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RESULTS ON THE BETA FUNCTION

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1. Introduction

The Beta function is usually defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

for $x, y > 0$, see for example Sneddon [4]. It then follows that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and this equation is used to define $B(x,y)$ for $x, y < 0$ and $x, y \neq -1, -2, \ldots$.

It was proved in [2] that

$$B(x,y) = \operatorname*{N-lim}_{\epsilon\to 0} \int_\epsilon^{1-\epsilon} t^{x-1}(1-t)^{y-1}\,dt$$

for $x, y \neq 0, -1, -2, \ldots$, where N is the neutrix, see van der Corput [1], having the domain $N' = \{\epsilon : 0 < \epsilon < \frac{1}{2}\}$ with negligible functions finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1}\epsilon, \quad \ln^r \epsilon \quad (\lambda < 0, \quad r = 1, 2, \ldots)$$

and all functions of ϵ which converge to zero in the usual sense as ϵ tends to zero.

Note that if a function $F(\epsilon) = \nu(\epsilon) + f(\epsilon)$, where $\nu(\epsilon)$ is the sum of the divergent negligible functions of $F(\epsilon)$, then p.f. $F(\epsilon)$, Hadamard's finite part of $F(\epsilon)$, is equal to $f(\epsilon)$ and so

$$\operatorname*{N-lim}_{\epsilon\to 0} F(\epsilon) = \lim_{\epsilon\to 0} f(\epsilon) = \lim_{\epsilon\to 0} \text{p.f.}\, F(\epsilon).$$

Thus, taking the neutrix limit of a function $F_n(\epsilon)$ as ϵ tends to 0 is equivalent to taking the normal limit of the function p.f. $F(\epsilon)$ as ϵ tends to 0.

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It was proved that this neutrix limit for $B(x,y)$ exists for all x,y and so was used to define $B(x,y)$ for all x,y. Note that if $x>0$, we could write

$$B(x,y) = \operatorname*{N-lim}_{\epsilon\to 0}\int_0^{1-\epsilon} t^{x-1}(1-t)^{y-1}\,dt$$

and if $y>0$, we could write

$$B(x,y) = \operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^{1} t^{x-1}(1-t)^{y-1}\,dt.$$

For example, it was proved in [2] that

$$B(0,0) = B(0,1) = 0 \tag{1}$$

and

$$B(0,r) = \sum_{i=1}^{r-1}\binom{r-1}{i}\frac{(-1)^i}{i+1} \tag{2}$$

for $r = 1,2,\ldots$.

More generally, it was proved in [3] that if

$$B_{p,q}(x,y) = \frac{\partial^{p+q}}{\partial x^p \partial x^q}B(x,y),$$

then

$$B_{p,q}(x,y) = \operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} t^{x-1}\ln^p t(1-t)^{y-1}\ln^q(1-t)\,dt$$

for $x,y \neq 0,-1,-2,\ldots$. It was proved that this neutrix limit exists for all x,y and p,q and so was used to define $B(x,y)$ for all x,y and $p,q = 0,-1,-2,\ldots$.

The following theorem was proved in [3].

Theorem 1.

$$B_{p,q}(\lambda,\mu) = B_{q,p}(\mu,\lambda)$$

for $p,q = 0,1,2,\ldots$ and all λ,μ.

The following results were also proved in [3].

$$B_{p,0}(0,1) = 0: \quad p = 1,2,\ldots. \tag{3}$$

$$B_{p,0}(0,0) = B_{p,0}(1,0) = (-1)^p p!\zeta(p+1): \quad p = 1,2,\ldots, \tag{4}$$

where ζ denotes the zeta function.

$$B_{p,0}(0,r+1) = \sum_{i=1}^{r}\binom{r}{i}\frac{(-1)^{p+i}p!}{i^{p+i}}: \quad p,r = 1,2,\ldots. \tag{5}$$

$$B_{p,0}(-1,0) = -p! + (-1)^p\zeta(p+1): \quad p = 1,2,\ldots. \tag{6}$$

$$B_{p,0}(-r-1,1) = -\frac{p!}{(r+1)^{p+1}}: \quad p,r = 0,1,2,\ldots. \tag{7}$$

2. Main results

We now prove the following generalization of equation (6).

