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DYNAMICAL SYSTEMS ASSOCIATED WITH QRT FAMILIES OF DEGREE FOUR BIQUADRATIC CURVES EACH OF THEM WITH GENUS ZERO

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ABSTRACT. We give some examples of QRT families of biquadratic curves which are of degree 4 (with high degree term x^2y^2), but such that every curve of such a family is singular, with genus 0, or is reducible (for some finite set of values of the parameter). This contrasts with classical examples of QRT families studied by many authors. We give also examples of QRT families of degree 4 whose every curve is reducible. Then we sketch out to study the dynamical systems associated to such QRT families of genus zero.

1. INTRODUCTION, THE RESULTS

When one reads some papers about QRT-maps defined by a family of biquadratics curves in the plane (which are of degree 2 in x and degree 2 in y) affinement depending of a parameter, it is possible to think that, except for a finite number of values of the parameter, the curves of such a family are of genus 1, *id est* elliptic : see [7], [8], [10] . . . And many examples of such families are given in many papers: see for instances [1], [4], [5], [6], [7], [12]. . .

Examples of QRT families of curves each of them with genus 0 are given, but there are generally examples of families of conics: see [2], [3]. It seems to us that there is no examples of QRT families of degree exactly 4, that is with the term x^2y^2 in the equation of every curve of the family, but such that each curve of it is of genus 0. The goal of this paper is to give such examples, and to study the dynamical systems associated to such QRT families.

We recall first that is a QRT-map (see [7]). Let be two biquadratic curves with equations $Q_1(x,y)=0$ and $Q_2(x,y)=0$, that is which are of degree 2 in x and in y. The QRT family of quartic curves with equation $B_\lambda(x,y) := Q_1(x,y)-\lambda Q_2(x,y) = 0$ defines a map by the following method: if M is a point of the plane (with $Q_2(M)\neq 0$), the horizontal line passing through M cuts again the curve $\mathcal{C}_\lambda$ of the family passing through M (that is with $\lambda = \frac{Q_1(M)}{Q_2(M)}$) at a point M_1, and the vertical

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line through M_1 cuts again the same curve at a point M'. The map $M \mapsto M'$ is the QRT-map T associated to the QRT family B_λ.

These maps were introduced in [11] for reasons that come from physical problems. In many papers (see for example [1], [4], [5], [12]) authors study the dynamical system associated to the QRT family, that is they study the behavior of the sequence $M_n = T^n(M_0)$, in particular they find periods of periodic orbits, and prove the density of these orbits.

We start with the special family of biquadratic curves of [4] :

$$x^2y^2 - 5xy(x+y) + d(x^2+y^2) - 20(x+y) + 16 - Kxy = 0, \tag{1.1}$$

where $K \in \mathbb{R}$ and d is a real parameter. It is shown in [4] that for $d > d_0 = \frac{25}{4}$ these curves are elliptic except for the levels $K = -2d-8, K = 2d-33, K = 2d-32$ and $K = 2d+48$, where the curves are of genus 0 or reducible. Substituting the corresponding values for d, and putting $-\frac{K}{2} = \lambda$ or $\frac{K}{2} = \lambda$, depending on the case, we find four QRT-families of degree 4 biquadratic curves, symmetric with respect to the diagonal:

$$x^2y^2 - 5xy(x+y) - 4(x^2+y^2) - 20(x+y) + 16 - \lambda(x+y)^2 = 0, \tag{1.2}$$

$$x^2y^2 - 5xy(x+y) + \frac{33}{2}(x^2+y^2) - 20(x+y) + 16 - \lambda(x-y)^2 = 0, \tag{1.3}$$

$$x^2y^2 - 5xy(x+y) + 16(x^2+y^2) - 20(x+y) + 16 - \lambda(x-y)^2 = 0, \tag{1.4}$$

$$x^2y^2 - 5xy(x+y) - 24(x^2+y^2) - 20(x+y) + 16 - \lambda(x-y)^2 = 0, \tag{1.5}$$

