infty}\frac{[p+\mu(k-p-\lambda)]\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)}a_k \leq 1$$

which readily yields

$$\sum_{k=j+p}^{\infty} k!a_k \leq \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)(n+p)!}{[p+\mu(n-\lambda)]\Gamma(n+p+1)\Gamma(p+1-\lambda)}. \quad (3.3)$$

Now, by differentiating both sides of (1.1) j times, we obtain

$$f^{(j)}(z) = \frac{p!}{(p-j)!}z^{p-j} - \sum_{k=n+p}^{\infty}\frac{k!}{(k-j)!}a_k z^{k-j}, \quad (3.4)$$

where $k \geq n+p;\ p,n \in \mathbb{N};\ j \in \mathbb{N}_0;\ p > j$.

Theorem 2 follows readily from (3.3) and (3.4). □

Finally, it is easy to see that the bounds in (3.1) are attained for the function $f(z)$ given by (3.2).

Putting (i) $p = 1$ and $\lambda = j = 0$ (ii) $p = j = 1$ and $\lambda = 0$ in Theorem 2, we obtain the following results.

Corollary 3. *If a function $f(z)$ defined by* (1.1) *(with $p = 1$) is in the class $F_0(n, 1, \alpha, \mu) = F_0(n, \alpha, \mu)$, then*

$$|z| - \frac{(1-\alpha)}{(1+\mu n)}|z|^{n+1} \leq |f(z)| \leq |z| + \frac{(1-\alpha)}{(1+\mu n)}|z|^{n+1} \quad (n \in \mathbb{N};\ z \in U). \tag{3.5}$$

The result is sharp.

Corollary 4. *If a function $f(z)$ is defined by* (1.1) *(with $p = 1$) is in the class $F_0(n, 1, \alpha, \mu) = F_0(n, \alpha, \mu)$, then*

$$1 - \frac{(1-\alpha)(n+1)}{(1+\mu n)}|z|^{n} \leq \left|f'(z)\right| \leq 1 + \frac{(1-\alpha)(n+1)}{(1+\mu n)}|z|^{n} \quad (n \in \mathbb{N}; z \in U). \tag{3.6}$$

The result is sharp.

Remark 2. We note that the results obtained by Altintas et al. [1, Corollary 8 and Corollary 9] are not correct. The correct results are given by (3.5) and (3.6), respectively;

Putting (i) $n = 1$ and $\lambda = j = 0$ (ii) $n = j = 1$ and $\lambda = 0$ in Theorem 2, we obtain the result obtained by Lee at al. [6, Theorem 3].

4. Radii of close-to-convexity, starlikeness and convexity

Theorem 3. *Let the function $f(z)$ defined by* (1.1) *be in the class $F_\lambda(n, p, \alpha, \mu)$, then*

(i) *$f(z)$ is p-valently close-to-convex of order $\varphi\,(0 \leq \varphi < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_k \left\{ \frac{[p+\mu(k-p-\lambda)]\,\Gamma(k)\Gamma(p+1-\lambda)(p-\varphi)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)} \right\}^{\frac{1}{k-p}} \tag{4.1}$$

for $k \geq n+p$; $p, n \in \mathbb{N}$,

(ii) *$f(z)$ is p-valently starlike of order $\varphi\,(0 \leq \varphi < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_k \left\{ \frac{[p+\mu(k-p-\lambda)]\,\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)} \left(\frac{p-\varphi}{k-\varphi}\right) \right\}^{\frac{1}{k-p}} \tag{4.2}$$

for $k \geq n+p$; $p, n \in \mathbb{N}$,

(iii) *$f(z)$ is p-valently convex of order $\varphi\,(0 \leq \varphi < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_k \left\{ \frac{[p+\mu(k-p-\lambda)]\,\Gamma(k)\Gamma(p+1-\lambda)(p-\varphi)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)} \left(\frac{p-\varphi}{k-\varphi}\right) \right\}^{\frac{1}{k-p}} \tag{4.3}$$

for $k \geq n+p;\ p,n \in \mathbb{N}$. *Each of these results is sharp for the function* $f(z)$ *given by (2.7).*