Theorem 2.

$$B_{p,0}(-r,0) = -\sum_{i=0}^{r-1} \frac{p!}{(r-i)^{p+1}} + (-1)^p p!\zeta(p+1), \tag{8}$$

for $p, r = 1, 2, \ldots$.

Proof. We have

$$\int_\epsilon^{1-\epsilon} t^{-r-1} \ln^p t(1-t)^{-1}\, dt = \int_\epsilon^{1-\epsilon} t^{-r} \ln^p t[t^{-1} + (1-t)^{-1}]\, dt$$

and it follows that

$$\begin{aligned} B_{p,0}(-r,0) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_\epsilon^{1-\epsilon} t^{-r-1} \ln^p t(1-t)^{-1}\, dt \\ &= B_{p,0}(-r,1) + B_{p,0}(-r+1,0) \\ &= -\frac{p!}{r^{p+1}} + B_{p,0}(-r+1,0), \end{aligned} \tag{9}$$

on using equation (7) for $r, p = 1, 2, \ldots$. □

Now assume that equation (8) holds for some r and $p = 1, 2, \ldots$. This is true when $r = 1$ and $p = 1, 2, \ldots$ by equation (4). Then using equation (9) and our assumption, we have

$$\begin{aligned} B_{p,0}(-r-1,0) &= -\frac{p!}{(r+1)^{p+1}} - \sum_{i=0}^{r-1} \frac{p!}{(r-i)^{p+1}} + (-1)^p p!\zeta(p+1) \\ &= -\sum_{i=0}^{r} \frac{p!}{(r-i)^{p+1}} + (-1)^p p!\zeta(p+1) \end{aligned}$$

and so equation (8) holds for $r+1$. Equation (8) now follows by induction.

Theorem 3.

$$B_{p+1,0}(2,-1) = (-1)^{p+1}(p+1)![\zeta(p+2) - \zeta(p+1)] \tag{10}$$

for $p = 1, 2, \ldots$ *and*

$$B_{1,0}(2,-1) = \zeta(2) - 1. \tag{11}$$

Proof. We have

$$\begin{aligned}\int_\epsilon^{1-\epsilon} \ln^p t(1-t)^{-1}\,dt &= \frac{1}{p+1}\int_\epsilon^{1-\epsilon} t(1-t)^{-1}\,d\ln^{p+1} t \\ &= \frac{1}{p+1}[(1-\epsilon)\epsilon^{-1}\ln^{p+1}(1-\epsilon) - \epsilon(1-\epsilon)^{-1}\ln^{p+1}\epsilon] \\ &\quad - \frac{1}{p+1}\int_\epsilon^{1-\epsilon}[(1-t)^{-1}\ln^{p+1} t + t(1-t)^{-2}\ln^{p+1} t]\,dt\end{aligned}$$

and it follows that

$$\begin{aligned}B_{p,0}(1,0) &= \operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} \ln^p t(1-t)^{-1}\,dt \\ &= \frac{1}{p+1}\operatorname*{N-lim}_{\epsilon\to 0}[(1-\epsilon)\epsilon^{-1}\ln^{p+1}(1-\epsilon) - \epsilon(1-\epsilon)^{-1}\ln^{p+1}\epsilon] \\ &\quad - \frac{1}{p+1}\operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^{1-\epsilon}[(1-t)^{-1}\ln^{p+1} t + t(1-t)^{-2}\ln^{p+1} t]\,dt \\ &= 0 - \frac{1}{p+1}[B_{p+1,0}(1,0) + B_{p+1,0}(2,-1)],\end{aligned}$$

for $p = 1, 2, \ldots$.

Using equation (4), we now have

$$(-1)^p p!\zeta(p+1) = (-1)^p p!\zeta(p+2) - \frac{1}{p+1}B_{p+1,0}(2,-1)$$

and equation (10) follows for $p = 1, 2, \ldots$.