The results are the following:

Theorem 1.1. *Every curve of each QRT families (4) and (5) is not reducible and is of genus 0, except for two values of λ (which depend on the family) for which the curves are reducible. Every curve of families (2) and (3) is reducible as the union of two conics.*

Define the two functions

$$\begin{aligned} G_4(x,y) &:= \frac{x^2y^2 - 5xy(x+y) + 16(x^2+y^2) - 20(x+y) + 16}{(x-y)^2}, \\ G_5(x,y) &:= \frac{x^2y^2 - 5xy(x+y) - 24(x^2+y^2) - 20(x+y) + 16}{(x-y)^2}. \end{aligned} \tag{1.6}$$

Theorem 1.2. *For the family (4), suppose $G_4(M_0) \notin \{10, 11\}$. If $G_4(M_0) < 11$, then the sequence of points $T^n(M_0)$ converges to the point $D = (2,2)$; if $G_4(M_0) > 11$, then the sequence of points $T^n(M_0)$ is periodic or is dense in the curve $\mathcal{C}_\lambda$ which passes through M_0.*

Theorem 1.3. *For the family (5), suppose $G_5(M_0) \notin \{-19, -10\}$. If $G_5(M_0) < -19$, then the sequence of points $T^n(M_0)$ converges to $D' = (-2,-2)$; if $G_5(M_0) >$*

-19*, then the sequence of points* $T^n(M_0)$ *is periodic or is dense in the curve* $\mathcal{C}_\lambda$ *which passes through* M_0.

2. Study of the QRT family (4)

First, we will determine the singular points of the curves $\mathcal{C}_\lambda$ with equation $B_\lambda(x,y) = 0$. Our goal is to use the following formula of algebraic geometry for the genus g of a degree d algebraic curve which is *not reducible* and whose *singular points p are ordinary*, that is of multiplicity μ_p with μ_p distinct tangents:

$$g = \frac{(d-1)(d-2)}{2} - \sum_{p\in P} \frac{\mu_p(\mu_p-1)}{2}. \tag{2.1}$$

For this formula, see [9]. In our case, $d = 4$, and the formula becomes $3 - \sum_{p\in P} \frac{\mu_p(\mu_p-1)}{2}$.

2.1. **The singular points at infinity**

We start with the infinite points H and V in horizontal and vertical directions (after use of homogeneous variables X,Y,T we make $T = 0$: these points are double), and search the vertical and horizontal asymptotes, by solving the quadratic equations (4) $B_\lambda(x,y) = 0$ in y and in x: we find

$$\begin{aligned} &y^2(x^2-5x+16-\lambda)+y(-5x^2+2\lambda x-20)+(16-\lambda)x^2-20x+16=0\\ &x^2(y^2-5y+16-\lambda)+x(-5y^2+2\lambda y-20)+(16-\lambda)y^2-20y+16=0. \end{aligned} \tag{2.2}$$

The equations from vertical asymptotes are $x^2 - 5x + 16 - \lambda = 0$, whose roots are distinct if and only if

$$\lambda \neq \frac{39}{4}, \tag{2.3}$$

and the same condition is this one for having to distinct horizontal asymptotes. So under condition (2.3) H and V are two ordinary singular double points (with 2 distinct tangent lines). The four asymptotes are real if $\lambda > \frac{39}{4}$, and complex if $\lambda < \frac{39}{4}$.

2.2. **The finite singular points**

Now we search finite singular points, and so we shall resolve $B_\lambda = 0$, $\frac{\partial B_\lambda}{\partial x} = 0$ and $\frac{\partial B_\lambda}{\partial y} = 0$, where

$$\begin{aligned} \frac{\partial B_\lambda}{\partial x} &= 2xy^2 - 10xy - 5y^2 + 32x - 20 - 2\lambda(x-y)\\ \frac{\partial B_\lambda}{\partial y} &= 2yx^2 - 10yx - 5x^2 + 32y - 20 + 2\lambda(x-y), \end{aligned} \tag{2.4}$$

which are of degree one in respectively x and y.

