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I–CONVERGENCE ON CONE METRIC SPACES

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Abstract. The concept of I–convergence is an important generalization of statistical convergence which depends on the notion of an ideal I of subsets of the set $\mathbb{N}$ of positive integers. In this paper we introduce the ideas of I–Cauchy and I^*–Cauchy sequences in cone metric spaces and study their properties. We also investigate the relation between this new Cauchy type condition and the property of completeness.

1. Introduction

Since 1951 when Steinhaus [14] and Fast [5] defined statistical convergence for sequences of real numbers, several generalizations and applications of this notion have been investigated. In particular two interesting generalizations of statistical convergence were introduced by Kostyrko et al [7] using the notion of ideals of the set $\mathbb{N}$ of positive integers who named them as I and I^*-convergence. Corresponding I-Cauchy condition was first introduced and studied by Dems [4] . I^*-Cauchy sequences has been very recently introduced by Nabiev et al [11] in metric spaces and further investigated by Das et al [2].

The concept of a cone metric space is a very recent and interesting generalization of the notion of an usual metric space where the distance between two points is given by an element of a Banach space endowed with a suitable partial order. after the initial introduction of this space by Guang and Xian [10, 11], a lot of work have been done on this structure, in particular in fixed point theory. In this paper we proceed in a different direction and study ideal convergence and related Cauchy conditions in cone metric spaces. Our results automatically extend the results of [2], [4], [7], [11]. However it should be noted that due to the absence of real numbers (which are replaced by elements of a Banach space), the methods of proofs are not always analogous to the usual metric case.

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2. Preliminaries

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. A family $I \subset 2^Y$ of subsets of a non-empty set Y is said to be an ideal in Y if (i) $A, B \in I$ implies $A \cup B \in I$; (ii) $A \in I, B \subset A$ implies $B \in I$, while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. If I is a proper ideal in Y (i.e. $Y \notin I$, $Y \neq \phi$) then the family of sets $F(I) = \{M \subset Y : \text{there exists } A \in I : M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal I. Throughout I will denote an admissible ideal of $\mathbb{N}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, d) is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : d(x_n, x) \geq \epsilon\}$ belongs to I [5].

An admissible ideal $I \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for any sequence $\{A_1, A_2, \dots,\}$ of mutually disjoint sets in I, there is a sequence $\{B_1, B_2, \dots\}$ of subsets of $\mathbb{N}$ such that $A_i \Delta B_i$ $(i = 1, 2, 3, \dots)$ is finite and $B = \bigcup_{j\in\mathbb{N}} B_j \in I$.

We now recall the following basic concepts from [10] which will be needed throughout the paper. E will always denote a real Banach space and let P be a subset of E. P is called a cone if and only if: (i) P is closed, nonempty, and $P \neq \{0\}$; (ii) $a, b \in \mathbb{R},\ a, b \geq 0,\ x, y \in P$ implies $ax + by \in P$; (iii) $x \in P$ and $-x \in P$ implies $x = 0$.

Given a cone $P \subset E$, we can define a partial ordering $\leq$ with respect to P by defining $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}\,P$ stands for the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E, 0 \leq x \leq y$ we have $||x|| \leq K||y||$. The least positive number K satisfying above is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent, i.e. if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $||x_n - x|| \to 0$ as $n \to \infty$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Throughout we always assume that E is a Banach space, P is a cone in E with $\text{int}P \neq \phi$ and $\leq$ stand for the partial ordering with respect to P.

Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies $(d1)$ $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$; $(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$; $(d3)$ $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$. Then d is called a cone metric on X, and (X, d) is called a cone metric space. It is obvious that the notion of cone metric spaces generalizes the notion of metric spaces.

We now give an example of a cone metric space. Let $E = \mathbb{R}^2, P = \{(x, y) \in E : x, y \geq 0\} \subset \mathbb{R}^2, X = \mathbb{R}$ and $d : X \times X \to E$ is given by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then it is easy to verify that (X, d) is a cone metric space.

Let (X, d) be a cone metric space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. If for every $c \in E$ with $0 \ll c$ there is $J \in \mathbb{N}$ such that for all $n > J$, $d(x_n, x) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to x and x is called the limit of the sequence $\{x_n\}_{n\in\mathbb{N}}$. We denote this by $\lim\limits_{n\to\infty} x_n = x$.

If for any $c \in E$ with $0 \ll c$, there is $J \in \mathbb{N}$ such that for all $n, m > J, d(x_n, x_m) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in X. If every Cauchy sequence in X is convergent in X then X is called a complete cone metric space [10].