Proof. (i) It is sufficient to show that

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \varphi \quad (|z| < r_1; 0 \leq \varphi < p;\ p \in \mathbb{N}), \tag{4.4}$$

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \varphi \quad (|z| < r_2; 0 \leq \varphi < p;\ p \in \mathbb{N}), \tag{4.5}$$

and that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \varphi \quad (|z| < r_3;\ 0 \leq \varphi < p;\ p \in \mathbb{N}) \tag{4.6}$$

for a function $f(z) \in F_\lambda(n,p,\alpha,\mu)$, where r_1, r_2 and r_3 are defined by (4.1), (4.2) and (4.3), respectively. □

Remark 3. (i) We note that the results obtained by Altintas et al. [1, Theorems 6 and 7 and Corollary 1] are not correct. The correct results are given by (4.1), (4.2) and (4.3) (with $p = 1$), respectively;

(ii) Putting $n = 1$ and $\lambda = 0$ in Theorem 3, we obtain the results obtained by Aouf and Darwish [2, Theorems 6 and 7, Corollary 2, respectively].

5. Applications of fractional calculus

In this section, we shall investigate the growth and distortion properties of the operators $J_{c,p}$ and D_z^λ. In order to derive our results, we need the following lemma given by Chen et al. [5].

Lemma 1. (see [5], Chen et al.) *Let the function* $f(z)$ *defined by* (1.1). *Then*

$$D_z^\lambda\{(J_{c,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\lambda)}z^{p-\lambda} - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)}{(c+k)\Gamma(k+1-\lambda)}a_k z^{k-\lambda} \tag{5.1}$$

where $\lambda \in R;\ c > -p;\ p,n \in \mathbb{N}$ *and*

$$J_{c,p}(D_z^\lambda\{f(z)\} = \frac{(c+p)\Gamma(p+1)}{(c+p-\lambda)\Gamma(p+1-\lambda)}z^{p-\lambda} - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)}{(c+k-\lambda)\Gamma(k+1-\lambda)}a_k z^{k-\lambda}, \tag{5.2}$$

where $\lambda \in R;\ c > -p;\ p,n \in \mathbb{N}$, *provided that no zeros appear in the denominators in* (5.1) *and* (5.2).

Theorem 4. *Let the function $f(z)$ defined by (1.1) be in the class $F_\lambda(n, p, \alpha, \mu)$. Then*

$$\left|D_z^{-\delta}\{(J_{c,p}f)(z)\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} - \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\delta} \tag{5.3}$$

for $z \in U$; $0 \leq \alpha < p$; $0 \leq \lambda \leq 1$; $\alpha + \lambda < p$; $p, n \in \mathbb{N}$; $0 \leq \mu \leq 1$; $c > -p$ and

$$\left|D_z^{-\delta}\{(J_{c,p}f)(z)\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} + \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\delta} \tag{5.4}$$

for $z \in U$; $0 \leq \alpha < p$; $0 \leq \lambda \leq 1$; $\alpha + \lambda < p$; $p, n \in \mathbb{N}$; $0 \leq \mu \leq 1$; $c > -p$.

Each of the assertions (5.3) and (5.4) is sharp.

Proof. In view of Theorem 1, we have

$$\frac{[p+\mu(n-\lambda)]\Gamma(n+p+1)\Gamma(p+1-\lambda)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}\sum_{k=n+p}^{\infty} a_k \leq \sum_{k=n+p}^{\infty}\frac{[p+\mu(k-p-\lambda)]\Gamma(k+1)\Gamma(p+1-\lambda)}{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(k+1-\lambda)}a_k \leq 1\,, \tag{5.5}$$

which readily yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\Gamma(n+p+1)\Gamma(p+1-\lambda)}\,. \tag{5.6}$$