From equations $\frac{\partial B_\lambda}{\partial x} = 0$ and $\frac{\partial B_\lambda}{\partial y} = 0$ one deduces easily (by difference and by linear combination with coefficients x and $-y$) the relations

$$\begin{aligned} &(y-x)(2xy - 5(x+y) + 4\lambda - 32) = 0, \\ &(y-x)(5xy + 2(\lambda - 16)(x+y) + 20) = 0. \end{aligned} \tag{2.5}$$

(a) We have first the solutions $x = y := z$; the equations $\frac{\partial B_\lambda}{\partial x} = 0$ and $\frac{\partial B_\lambda}{\partial y} = 0$ become the same, and we have also equation $B_\lambda(z,z) = 0$, that is

$$\begin{aligned} &(z-2)(2z^2 - 11z + 10) = 0, \\ &(z-2)^2(z^2 - 6z + 4) = 0. \end{aligned} \tag{2.6}$$

So the point $D = (2,2)$ is a singular point. Remark that it is independant of λ, it is on every curve $\mathcal{C}_\lambda$ (it is also the case of the two points on the diagonal with abscisses $3 \pm \sqrt{5}$). Now we make in $B_\lambda(x,y)$ the change of variables $x = 2+u, y = 2+v$, and obtain $u^2v^2 - uv(u+v) + 2(\lambda - 12)uv + (10-\lambda)(u^2 + v^2)$, whose terms of degree 2 give the multiplicity of D and the tangent lines at D. So if we have the condition

$$\lambda \neq 11 \tag{2.7}$$

the point D is an ordinary singular point (with two distinct tangents, real if $\lambda < 11$, or complex if $\lambda > 11$).

(b) Now we search singular points out of the diagonal. If we put

$$s := x + y \text{ and } p := xy,$$

and from (4) and (11), the equations in s and p of a singular point out of the diagonal are

$$\begin{aligned} &2p - 5s + 4\lambda - 32 = 0 \\ &5p + 2(\lambda - 16)s + 20 = 0 \\ &p^2 - 5ps + 16(s^2 - 2p) - 20s + 16 - \lambda(s^2 - 4p) = 0. \end{aligned} \tag{2.8}$$

By solving the two first equations and putting the values for s and p in the third one, we find

$$\frac{(\lambda - 10)(\lambda - 11)^2}{\lambda - \frac{39}{4}} = 0. \tag{2.9}$$

So if $\lambda \neq 10$, $\lambda \neq 11$, $\lambda \neq \frac{39}{4}$, there is no singular point out of the diagonal, and only three double ordinary points. Before to conclude from (7) that the genus of the curve $\mathcal{C}_\lambda$ is zero for $\lambda \notin \{10, 11, \frac{39}{4}\}$, *it is necessary to know if the curve $\mathcal{C}_\lambda$ is not reducible*.

2.3. **Is the curve $\mathcal{C}_\lambda$ reducible?**

If the curve is reducible, it is not difficult (use the symmetry with respect to the diagonal and the biquadratic property) to see that there is only two cases:
(a) the curve is composed of a vertical line and an horizontal line which are symmetric, and of a symmetric hyperbola with horizontal and vertical asymptotes;
(b) the curve is composed of two hyperbolas with vertical and horizontal asymptotes.