We first consider the following definitions.

Definition 1. *Let (X, d) be a cone metric space. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and let $x \in X$. If for every $c \in E$ with $0 \ll c$ (i.e. $c - 0 \in \operatorname{int} P$) the set $\{n \in \mathbb{N} : c - d(x_n, x) \notin \operatorname{int} P\} \in I$ then $\{x_n\}_{n\in\mathbb{N}}$ is said to be I-convergent to x and we write $I - \lim_{n\to\infty} x_n = x$.*

Definition 2. *A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be I^*-convergent to $x \in X$ if and only if there exists a set $M \in F(I), M = \{m_1 < m_2 < \cdots < m_k < \dots\}$ such that $\lim_{k\to\infty} x_{m_k} = x$. i.e. for every $c \in E$ with $0 \ll c$, there exists $p \in \mathbb{N}$ such that $c - d(x_{m_k}, x) \in intP$, for all $k \geq p$.*

It is known [16] that any cone metric space is a first countable Hausdorff topological space with the topology induced by the open balls defined naturally for each element z in X and for each element c in $\operatorname{int} P$. So as in [9] we can show that I^*-convergence always implies I-convergence but the converse is not true. The two concepts are equivalent if and only if the ideal I has condition (AP).

Further, throughout our paper we consider only cone metric spaces X with normal cone P and it is also known that

$$\text{For } a, b, c \in X, a \leq b, b \ll c \Rightarrow a << c. \qquad (*)$$

Note that the example of the cone metric space given above (see also [10]) is such a cone metric space.

3. I and I^*-Cauchy conditions in cone metric spaces

We first introduce the following definitions.

Definition 3. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is said to be* $I-$*Cauchy if for every* $c \in E$ *with* $0 \ll c$ *there exists* J *such that the set* $\{n \in \mathbb{N} : c - d(x_n, x_J) \notin \operatorname{int} P\} \in I$.

Definition 4. *The sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *is said to be an* I^**-Cauchy sequence if there exists a set* $M = \{m_1 < m_2 < \cdots < m_k < \dots\} \subset \mathbb{N}$ *with* $M \in F(I)$ *such that the subsequence* $\{x_{m_k}\}_{k\in\mathbb{N}}$ *is an ordinary Cauchy sequence in* X.

Theorem 1. *Let* I *be an arbitrary admissible ideal. Then* $I-\lim_{n\to\infty} x_n = \xi$ *implies that* $\{x_n\}_{n\in\mathbb{N}}$ *is an* I*-Cauchy sequence.*

Proof. Let $I - \lim_{n\to\infty} x_n = \xi$. Then for each $c \in E$ with $0 \ll c$ we have $A(c) = \{n \in \mathbb{N} : c - d(x_n, \xi) \notin \operatorname{int}P\} \in I$. Since I is an admissible ideal, there exists an $n_0 \in \mathbb{N}$ such that $n_0 \notin A(c)$. Let us put $B(c) = \{n \in \mathbb{N} : 2c \leq d(x_n, x_{n_0})\}$. Then if $n \in B(c)$ it follows that $d(x_n, \xi) + d(x_{n_0}, \xi) \geq d(x_n, x_{n_0}) \geq 2c$ and $d(x_{n_0}, \xi) \ll c$ and so we must have $d(x_n, \xi) \geq c$. This implies $c - d(x_n, \xi) \notin \operatorname{int}P$. Hence $n \in A(c)$. Thus $B(c) \subset A(c) \in I$, for each $0 \ll c$. Therefore it follows that $B(c) \in I$ and consequently $\{x_n\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence. □

Theorem 2. *Let* (X, d) *be a cone metric space and let* I *be an admissible ideal of* $\mathbb{N}$. *If* $\{x_n\}_{n\in\mathbb{N}}$ *is an* I^**-Cauchy sequence in* X *then it is an* I*-Cauchy sequence.*

Proof is omitted.

Lemma 1. *Let* $\{A_i\}_{i\in\mathbb{N}}$ *be a countable family of subsets of* $\mathbb{N}$ *such that* $A_i \in F(I)$ *for each* i *where* $F(I)$ *is the filter associated with an admissible ideal* I *with the property (AP). Then there is a set* $A \in \mathbb{N}$ *such that* $A \in F(I)$ *and the sets* $A \setminus A_i$ *is finite for all* i.

