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SOLVABILITY OF A SYSTEM OF NONCONVEX GENERAL VARIATIONAL INEQUALITIES

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Abstract. Using the prox-regularity notion, we introduce a system of nonconvex general variational inequalities and a general three step algorithm for approximate solvability of this system. We establish the convergence of three-step projection method for a general system of nonconvex variational inequality problem. We obtain as a particular case some known results.

1. Introduction and preliminaries

In 1988, Noor [4] introduced and studied general variational inequality. It was a significant generalization of the variational inequalities, which was introduced and studied by Stampacchia [10] in 1964. In the recent years, much attention has been given to study the system of variational inequalities, which plays a significant role in the interdisciplinary research between different branches of mathematics, biomedical sciences and mathematical physics. In almost all the results regarding the existence and iterative solvability of variational inequalities and related optimizations problems the set considered is convex. Moreover, all the techniques are based on projection method which was mainly due to Sibony [9]. The projection method may not hold in general, when the sets are nonconvex. For application point of view, getting convex sets is itself a difficult problem. To overcome this difficulty Bounkhel et al. [1] considered uniform prox-regular sets and studied iterative schemes to solve nonconvex variational problems. Noor [7] introduced a new class of variational inequality called the general nonconvex variational inequality. The prox-regular sets are nonconvex and include the convex sets as special cases; (see [2]). It was also found that the two-step and three-step iteration method performs better than the one-step method, see [5, 6].

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The basic idea behind the methods for approximation solvability of the system of variational inequalities has arisen from the equivalence between the variational inequality problem and fixed point problem. Using this equivalence several projection iterative methods have been developed for solving variational inequality problems. Motivated and inspired by the research going on in this area, in this paper we study a system of nonconvex general variational inequalities (SNGVI), we also propose a three step projection iterative algorithm to solve (SNGVI).

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty subset of H. We denote by $d(.,K)$ the usual distance function to the subset K, i.e. $d(x,K) = \inf_{y\in K} \|x-y\|$. Now we recall some well-known definitions and results of nonlinear convex analysis and nonsmooth analysis.

Definition 1.1. [2] *Let $x \in H$ be a point not lying in K. Let $y \in K$ is a point whose distance to x is minimal, i.e. $d(x,K) = \|x-y\|$, then y is called a closest point or a projection of x onto K. The set of all such closest points is denoted by* $\mathrm{proj}_K(x)$*; that is,*

$$\mathrm{proj}_K(x) = \{y \in K \,:\, d(x,K) = \|x-y\|\} \,.$$

Also, $y \in \mathrm{proj}_K(x)$ if and only if $\{y\} \subset K \cap \bar{B}\{x; \|x-y\|\}$ and $K \cap B\{x; \|x-y\|\} = \emptyset$. The vector $x-y$ is called a proximal normal direction to K at y. Any nonnegative multiple $z = \alpha(x-y)$, $\alpha \geq 0$ of such a vector is called a proximal normal to K at x. The set of all z obtainable in this manner is called the proximal normal cone to K at x and is denoted by $N_K^p(x)$.

Definition 1.2. [8] *The proximal normal cone to K at $x \in H$ is given by*

$$N_K^p(x) = \{z \in H : \exists\, \alpha > 0\,;\; x \in \mathrm{proj}_K\,(x + \alpha z)\} \,.$$

The proximal normal cone $N_K^p(x)$ has the following characterization.

Lemma 1.3. [8, Proposition 1.5] *Let K be a nonempty subset of H. Then a vector $z \in N_K^p(x)$ if and only if there exists a constant $\alpha = \alpha(z,x) \geq 0$ such that*

$$\langle z, y-x \rangle \;\leq\; \alpha\, \|y-x\|^2 \,,\; \forall\, y \in K \,.$$

Clarke et al. [2] and Poliquin et al. [8] introduced and studied a new class of nonconvex sets called uniformly prox-regular sets.

Definition 1.4. *For a given $r \in (0,\infty]$, a subset K of H is said to be uniformly prox-regular with respect to r (or r-uniformly prox-regular) if and only if, for all $x \in K$ and for all $0 \neq z \in N_K^p(x)$, one has*

$$\left\langle \frac{z}{\|z\|}\,, y-x \right\rangle \;\leq \frac{1}{2r}\, \|y-x\|^2 \quad \textit{for all } y \in K \,.$$

We use the convention that $\frac{1}{r} = 0$ when $r = +\infty$.
A closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in view of Definition 1.4, for the case of $r = \infty$, the notion of uniform prox-regularity and convexity of K coincide. It is known that the class of uniformly prox-regular set is sufficiently large to include the class of convex sets, $p-$convex sets, $C^{1,1}$ submanifolds of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets.

Now recall the well known proposition which summarizes some important properties of the uniformly prox-regularity.

Lemma 1.5. *Let K be nonempty closed subset of H, $r \in (0, \infty]$ and $K_r = \{x \in H : d(x, K) < r\}$. If K is uniformly r-prox-regular, then the following holds:*

(i) *For all $x \in K_r$, set* $\mathrm{proj}_K(x) \neq \emptyset$.
(ii) *For all $s \in (0, r)$,* proj_K *is Lipschitz continuous with constant $\frac{r}{r-s}$ on K_s.*
(iii) *The proximal normal cone $N_K^p(x)$ is closed as a set valued mapping.*

Let H be a Hilbert space and K a closed convex subset of H. Consider the following problem:

Find $x^*, y^*, z^* \in K$ such that $g(x^*), g(y^*), g(z^*) \in K$ and

$$\left.\begin{aligned} \langle \rho T_1(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0\,, \\ \langle \eta T_2(z^*) + g(y^*) - g(z^*), x - g(y^*)\rangle \geq 0\,, \\ \langle \sigma T_3(x^*) + g(z^*) - g(x^*), x - g(z^*)\rangle \geq 0, \end{aligned}\right\} \tag{1.1}$$

for all $x \in K$, where $g : H \to H$ be a given mapping, $T_1, T_2, T_3 : K \to H$ are nonlinear operators and ρ, η, σ are nonnegative real numbers. We call problem (1.1) as system of general variational inequalities (SGVI).