Consider the function $F(z)$ defined in U by

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta}\{(J_{c,p}f)(z)\} \\ &= z^p - \sum_{k=n+p}^{\infty}\frac{(c+p)\Gamma(k+1)\Gamma(p+1+\delta)}{(c+k)\Gamma(k+1+\delta)\Gamma(p+1)}a_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty}\Phi(k)a_k z^k \qquad (z \in U)\,, \end{aligned}$$

where

$$\Phi(k) = \frac{(c+p)\Gamma(k+1)\Gamma(p+1+\delta)}{(c+k)\Gamma(k+1+\delta)\Gamma(p+1)} \tag{5.7}$$

for $k \geq n+p;\ p,n \in \mathbb{N};\ \delta > 0$. Since $\Phi(k)$ is a decreasing function of k when $\delta > 0$, we get

$$0 < \Phi(k) \leq \Phi(n+p) = \frac{(c+p)\Gamma(n+p+1)\Gamma(p+1+\delta)}{(c+n+p)\Gamma(n+p+1+\delta)\Gamma(p+1)} \tag{5.8}$$

for $c > -p;\ p,n \in \mathbb{N};\ \delta > 0$.

Thus, by using (5.6) and (5.8), we deduce that

$$|F(z)| \geq |z|^p - \Phi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} a_k \geq |z|^p$$
$$- \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1+\delta)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z \in U)\,,$$

and

$$|F(z)| \leq |z|^p + \Phi(n+p)\,|z|^{n+p} \sum_{k=n+p}^{\infty} a_k \leq |z|^p$$
$$+ \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1+\delta)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)}\,|z|^{n+p} \quad (z \in U)\,,$$

which yield the inequalities (5.3) and (5.4) of Theorem 4. The equalities in (5.3) and (5.4) are attained for the function $f(z)$ given by

$$D_z^{-\delta}\{(J_{c,p}f)(z)\} = \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} \right.$$
$$\left. - \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1+\delta)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)} z^n \right\} z^{p+\delta} \tag{5.9}$$

or, equivalently, by

$$(J_{c,p}f)(z) = z^p - \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1)\Gamma(p+1-\lambda)} z^{n+p}. \tag{5.10}$$

Thus we complete the proof of Theorem 4. □

Using arguments similar to those in the proof of Theorem 4, we obtain the following result.

Theorem 5. *Let the function $f(z)$ defined by* (1.1) *be in the class $F_\lambda(n,p,\alpha,\mu)$. Then*

$$\left|D_z^\delta\left\{(J_{c,p}f)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\delta)} - \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1-\delta)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-\delta}, \tag{5.11}$$

where $z\in U;\ 0\leq\alpha<p;\ 0\leq\lambda\leq 1;\ \alpha+\lambda<p;\ p,n\in\mathbb{N};\ 0\leq\mu\leq 1;\ c>-p$ *and*

$$\left|D_z^\delta\left\{(J_{c,p}f)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\delta)} + \frac{(c+p)(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{(c+n+p)[p+\mu(n-\lambda)]\Gamma(n+p+1-\delta)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p-\delta}, \tag{5.12}$$

where $z\in U;\ 0\leq\alpha<p;\ 0\leq\lambda\leq 1;\ \alpha+\lambda<p;\ p,n\in\mathbb{N};\ 0\leq\mu\leq 1;\ c>-p$. *Each of the assertion* (5.11) *and* (5.12) *is sharp.*

Remark 4. (i) Putting $\lambda=0$ and $n=1$ in Theorems 4 and 5, we obtain the corresponding results for the class $F_0(p,\alpha,\mu)$;

(ii) Putting $p=1$ in Theorems 4 and 5, we obtain the corresponding results for the class $F_\lambda(n,\alpha,\mu)$.