(a) First case. We can write for the equation of the curve

$$(x+b)(y+b)(xy+a(x+y)+\gamma)=0,$$

and we must identify this equation with equation (4). So we find

$$\begin{aligned}
&\gamma=\frac{16}{b^2},\\
&b+a=-5,\\
&ab=16-\lambda,\\
&b^2+\frac{16}{b^2}+2ab=2\lambda,\\
&\frac{16}{b}+ab^2=-20.
\end{aligned}\tag{2.10}$$

We put $a=-5-b$ in the fifth equation and obtain $b^4+5b^3-20b-16=0$, that is $(b+4)(b+2)(b+1)(b-2)=0$. This gives the possible solutions for (a,b):

$$(-1,-4),\ (-3,-2),\ (-4,-1),\ (-7,2).$$

So we compute the values of λ given by the third and the fourth equations, and they must be the same. But it is the case only for $a=-3,\ b=-2$, and the curve is reducible with the two lines $x=2$ and $y=2$, if and only if

$$\lambda=10.\tag{2.11}$$

(b) Second case. We suppose now that the curve splits into two conics. It is easy to see that, with the symmetry, there is only two possibility for the form of the equation:

$$(xy+ax+by+c)(xy+bx+ay+\frac{16}{c}),$$
$$(xy+a(x+y)+c)(xy+b(x+y)+\frac{16}{c}).$$

(b1) In the first possibility, we have

$$\begin{aligned}
&a+b=-5,\\
&ab=16-\lambda,\\
&\frac{16}{c}+c+a^2+b^2=2\lambda,\\
&\frac{16a}{c}+cb=\frac{16b}{c}+ca=-20.
\end{aligned}\tag{2.12}$$

If $a \neq b$, the symmetry implies that $\frac{16}{c} = c$, and so we have $-20 = c(a+b) = -5c$, and $c = 4$. Then $a^2 + b^2 = 2\lambda - 8$, but the first two equations give $a^2 + b^2 = 2\lambda - 7$, which does not agree with the previous relation. So we have $a = b$, which leads to the value $\lambda = \frac{39}{4}$.

In this case, we put $x = 2 + \frac{1}{u}$ and $y = 2 + \frac{1}{v}$, and obtain for the equation (after exclusion of points H and V):

$$u^2 + v^2 - 18uv - (u+v) + 1 = 0.$$

So the curve $\mathcal{C}_{\frac{39}{4}}$ is equivalent by a birational transformation to a conic not reducible. This prove that the biquadratic curve is not reducible and of genus zero.

(b2) In the second possibility, we have

$$\begin{aligned} a+b &= -5, \\ ab &= 16 - \lambda, \\ \frac{16a}{c} + bc &= -20, \\ 2ab + \frac{16}{c} + c &= 2\lambda. \end{aligned} \tag{2.13}$$

If $c = 4$, one find $\lambda = 10$, previously obtained, and if $c \neq 4$, some calculations give

$$\lambda = 11, \tag{2.14}$$

with $c = 6 \pm 2\sqrt{5}$.

So at the end of this study, we see that the final result proves a part of theorem 1.1:

Proposition 2.1. *If $\lambda \notin \{10, 11\}$, then the curve $\mathcal{C}_\lambda$ is not reducible, and is genus is exactly zero. It is also the case if $\lambda = \frac{39}{4}$, but in this case there is only a vertical asymptote and an horizontal one.*

In the followinf figure we give two examples of the forms of curves $\mathcal{C}_\lambda$:

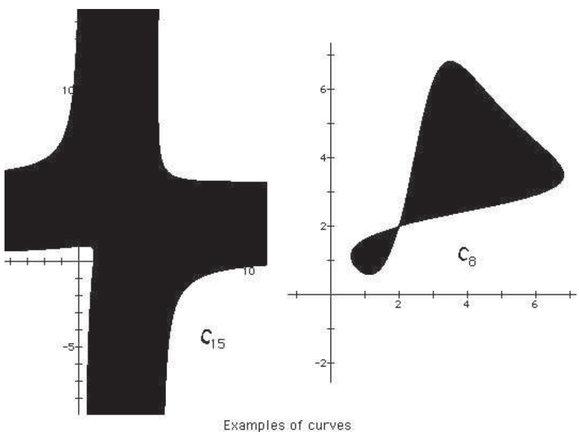

Figure 1: Examples of curves

2.4. **The two reducible curves**

So with the results of previous calculations, it is not difficult to prove the

Proposition 2.2. *If* $\lambda = 11$*, the equation of* $\mathcal{C}_\lambda$ *is*