Theorem 3. *If* I *has property (AP) then the concepts of* I *and* I^**-Cauchy conditions coincide.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence in X. Then from the definition there exists an $J = J(c)$ such that the set $A(c) = \{n \in \mathbb{N} : c - d(x_n, x_J) \notin \operatorname{int}P\} \in I$ for every $c \in E$ with $0 \ll c$. Let $x \in P$ with $x \neq 0$. Now define $A_i = \{n \in \mathbb{N} : d(x_n, x_{m_i}) < \frac{x}{i}\},\ i = 1, 2, 3, \dots$ where $m_i = J(\frac{x}{i})$. It is clear that $A_i \in F(I)$ for $i = 1, 2, \dots$. Since I has the property (AP) there exists a set $Q \subset \mathbb{N}$ such that $Q \in F(I)$ and $Q \setminus A_i$ is finite for all i.

Let $c \in E$ with $0 \ll c$ and let $j \in \mathbb{N}$ be such that $\frac{2x}{j} \ll c$. As $Q \setminus A_j$ is finite, there exists $k = k(j)$ such that for all $m, n \in A_j$, we have $m, n > k(j)$. Therefore $d(x_n, x_{m_j}) < \frac{x}{j}$ and $d(x_m, x_{m_j}) < \frac{x}{j}$ for all $m, n > k(j)$. Then it readily follows that $d(x_n, x_m) \leq d(x_n, x_{m_j}) + d(x_m, x_{m_j}) \ll c$ for $m, n > k(j)$. Thus $\{x_n\}_{n\in Q}$ is an ordinary Cauchy sequence.

The following example shows that in general I-Cauchy condition does not imply I^*-Cauchy condition. □

Example 1. Let (X, d) be a cone metric space containing a Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct elements. Let $\mathbb{N} = \bigcup_{j\in\mathbb{N}}\Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is finite and $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Let I be the class of all those subsets A of $\mathbb{N}$ which intersects only finite number of Δ_j's. Then I is a non trivial admissible ideal of $\mathbb{N}$.

Define a sequence $\{y_n\}_{n\in\mathbb{N}}$ as $y_n = x_j$ if $n \in \Delta_j$. Then clearly $\{y_n\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence. If possible assume that $\{y_n\}_{n\in\mathbb{N}}$ is also I^*-Cauchy. Then there is an $A \in F(I)$ such that $\{y_n\}_{n\in A}$ is a Cauchy sequence. As $\mathbb{N} \setminus A \in I$ so there exists $l \in \mathbb{N}$ such that $\mathbb{N} \setminus A \subset \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_l$. Let us put $d(x_{l+1}, x_{l+2}) = \epsilon_0$. Then $0 < \epsilon_0$. From the construction of Δ_j's it clearly follows that for any given $k \in \mathbb{N}$ there are $m \in \Delta_{l+1}$ and $n \in \Delta_{l+2}$ such that $m, n \geq k$. Hence there is no $k \in \mathbb{N}$ such that whenever $m, n \in A$ with $m, n \geq k$ then $d(y_m, y_n) \ll \epsilon$ where $\epsilon = \frac{\epsilon_0}{2}$. This contradicts the fact that $\{y_n\}_{n\in A}$ is Cauchy.

Theorem 4. *Let (X, d) be a cone metric space containing at least one accumulation point. If for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X, I-Cauchy condition implies I^*-Cauchy condition then I satisfies the condition(AP).*

Proof. Let x_0 be an accumulation point of X. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct points of X which converges to x_0 and $x_n \neq x_0$ for all $n \in \mathbb{N}$. Let $\{A_i : i = 1, 2, 3, \dots\}$ be a sequence of mutually disjoint non empty sets from I. define a sequence $\{y_n\}_{n\in\mathbb{N}}$ as $y_n = x_j$ if $n \in A_j$ and $y_n = x_0$ if $n \notin A_j$ for any $j \in \mathbb{N}$. Let $c \in E$ with $0 \ll c$. Then there exists $l \in \mathbb{N}$ such that $d(x_0, x_n) \ll \frac{c}{2}$ for all $n \geq l$. Then $A(\frac{c}{2}) = \{n \in \mathbb{N} : \frac{c}{2} - d(x_0, x_n) \notin \text{int}P\} \subset A_1 \cup A_2 \cup \cdots \cup A_l \in I$. Now clearly $i, j \notin A(\frac{c}{2})$ implies that $d(x_0, y_i) \ll \frac{c}{2}$ and $d(x_0, y_j) \ll \frac{c}{2}$ which shows that $d(y_i, y_j) \ll c$ and consequently $\{y_n\}_{n\in\mathbb{N}}$ is an I-Cauchy sequence. By our assumption $\{y_n\}_{n\in\mathbb{N}}$ is then also I^*-Cauchy. Hence there exists $H \in I$ such that $B = \mathbb{N}\setminus H \in F(I)$ and $\{y_n\}_{n\in B}$ is an ordinary Cauchy sequence.