Theorem 6. *Let the function $f(z)$ defined by* (1.1) *be in the class $F_\lambda(n,p,\alpha,\mu)$. Then we have*

$$\left|D_z^{-\delta}f(z)\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\delta} \tag{5.13}$$

and

$$\left|D_z^{-\delta}f(z)\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} + \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)}|z|^n\right\}|z|^{p+\delta} \tag{5.14}$$

for $\delta>0$ and $z\in U$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\delta)}{\Gamma(p+1)} z^{-\delta} D_z^{-\delta} f(z) \\ &= z^p - \sum_{k=n+p}^{\infty} \frac{\Gamma(k+1)\Gamma(p+1+\delta)}{\Gamma(k+1+\delta)\Gamma(p+1)} a_k z^k \\ &= z^p - \sum_{k=n+p}^{\infty} D(k) a_k z^k \qquad (z \in U) \ , \end{aligned}$$

where

$$D(k) = \frac{\Gamma(k+1)\Gamma(p+1+\delta)}{\Gamma(k+1+\delta)\Gamma(p+1)} \qquad (k \geq n+p;\ p,n \in \mathbb{N};\ \delta > 0)\,.$$

Since $D(k)$ is a decreaing function of k when $\delta > 0$, we get

$$0 < D(k) \leq D(n+p) = \frac{\Gamma(n+p+1)\Gamma(p+1+\delta)}{\Gamma(n+p+1+\delta)\Gamma(p+1)} \ . \tag{5.15}$$

Thus, by using (5.6) and (5.15), we deduce that

$$\begin{aligned} |F(z)| &\geq |z|^p - D(n+p)\, |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \\ &\geq |z|^p - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1+\delta)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)} |z|^{n+p} \quad (z \in U) \end{aligned}$$

and

$$\begin{aligned} |F(z)| &\leq |z|^p + D(n+p)\, |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \\ &\leq |z|^p + \frac{(p-\mu\lambda-\alpha)\Gamma(p+1+\delta)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)} |z|^{n+p} \quad (z \in U) \end{aligned}$$

which yield the inequalities (5.13) and (5.14) of Theorem 6. The equalities in (5.13) and (5.14) are attained for the function $f(z)$ given by

$$\begin{aligned} D_z^{-\delta} f(z) = \bigg\{ & \frac{\Gamma(p+1)}{\Gamma(p+1+\delta)} \\ & - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1+\delta)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\Gamma(n+p+1+\delta)\Gamma(p+1-\lambda)} z^n \bigg\} z^{p+\delta} \end{aligned} \tag{5.16}$$

or, equivalently, by the function $f(z)$ given by (3.2). □

Putting $p = 1$ in Theorem 6, we obtain the following result.

Corollary 5. *Let the function $f(z)$ defined by* (1.1) *(with $p = 1$) be in the class $F_\lambda(n, \alpha, \mu)$. Then we have*

$$\left|D_z^{-\delta} f(z)\right| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{1 - \frac{(1-\mu\lambda-\alpha)\Gamma(2+\delta)\Gamma(n+2-\lambda)}{[1+\mu(n-\lambda)]\Gamma(n+2+\delta)\Gamma(2-\lambda)} |z|^n\right\} \quad (5.17)$$

and

$$\left|D_z^{-\delta} f(z)\right| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{1 + \frac{(1-\mu\lambda-\alpha)\Gamma(2+\delta)\Gamma(n+2-\lambda)}{[1+\mu(n-\lambda)]\Gamma(n+2+\delta)\Gamma(2-\lambda)} |z|^n\right\} \quad (5.18)$$

for $\delta > 0$ and $z \in U$. The result is sharp.

Remark 5. (i) We note that the result obtained by Altintas et al. [1, Theorem 3] is not correct. The correct result is given by Corollary 5;

(ii) Putting $\lambda = 0$ and $n = 1$ in Theorem 6, we obtain the result obtained by Aouf and Darwish [2, Theorem 9].

Using arguments similar to those in the proof of Theorem 6, we obtain the following result.