In the particular case when $p = 0$, we have

$$\begin{aligned}\int_\epsilon^{1-\epsilon}(1-t)^{-1}\,dt &= \int_\epsilon^{1-\epsilon} t(1-t)^{-1}\,d\ln t \\ &= [(1-\epsilon)\epsilon^{-1}\ln(1-\epsilon) - \epsilon(1-\epsilon)^{-1}\ln\epsilon] \\ &\quad - \int_\epsilon^{1-\epsilon}[(1-t)^{-1}\ln t + t(1-t)^{-2}\ln t]\,dt\end{aligned}$$

and it follows that

$$\begin{aligned}B(1,0) &= \operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^{1-\epsilon}\ln t(1-t)^{-1}\,dt \\ &= \operatorname*{N-lim}_{\epsilon\to 0}[(1-\epsilon)\epsilon^{-1}\ln(1-\epsilon) - \epsilon(1-\epsilon)^{-1}\ln\epsilon] \\ &\quad - \operatorname*{N-lim}_{\epsilon\to 0}\int_\epsilon^{1-\epsilon}[(1-t)^{-1}\ln t + t(1-t)^{-2}\ln t]\,dt \\ &= -1 - [B_{1,0}(1,0) + B_{1,0}(2,-1)] \\ &= -1 + \zeta(2) - B_{1,0}(2,-1) = 0\end{aligned}$$

on using equations (1) and (4), proving equation (11). □

Theorem 4.

$$B_{p,0}(r,0) = (r-1)B_{p,1}(r-1,1) + pB_{p-1,1}(r-1,1), \tag{12}$$

for $p = 1, 2, \ldots$ *and* $r = 2, 3, \ldots$.

Proof. Note that $B_{p,1}(r-1,1)$ and $B_{p-1,1}(r-1,1)$ are standard forms of the Beta function.

We have

$$\begin{aligned}\int_\epsilon^{1-\epsilon} t^{r-1}\ln^p t(1-t)^{-1}\,dt &= -\int_\epsilon^{1-\epsilon} t^{r-1}\ln^p t\,d\ln(1-t)\\ &= -(1-\epsilon)^{r-1}\ln^p(1-\epsilon)\ln\epsilon + \epsilon^{r-1}\ln^p\epsilon\ln(1-\epsilon)\\ &\quad+\int_\epsilon^{1-\epsilon}[(r-1)t^{r-2}\ln^p t + pt^{r-2}\ln^{p-1} t]\ln(1-t)\,dt\end{aligned}$$

and it follows that

$$\begin{aligned}B_{p,0}(r,0) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} t^{r-1}\ln^p t(1-t)^{-1}\,dt\\ &= (r-1)B_{p,1}(r-1,1) + pB_{p-1,1}(r-1,1),\end{aligned}$$

proving equation (12). □

In the next theorem, the constants $c_{p,r}(i)$ are defined by the expansion

$$(1-\epsilon)^{r-1}\ln^p(1-\epsilon) = \sum_{i=1}^{\infty} c_{p,r}(i)\epsilon^i$$

for $p, r = 1, 2, \ldots$. In particular

$$c_{p,r}(s) = \begin{cases} 0, & s < p,\\ (-1)^s, & s = p,\\ r-s, & s = p+1.\end{cases} \tag{13}$$

Theorem 5.

$$B_{p,0}(r,-s) = (r-1)B_{p,0}(r-1,-s+1) + pB_{p-1,0}(r-1,-s+1) + \frac{c_{p,r}(s)}{s}, \tag{14}$$

for $p, s = 1, 2, \ldots$ *and* $r = 2, 3, \ldots$.

In particular

$$B_{p,0}(r,-s) = (r-1)B_{p,0}(r-1,-s+1) + pB_{p-1,0}(r-1,-s+1), \tag{15}$$

for $s = 1, 2, \ldots, p-1$ *and* $p, r = 2, 3, \ldots$,

$$B_{s,0}(r,-s) = (r-1)B_{s,0}(r-1,-s+1) + sB_{s-1,0}(r-1,-s+1) + \frac{(-1)^s}{s}, \tag{16}$$

for $s = 1, 2, \dots$ *and* $r = 2, 3, \dots$ *and*

$$B_{s-1,0}(r, -s) = (r-1)B_{s-1,0}(r-1, -s+1) + (s-1)B_{s-2,0}(r-1, -s+1) - \frac{(-1)^s(r-s)}{s}, \quad (17)$$

for $s = 2, 3, \dots$ *and* $r = 2, 3, \dots$.