Now let $B_j = A_j \cap H$ for $j \in \mathbb{N}$. Then each $B_j \in I$. Further $\cup B_j = H \cap (\cup A_j) \subset H$. Therefore $\cup B_j \in I$. Now for the sets $A_i \cap B$, $i \in \mathbb{N}$ following three cases may arise.

Case (I) : Each $A_i \cap B$ is included in a finite subset of $\mathbb{N}$.

Case (II) : Only one of $A_i \cap B'$s, namely, $A_k \cap B$ is not included in a finite subset of $\mathbb{N}$.

Case (III) : More than one of $A_i \cap B'$s are not included in finite subsets of $\mathbb{N}$.

If (I) holds then $A_j \Delta B_j = A_j \setminus B_j = A_j \setminus H = A_j \cap B$ is included in a finite subset of $\mathbb{N}$ which shows that I has the property (AP). If (II) holds then we redefine $B_k = A_k$ and $B_j = A_j \cap H$ for $j \neq k$. Then $\bigcup_{n\in\mathbb{N}} B_n = [H \cap (\bigcup_{j\neq k} A_j)] \subset H \cup A_k$ and so $\cup B_j \in I$. Also since $A_i \Delta B_i = A_i \cap B$ for $i \neq k$ and $A_k \Delta B_k = \emptyset$ so as in Case (I) the criterion for (AP) condition is satisfied.

If (III) holds then there exists $k, l \in \mathbb{N}$ with $k \neq l$ such that $A_k \cap B$ and $A_l \cap B$ are not included in any finite subset of $\mathbb{N}$. Let us put $\frac{d(x_k,x_l)}{2} = \epsilon_0$. Then $0 < \epsilon_0$. Let $\epsilon > 0$ be given. Choose $c \in E$ such that $K||c|| < \epsilon$. As $\{y_n\}_{n\in B}$ is a Cauchy sequence so there exists $k_0 \in N$ such that $d(y_i, y_j) \ll c$ for all $i, j \geq k_0; i, j \in B$. Then $||d(y_i, y_j)|| << K||c|| < \varepsilon$ and so $d(y_i, y_j) \to 0$ as $i, j \to \infty, i, j \in B$.

Now since $A_k \cap B$ and $A_l \cap B$ are not included in finite subsets of $\mathbb{N}$ so we can choose $i \in A_k \cap B$ and $j \in A_l \cap B$ with $i, j \geq k_0$. But $y_i = x_k$ and $y_j = x_l$ and hence $0 < \epsilon_0 < d(x_k, x_l) = d(y_i, y_j)$ (in fact there are infinitely many indices with that property). This contradicts the fact that $\{y_n\}_{n\in B}$ is Cauchy. Therefore case(III) can not arise and in view of Case (I) and Case (II) I satisfies (AP) condition. □

4. I-Cauchy condition and completeness

Let us denote,

$m(X) \equiv \{$the space of all bounded sequence in X endowed with sup-norm$\}$.

$$\mathcal{F}(I) = \{x = \{x_n\}_{n\in\mathbb{N}} \in m(X) : \text{ there is } I - \lim x_n \in X\}$$
$$\mathcal{F}(I^*) = \{\{x_n\}_{n\in\mathbb{N}} \in m(X) : \text{ there is } I^* - \lim x_n \in X\}.$$

We first show the following interesting implication of the completeness of the space.

Theorem 5. *Let (X, d) be a complete cone metric space and let I be an admissible ideal of $\mathbb{N}$. Then $\mathcal{F}(I)$ is a closed subspace of $m(X)$.*

Proof. Let $x^{(m)} = \{x_k^{(m)}\}_{k\in\mathbb{N}} \in F(I)$ $(m = 1, 2, 3, \dots)$ and $\lim_{m\to\infty} x^{(m)} = x$ where $x = \{x_k\} \in m(X)$. That is, $\lim_{m\to\infty} \overline{d}(x^{(m)}, x) = 0$ (where $\overline{d}$ denoted the sup metric in $m(X)$). We show that $x \in F(I)$. By our assumption each $x^{(m)}$ is I-convergent in X and let $I - \lim_{m\to\infty} x^{(m)} = \xi_m$, $m = 1, 2, 3, \dots$.