$$\left(xy + \frac{-5+\sqrt{5}}{2}(x+y) + 6 - 2\sqrt{5}\right)\left(xy + \frac{-5-\sqrt{5}}{2}(x+y) + 6 + 2\sqrt{5}\right) = 0;$$

if $\lambda = 10$*, the equation is*

$$(x-2)(y-2)\big(xy - 3(x+y) + 4\big) = 0.$$

This achieves the proof of theorem 1.1 for the QRT family (4).

Remark 2.1. In the notations of paper [10], we have $\alpha_2 = x^2 - 5x + 16 - \lambda$ and $\beta_2 = y^2 - 5y + 16 - \lambda$, and so we have

$$discr_x(\alpha_2) = discr_y(\beta_2) = 4\lambda - 39.$$

So if $\lambda \neq \frac{39}{4}$, the ASSUMPTION 1 of this paper is true (note that ASSUMPTION 2 of [10] is also true for every λ: the two quantities to evaluate in [10] are equal to 245). But we have, in the QRT family (4), $discr_{yx}(B_\lambda) \equiv 0$, and every $\mathcal{C}_\lambda$ is singular. So this QRT family (4) contrasts with [10], lemma 4, which asserts that only for at most 12 values of λ the curve $\mathcal{C}_\lambda$ is singular.

Remark 2.2. Another method to study the genus of the curves of QRT family (4) follows this one of [4]: we put in the equation $B_\lambda(x,y) = 0$ of the curves of (4)

$$x = \frac{1-2X}{Y} \quad and \quad y = \frac{1-2Y}{X},$$

and obtain the new equation

$$(2X + 2Y - 1)^2\Big((\lambda - 16)(X^2 + Y^2) + 2(6-\lambda)XY + 5(X+Y) - 1\Big) = 0.$$

So by a birational transformation the cuve split in a double right line and a conic (we have make split the double point $D = (2,2)$); so the genus is easy to find ... We shall use this method in section 4.

3. The other families

Analogous calculations (with the aid of *Maple*) give the case of families (2), (3) and (5). For the family (5), the calculations give three ordinary singular points on every curve if $\lambda \neq -10$, $\lambda \neq -19$ and $\lambda \neq -\frac{121}{4}$: $H, V, D' = (-2,-2)$. The final result is:

Proposition 3.1. (a) *Every curve of the family* (2) *is reducible :*

$$B_\lambda = \left(xy - \frac{5}{2}(x+y) + 4 + \frac{\sqrt{4\lambda+41}}{2}(x+y)\right)\left(xy - \frac{5}{2}(x+y) + 4 - \frac{\sqrt{4\lambda+41}}{2}(x+y)\right);$$

if $\lambda = -\frac{41}{4}$ *the curve is a double hyperbola.*

(b) *Every curve of the family (3) is reducible :*

$$B_\lambda=\Big(xy-\frac{5}{2}(x+y)+4+\frac{\sqrt{4\lambda-41}}{2}(x-y)\Big)\Big(xy-\frac{5}{2}(x+y)+4-\frac{\sqrt{4\lambda-41}}{2}(x-y)\Big);$$

if $\lambda=\frac{41}{4}$ *the curve is a double hyperbola.*
(c) *Every curve of the family* (5) *is not reducible and of genus zero, except for the values* $\lambda=-10$ *and* $\lambda=-19$ *for which* $\mathcal{C}_\lambda$ *is reducible: if* $\lambda=-10$ *we have*

$$B_\lambda=(x+2)(y+2)\big(xy-7(x+y)+4\big);$$

if $\lambda=-19$ *we have*

$$B_\lambda=\Big(xy-\frac{5+3\sqrt{5}}{2}(x+y)-14-6\sqrt{5}\Big)\Big(xy-\frac{5-3\sqrt{5}}{2}(x+y)-14+6\sqrt{5}\Big).$$

If $\lambda=-\frac{121}{4}$ *there is only a vertical asymptote and only a vertical one, but the curve is not reducible and is of genus zero.*

So we have completely proved theorem 1.1.