Theorem 7. *Let the function $f(z)$ defined by* (1.1) *be in the class $F_\lambda(n, p, \alpha, \mu)$. Then we have*

$$\left|D_z^{\delta} f(z)\right| \geq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\delta)} - \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\Gamma(n+p+1-\delta)\Gamma(p+1-\lambda)} |z|^n\right\} z^{p-\delta} \quad (5.19)$$

and

$$\left|D_z^{\delta} f(z)\right| \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+1-\delta)} + \frac{(p-\mu\lambda-\alpha)\Gamma(p+1)\Gamma(n+p+1-\lambda)}{[p+\mu(n-\lambda)]\Gamma(n+p+1-\delta)\Gamma(p+1-\lambda)} |z|^n\right\} z^{p-\delta} \quad (5.20)$$

for $0 \leq \delta < 1$ and $z \in U$. The result is sharp.

Putting $p = 1$ in Theorem 7, we obtain the following result.

Corollary 6. *Let the function* $f(z)$ *defined by* (1.1) *(with* $p = 1$*) be in the class* $F_\lambda(n, \alpha, \mu)$*. Then we have*

$$\left|D_z^\delta f(z)\right| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{1 - \frac{(1-\mu\lambda-\alpha)\Gamma(2+\delta)\Gamma(n+2-\lambda)}{[1+\mu(n-\lambda)]\Gamma(n+2-\delta)\Gamma(2-\lambda)} |z|^n\right\}$$

and

$$\left|D_z^\delta f(z)\right| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{1 + \frac{(1-\mu\lambda-\alpha)\Gamma(2+\delta)\Gamma(n+2-\lambda)}{[1+\mu(n-\lambda)]\Gamma(n+2-\delta)\Gamma(2-\lambda)} |z|^n\right\}$$

for $0 \leq \delta < 1$ *and* $z \in U$*. The result is sharp.*

Remark 6. (i) We note that the result obtained by Altintas et al. [1, Theorem 4] is not correct. The correct result is given by Corollary 6;

(ii) Putting $\lambda = 0$ and $n = 1$ in Theorem 7, we obtain the result obtained by Aouf and Darwish [2, Theorem 10].

Acknowledgements. The authors are thankful to the referee for his comments and suggestions.

References

[1] O. Altintas, H. Irmark and H. M. Srivastava, *A subclass of analytic functions defined by using certain operators of fractional calculus*, Comput. Math. Appl., 30 (1) (1995), 1–9.

[2] M. K. Aouf and H. E. Darwish, *Basic properties and characterizations of a certain class of analytic functions with negative coefficients, II*, Utilitas Math., 46 (1994), 167–177.

[3] S. D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135(1969), 429–446.

[4] S. S. Bhoosnurmath and S. R. Swamy, *Certain classes of analytic functions with negative coefficients*, Indian J. Math., 27 (1985), 89–89.

[5] M. -P. Chen, H. Irmak and H. M. Srivastava, *Some families of multivalently analytic functions with negative coefficients*, J. Math. Anal. Appl., 214 (1997), 674–490.

[6] S. K. Lee, S. Owa and H. M. Srivastava, *Basic properties and characterizations of a certain class of analytic functions with negative coefficients*, Utilitas Math. 36 (1989), 121–128.

[7] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., 16 (1969), 755–758.

[8] A. E. Livingston, *On the radius of univalence of cerain analytic functions*, Proc. Amer. Math. Soc., 17 (1966), 352–357.

[9] S. Owa, *On the distortion theorems, I*, Kyungpook Math. J., 18 (1978), 55–59.

[10] H. M. Srivastava and M. K. Aouf, *A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients, I and II*, J. Math. Anal. Appl., 171 (1992), 1–13; ibid. 192 (1995), 973–688.

[11] H. M. Srivastava and S. Owa (Editors), *Univalent Functions, Fractional Calculus and Their Applivations*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[12] H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

(Received: September 3, 2011)
(Revised: January 17, 2012)

M. K. Aouf
Faculty of Science
Mansoura University
Mansoura 35516, Egypt
E-mail: mkaouf127@yahoo.com

B. A. Frasin
Department of Mathematics
Al- Al-Bayt University
P. O. Box: 130095 Mafraq
Jordan
E-mail: bafrasin@yahoo.com