Since $\lim_{m\to\infty} x^{(m)} = x$, so $\{x^{(m)}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $m(X)$. Let $\epsilon \in E$ with $0 \ll \epsilon$. Then there exists m_0 such that for all $u, v > m_0$, $\overline{d}(x^{(u)}, x^{(v)}) \ll \frac{\epsilon}{3}$. Fix $u, v > m_0$. Note that the set $U(\frac{\epsilon}{3}) = \{j : d(x_j^{(u)}, \xi_u) \ll \frac{\epsilon}{3}\}$, $V(\frac{\epsilon}{3}) = \{j : d(x_j^{(v)}, \xi_v) \ll \frac{\epsilon}{3}\}$ belongs to $F(I)$. Thus their intersection is non-void. For any element $s \in U(\frac{\epsilon}{3}) \cap V(\frac{\epsilon}{3})$ we have $d(x_s^{(u)}, \xi_u) < \frac{\epsilon}{3}$ and $d(x_s^{(v)}, \xi_v) < \frac{\epsilon}{3}$. Thus $d(\xi_u, \xi_v) \leq d(\xi_u, x_s^{(u)}) + d(x_s^{(u)}, x_s^{(v)}) + d(x_s^{(v)}, \xi_v) \ll \epsilon$

(since $\overline{d}(x^{(u)}, x^{((v))}) \ll \frac{\epsilon}{3}$ so this implies that $\sup_s d(x_s^{(u)}, x_s^{(v)}) \ll \frac{\epsilon}{3}$ which implies that $d(x_s^{(u)}, x_s^{(v)}) \ll \frac{\epsilon}{3}$, for any s).

Hence $\{\xi_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in X. Since X is complete there exists $\xi = \lim_{m\to\infty}\xi_m \in X$. Let $\eta \in E$ with $0 \ll \eta$. Choose v_0 such that for $v > v_0$ we have simultaneously $d(\xi_v, \xi) \ll \frac{\eta}{3}$ and $\overline{d}(x^{(v)}, x) \ll \frac{\eta}{3}$. Then for each $n \in \mathbb{N}$ we have $d(x_n, \xi) \leq d(x_n, x_n^{(v)}) + d(x_n^{(v)}, \xi_v) + d(\xi_v, \xi)$. Let us put

$$A(\eta) = \{n : \eta - d(x_n, \xi) \notin \text{int}P\}, A(\eta)^c = \{n : \eta - d(x_n, \xi) \ll \eta\}$$

$$A_v\left(\frac{\eta}{3}\right) = \left\{n : \frac{\eta}{3} - d(x_n^v, \xi_v) \notin \text{int}P\right\}, A_v\left(\frac{\eta}{3}\right)^c = \left\{n : d(x_n^v, \xi_v) \ll \frac{\eta}{3}\right\}.$$

Thus for $n \in A_v(\eta)^c$ the inequality $d(x_n, \xi) \ll \eta$ and the inclusion $A_v(\frac{\eta}{3})^c \subset A(\eta)^c$ holds. We now observe that $A_v(\frac{\eta}{3}) \in I$. Also we have $A(\eta) \subset A_v(\frac{\eta}{3})$ which shows that $A(\eta) \in I$. Hence $x \in m(X)$. □

Theorem 6. *For every admissible ideal I of $\mathbb{N}$ we have $\overline{\mathcal{F}(I^*)} = \mathcal{F}(I)$.*

The proof is straight forward and so is omitted.

We will need the following result.

Lemma 2. *Let (X, d) be a complete cone metric space and let $\{F_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of non empty closed subsets of X such that* $\text{diam}(F_n) \to 0$. *Then the intersection $\cap_{n=1}^{\infty} F_n$ contains exactly one point.*

Proof. Let us construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X by selecting a point $x_n \in F_n$ for each n. Since the sets F_n are nested, consequently $x_n \in F_m$ for all $n \geq m$. We now show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $c \in E$ with $0 \ll c$. Since $\text{diam}(F_n) \to 0$, there exists a positive integer N such that $\text{diam}(F_N) \ll c$. Note that $x_n, x_m \in F_N$ for all $n, m \geq N$ and so we have $d(x_n, x_m) \leq \text{diam}(F_N) \ll c$. Therefore we have $d(x_n, x_m) \ll c$ for all $n, m \geq N$ which implies that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since (X, d) is complete there exists $x \in X$ such that $x_n \to x$.