4. About the dynamical systems associated to families (4) and (5): proof of theorems 1.2 and 1.3

4.1. **Study of the dynamical system associated to the QRT family (4)**

As we said in remark 2.2 we put in the equation $B_\lambda(x,y)=0$ of (4)

$$x=\frac{1-2X}{Y}\quad\text{and}\quad y=\frac{1-2Y}{X},\tag{4.1}$$

and obtain the new equation

$$(2X+2Y-1)^2\Big((\lambda-16)(X^2+Y^2)+2(6-\lambda)XY+5(X+Y)-1\Big)=0.\tag{4.2}$$

Now we remark that the family of horizontal lines $y=C$ becomes the pencil of lines passing through the point $\widetilde{H}=(0,\frac{1}{2})$, and also the family of vertical lines $x=C$ becomes the pencil of lines passing through the point $\widetilde{V}=(\frac{1}{2},0)$.

In fact, we have made split the double point $D=(2,2)$ by the reciprocal map ϕ of the map defined by (21)

$$X=\frac{x-2}{xy-4},\;\; Y=\frac{y-2}{xy-4},\tag{4.3}$$

and this point gives the double line Δ with equation $X+Y=\frac{1}{2}$, the other points of the family of curves $\mathcal{C}_\lambda$ become the pencil of conic curves (symmetric with respect to the diagonal and the line $X+Y=\frac{1}{2}$) with equations

$$E_\lambda:=16(X^2+Y^2)-12XY-5(X+Y)+1-\lambda(X-Y)^2=0.\tag{4.4}$$

This pencil is defined, for example, by the double line $(X-Y)^2=0$ (for $\lambda=\infty$) and the conic reducible to two parallel lines (for $\lambda=11$) $X+Y=\frac{1}{2}\pm\frac{1}{10}\sqrt{5}$.

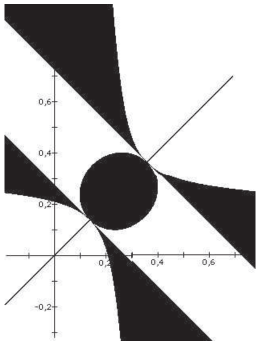

Figure 2: Pencil of conic curves

So, it is no difficult to prove that the QRT map T associated to the QRT family (4), restricted to the curve $\mathcal{C}_\lambda$, is conjugated to the map $\widetilde{T}_\lambda$ defined by the following way on E_λ: if $M \in E_\lambda$, the line passing by M and the point $\widetilde{H}$ cut E_λ at a point M_1, and the line passing by M_1 and $\widetilde{V}$ cuts again E_λ at M'; then $\widetilde{T}_\lambda$ is the map $M \mapsto M'$.

Except for $\lambda = 10$, the points $\widetilde{H}$ and $\widetilde{V}$ are not on the conic curve E_λ, so the map $\widetilde{T}_\lambda$ is an homography on the conic. So the dynamical of $\widetilde{T}_\lambda$ shall be explained by the following lemma, whose proof is easy with the aid of *Maple*:

Lemma 4.1. *Let be $a > 0, b > 0, \varepsilon = \pm 1, p > 0,\ p \neq a$, and consider the conic curve E with equation*

$$\frac{(y-b)^2}{b^2} + \varepsilon \frac{x^2}{a^2} - 1 = 0. \tag{4.5}$$

Put $A = (0,0), P_1 = (p,b), P_2 = (-p,b)$, and take for parameter of the points of the conic curve the slope t of the line $(AM(t))$.