Proof. We have

$$\int_\epsilon^{1-\epsilon} t^{r-1} \ln^p t (1-t)^{-s-1}\, dt = \frac{1}{s}\int_\epsilon^{1-\epsilon} t^{r-1} \ln^p t\, d(1-t)^{-s}$$
$$= \frac{(1-\epsilon)^{r-1}\ln^p(1-\epsilon)}{s\epsilon^s} + \frac{\epsilon^{r-1}\ln^p \epsilon}{s(1-\epsilon)^s} + \int_\epsilon^{1-\epsilon} [(r-1)t^{r-2}\ln^p t + pt^{r-2}\ln^{p-1} t](1-t)^{-s}\, dt$$

and it follows that

$$B_{p,0}(r, -s) = \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} t^{r-1}\ln^p t(1-t)^{-s-1}\, dt$$
$$= \frac{c_{p,r}(s)}{s} + 0 + (r-1)B_{p,0}(r-1, -s+1) + pB_{p-1,0}(r-1, -s+1),$$

proving equation (14).

Equations follow (15) to (17) on using equation (13). □

Corollary 5.1. *$B_{p,0}(r, -s)$ is a linear sum of the standard forms of the Beta function $B_{i,0}(j, 1)$ for $s = 1, 2, \dots$ and $p, r = s+2, s+3, \dots$.*

In particular,

$$B_{p,0}(r, -1) = (r-1)(r-2)B_{p,0}(r-2, 1) + p(2r-3)B_{p-1,0}(r-2, 1) + p(p-1)B_{p-2,0}(r-2, 1) \quad (18)$$

Proof. Equation (12) shows that the Corollary is true when $s = 0$.

Now assume the Corollary is true for some positive integer s. We then have from equation (14) on noting that $c_{p,r}(s) = 0$, since $p > s$,

$$B_{p,0}(r, -s-1) = (r-1)B_{p,0}(r-1, -s) + pB_{p-1,0}(r-1, -s)$$
$$= (r-1)[B_{p,0}(r-1, -s+1) + pB_{p-1,0}(r-1, -s+1)] + p[(r-1)B_{p-1,0}(r-1, -s+1) + (p-1)pB_{p-2,0}(r-1, -s+1)]$$

and it follows that the Corollary holds for $s+1$. The result now follows by induction.

When $s = 1$, we have from equation (15)

$$\begin{aligned} B_{p,0}(r,-1) &= (r-1)B_{p,0}(r-1,0) + pB_{p-1,0}(r-1,0) \\ &= (r-1)[(r-2)B_{p,0}(r-2,1) + pB_{p-1,0}(r-2,1)] \\ &\quad + p[(r-2)B_{p-1,0}(r-2,1) + (p-1)B_{p-2,0}(r-2,1)] \end{aligned}$$

and equation (18) follows. □

Theorem 6.

$$B_{p,1}(0,1) = (-1)^{p+1}p!\zeta(p+2), \tag{19}$$

for $p = 1, 2, \ldots$.

Proof. We have

$$\begin{aligned} \int_\epsilon^{1-\epsilon} t^{-1}\ln^p t \ln(1-t)\,dt &= \frac{1}{p+1}\int_\epsilon^{1-\epsilon} \ln(1-t)\,d\ln^{p+1} t \\ &= \frac{1}{p+1}\ln^{p+1}(1-\epsilon)\ln\epsilon - \ln^{p+1}\epsilon\ln(1-\epsilon) \\ &\quad + \frac{1}{p+1}\int_\epsilon^{1-\epsilon}(1-t)^{-1}\ln^{p+1} t\,dt \end{aligned}$$

and it follows that

$$\begin{aligned} B_{p,1}(0,1) &= \mathrm{N}\!-\!\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} t^{-1}\ln^p t\ln(1-t)\,dt \\ &= 0 + \frac{1}{p+1}B_{p+1,0}(1,0), \end{aligned}$$

and equation (19) follows on using equation (4). □

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