Now we claim that $x \in \cap_{n=1}^{\infty} F_n$. Let n be fixed. Then the subsequence $\{x_n, x_{n+1}, \dots\}$ of $\{x_n\}_{n\in\mathbb{N}}$ is contained in F_n and still converges to x. But F_n being a closed subspace of the complete cone metric space X is also complete and so $x \in F_n$. This is true for each $n \in \mathbb{N}$ and hence $x \in \cap_{n=1}^{\infty} F_n$.

Finally let $y \in \cap_{n=1}^{\infty} F_n$. Then $x, y \in F_n$ for each n. Therefore $0 \leq d(x, y) \leq \text{diam}(F_n)$. Now $\text{diam}(F_n) \to 0$ as $n \to \infty$. Thus for $c \in E$ with $0 \ll c$, there exists $m \in \mathbb{N}$ such that $\text{diam}(F_n) \ll c$ for all $n \geq m$ which shows that $d(x, y) \ll c$. Since $c \in E$ is arbitrary we have $d(x, y) = 0$ which implies that $x = y$. □

Theorem 7. *If (X, d) is a complete cone metric space then every I-Cauchy sequence in X is I-convergent in X.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be an I-Cauchy sequence in X. Let $y \in P$ and $y \neq 0$. Consider the sequence $\epsilon_m = \frac{y}{m}$, $m = 1, 2, 3, \dots$. Then $\{\epsilon_m\}_{m\in\mathbb{N}} \in P$ and $||\epsilon_m|| \to 0$. Now according to I-Cauchy condition pick numbers $k(m) \in \mathbb{N}, m \in \mathbb{N}$ such that $A_m = \{n \in \mathbb{N} : \frac{\epsilon_m}{2} - d(x_n, x_{k(m)}) \notin \text{int}P\} \in I$ for all $m \in \mathbb{N}$. Let us define inductively

$$B_1 = \text{cl}B(x_{k(1)}, \epsilon_1), B_{m+1} = B_m \cap \text{cl}B(x_{k(m+1)}, \epsilon_{m+1}), m \in \mathbb{N}.$$

Let us prove that $B_m \neq \emptyset$ for each $m \in \mathbb{N}$. We have $A_1 \in I$ and $x_n \in B_1$ for all $n \notin A_1$. Assume that $m \in \mathbb{N}$ and $C \in I$ is a set such that $x_n \in B_m$ for each $n \notin C$. We have $A_{m+1} \in I$ and $x_n \in B_{m+1}$ for all $n \notin C \cup A_{m+1}$. Thus $B_m \neq \emptyset$ for each m.

Since additionally $B_{m+1} \subset B_m$ for all m and the diameter of B_m tends to zero so there is an $x \in X$ such that $\cup_{m\in\mathbb{N}} B_m = \{x\}$ by Lemma 2. Now it suffices to show that $I - \lim x_n = x$. Let $\epsilon \in E$ with $0 \ll \epsilon$ and pick an $m \in \mathbb{N}$ such that $\epsilon_m < \frac{\epsilon}{2}$. Let us put $A(\epsilon) = \{n : \epsilon - d(x_n, x) \in \text{int}P\}$ and $B(\epsilon) = \{n : \epsilon - d(x_n, x_{k(m)}) - d(x_{k(m)}, x)\} \in \text{int}P$. Since $d(x_n, x) \leq d(x_n, x_{k(m)}) + d(x_{k(m)}, x)$ and P be a normal cone with normal constant $k < 1$ we have $B(\epsilon) \subset A(\epsilon)$ that is $A^c(\epsilon) \subset B^c(\epsilon)$. As $x \in B_m$ we get $d(x_{k(m)}, x) \leq \epsilon_m < \frac{\epsilon}{2}$. Therefore $A^c(\epsilon) \subset \{n : \frac{\epsilon}{2} - d(x_n, x_{k(m)}) \notin \text{int}P\} \subset \{n : \epsilon_m - d(x_n, x_{k(m)}) \notin \text{int}P\} \subset A_m \in I$. Thus $I - \lim_{n\to\infty} x_n = x$. □

Theorem 8. *If every I-Cauchy sequence in X is I-convergent in X then X is complete.*

The proof is parallel to the metric case and so is omitted.

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