Then the homographic map

$$F : E \to E \quad defined\ by \quad : M(t) \mapsto N := (M(t)P_1) \cap E \mapsto (NP_2) \cap E$$

whose images may be at infinity or complex, is traduced on the parameters as

$$t' = \mathcal{H}(t) := b \frac{(a^2 + \varepsilon p^2)t - 2\varepsilon bp}{(a^2 + \varepsilon p^2)b - 2a^2 pt}. \tag{4.6}$$

Figure 3 shows this lemma for the case of an ellipse.

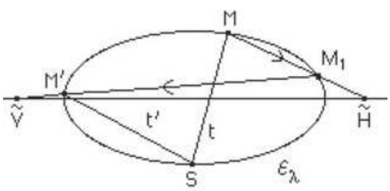

Figure 3. The map F in the case of an ellipse

If $\varepsilon = 1$, E is an ellipse, and if $\varepsilon = -1$, E is an hyperbola. The fixed points of $\mathcal{H}$ are

$$\alpha = \sqrt{\varepsilon}\,\frac{b}{a}, \quad \beta = -\alpha, \tag{4.7}$$

which are real if E is an ellipse and imaginary if E is an hyperbola. In the first case, the fixed points of F are the points (a,b) and $(-a,b)$, and the homography $\mathcal{H}$ satisfies the relation

$$\frac{\mathcal{H}(t)-\alpha}{\mathcal{H}(t)-\beta} = k\frac{t-\alpha}{t-\beta}, \tag{4.8}$$

where

$$k = \frac{(a+p)^2}{(a-p)^2}. \tag{4.9}$$

In the second case, the fixed points of $\mathcal{H}$ are $i\frac{b}{a}$ and $-i\frac{b}{a}$, and the number k' in the analogous formula to (29) is

$$k' = \frac{(a^2-p^2)^2 - 4a^2p^2 + i4ap(a^2-p^2)}{(a^2+p^2)^2}, \quad \text{and} \quad |k'| = 1. \tag{4.10}$$

So we have $k > 1$, and $k^n \to +\infty$ when $n \to +\infty$, and in the case of the ellipse the point $M_n = F^n(M(t)) \to M(-\alpha) = (-a,b)$.

In the case of the hyperbola, k' has is modulus equal to 1, then the behaviour of $M_n = F^n(M(t))$ depends on the rationality of the number

$$\frac{\theta}{\pi} = \frac{1}{\pi}\arctan\frac{4ap(a^2-p^2)}{(a^2-p^2)^2-4a^2p^2}. \tag{4.11}$$

If we return to the family of curves $\mathcal{C}_\lambda$, we see that we have proved the theorem 1.2.

In the following figure, we see the region where $G_4 < 11$.

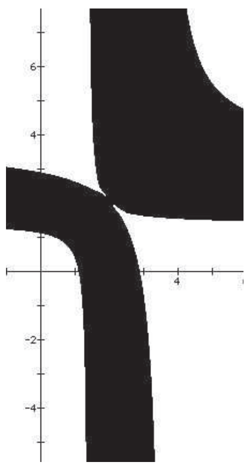

Figure 4: The region where $G_4 < 11$

4.2. **The possible minimal periods of the dynamical system (4) in** $\{G_4 > 11\}$

We have $\varepsilon = -1$, and so $k' = e^{i\theta}$. Some calculations give the value of the parameter $\frac{p}{a}$:

$$\frac{p}{a} = \sqrt{\lambda - 11}. \tag{4.12}$$

If we put $\psi = \arctan \frac{p}{a} \in \left]0, \frac{\pi}{2}\right[$, we have $\frac{2ap}{a^2-p^2} = \tan 2\psi$, and $\tan\theta = \frac{4ap(a^2-p^2)}{(a^2-p^2)^2-4a^2p^2}$ $= \frac{2\tan 2\psi}{1-\tan^2 2\psi} = \tan 4\psi$. So we have

$$\theta = 4\psi \bmod(\pi) = 4\arctan\sqrt{\lambda - 11} + r(\lambda)\pi,$$

where $r(\lambda) \in \mathbb{Z}$.

Now, it is necessary to determine the value of $r(\lambda)$. If λ is near 11, we have $r(\lambda) = 0$, and $r(\lambda)$ may be discontinue at the value where $\tan\theta = \frac{4ap(a^2-p^2)}{(a^2-p^2)^2-4a^2p^2} = \frac{4(12-\lambda)\sqrt{\lambda-11}}{\lambda^2-28\lambda+188}$ is discontinue, that is for $\lambda_1 = 14 - 2\sqrt{2}$ and $\lambda_2 = 14 + 2\sqrt{2}$.

An integer n is a minimal period if $e^{in\theta} = 1$, that is $n\theta = 2\pi s$ with $\gcd(n,s) = 1$. So $\frac{2}{\pi}\arctan\sqrt{\lambda - 11} = \frac{s}{n}$. We test first $s = 1$. We have then $\lambda = 11 + \tan^2\frac{\pi}{2n}$, which must be less than $14 - 2\sqrt{2}$, that is $n > \frac{\pi}{2\arctan(\sqrt{2}-1)} = 4$, and $n \geq 5$ is good.

So we have the following result:

Proposition 4.1. *Every integer $n \geq 5$ is the period of T on the curve $\mathcal{C}_{\lambda_n}$ for $\lambda_n = 11 + \tan^2\frac{\pi}{2n}$.*

We have the exact values for the number θ:

$$\sin\theta = \frac{4\sqrt{\lambda - 11}(12-\lambda)}{(\lambda - 10)^2}, \quad \cos\theta = \frac{\lambda^2 - 28\lambda + 188}{(\lambda - 10)^2}. \tag{4.13}$$

So for $\lambda > 11$ these two numbers are continuous; then it is easy to prove the following result

Proposition 4.2. *The points M_0 whose orbit is periodic are dense in the region $\{G_4 > 11\}$. Th points M_0 whose orbit is dense in the curve $\mathcal{C}_\lambda$ which passes to M_0 are also dense in $\{G_4 > 11\}$.*

In fact, the set of the first kind of points is the union of disjoint curves $\mathcal{C}_\lambda$, and the same is true for the set of the second kind of points.

4.3. **Study of the dynamical system associated to the QRT-family (5)**

The method is exactly the same as for the family (4) : we put $x = \frac{1+2X}{Y}$ and $y = \frac{1+2Y}{X}$. We obtain the double line with equation $(2X+2Y+1)^2 = 0$ and a pencil of conics with equations $(\lambda+24)(X^2+Y^2) - 2(\lambda+14)XY + 5(X+Y) - 1 = 0$, whose base is the double diagonal $(X-Y)^2 = 0$ (for $\lambda = \infty$) and the two lines $X+Y = -\frac{1}{2} \pm \frac{3\sqrt{5}}{10}$ (for $\lambda = -19$). And the pencils of horizontal and vertical lines become the pencils of lines passing through respectively the two points $(-\frac{1}{2}, 0)$ and

$(0,-\frac{1}{2})$. So by a change of coordinates we use once more lemma 4.1, and obtain exactly the theorem 1.3, with the critical value of λ, that is -19, corresponding to the conic defined by the two parallel lines. We leave the analogous results for family (5) to the reader.

In the following figure we see the region where $G_5 < -19$.

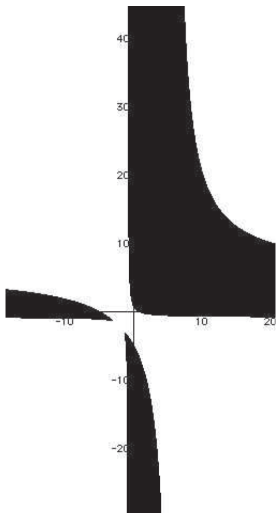

Figure 5 The region where $G_5 < -19$

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