By the definition of the normal cone, we now reformulate (SGVI) as follows:

$$\left.\begin{aligned} 0 \in \rho T_1(y^*) + g(x^*) - g(y^*) + N_K\left(g(x^*)\right)\,, \\ 0 \in \eta T_2(z^*) + g(y^*) - g(z^*) + N_K\left(g(y^*)\right)\,, \\ 0 \in \sigma T_3(x^*) + g(z^*) - g(x^*) + N_K\left(g(z^*)\right)\,. \end{aligned}\right\} \tag{1.2}$$

By replacing the usual normal cone by proximal normal cone, we now introduce the generalized version of problem (1.2) which we call system of nonconvex general variational inequalities (SNGVI).

Let H be a Hilbert space and K a uniformly r-prox-regular subset of H. We will consider the following problem (SNGVI):

Find $x^*, y^*, z^* \in K$ such that $g(x^*), g(y^*), g(z^*) \in K$ and

$$0 \in \rho T_1(y^*) + g(x^*) - g(y^*) + N_K^p\left(g(x^*)\right) , \tag{1.3}$$

$$0 \in \eta T_2(z^*) + g(y^*) - g(z^*) + N_K^p\left(g(y^*)\right) , \tag{1.4}$$

$$0 \in \sigma T_3(x^*) + g(z^*) - g(x^*) + N_K^p\left(g(z^*)\right) . \tag{1.5}$$

Lemma 1.6. *Let K a uniformly r-prox-regular subset of H, then $x^*, y^*, z^* \in H$ with $g(x^*)$, $g(y^*), g(z^*) \in K$ is a solution of (SNGVI) if and only if*

$$\begin{aligned} g(x^*) &= \mathrm{proj}_K(g(y^*) - \rho T_1(y^*)) \\ g(y^*) &= \mathrm{proj}_K(g(z^*) - \eta T_2(z^*)) \\ g(z^*) &= \mathrm{proj}_K(g(x^*) - \sigma T_3(x^*)) \end{aligned}$$

provided that

$$0 < \rho \leq \frac{s}{1 + \|T_1(y^*)\|}, \quad 0 < \eta \leq \frac{s}{1 + \|T_2(z^*)\|},$$
$$0 < \sigma \leq \frac{s}{1 + \|T_3(x^*)\|}, \quad s \in (0, r) .$$

Proof. Using (1.3), and the fact that $\mathrm{proj}_K = \left(I + N_K^p\right)^{-1}$, we have

$$\begin{aligned} &0 \in \rho T_1(y^*) + g(x^*) - g(y^*) + N_K^p\left(g(x^*)\right) \\ &\Leftrightarrow g(y^*) - \rho T_1(y^*) \in g(x^*) + N_K^p\left(g(x^*)\right) = \left(I + N_K^p\right)\left(g(x^*)\right) \\ &\Leftrightarrow g(x^*) = \mathrm{proj}_K(g(y^*) - \rho T_1(y^*)) , \end{aligned}$$

where I is the identity mapping.

Similarly, using (1.4) and (1.5), we have

$$\begin{aligned} g(y^*) &= \mathrm{proj}_K(g(z^*) - \eta T_2(z^*)) \\ g(z^*) &= \mathrm{proj}_K(g(x^*) - \sigma T_3(x^*)) . \end{aligned}$$

This completes the proof. □

Lemma 1.6 implies that (SNGVI) is equivalent to the fixed point problem. This alternative equivalent formulation is very useful from the numerical point of view. This fixed point formulation suggests the following iteration method to solve (SNGVI)

$$\left.\begin{aligned} g(z_k) &= \mathrm{proj}_K(g(x_k) - \sigma T_3(x_k)) , \\ g(y_k) &= \mathrm{proj}_K(g(z_k) - \eta T_2(z_k)) , \\ g(x_{k+1}) &= \mathrm{proj}_K(g(y_k) - \rho T_1(y_k)) . \end{aligned}\right\} \tag{1.6}$$

where ρ, η, σ are positive reals, satisfying certain conditions.

2. Main results

We now recall some definition, which will be used in the main result:

Definition 2.1. *An operator $T : H \to H$ with respect to an arbitrary operator g is said to be :*

(i) (g, t) *strongly monotone if there exists a constant $t > 0$ such that*

$$\langle T(x) - T(y), g(x) - g(y)\rangle \geq t \|g(x) - g(y)\|^2 , \quad \textit{for all } x, y \in H .$$

(ii) (g, μ) *Lipschitz continuous if there exists a constant $\mu > 0$ such that*

$$\|T(x) - T(y)\| \leq \mu \|g(x) - g(y)\| , \quad \textit{for all } x, y \in H .$$

We now present, a result for the approximation-solvability of the (SNGVI) problem using algorithm 1.6. In what follows we assume that K is a uniformly r-prox-regular subset of H with $r > 0$, also let $s \in (0, r)$ and set $\delta = \frac{r}{r-s}$.

Theorem 2.2. *Assume that $g : H \to H$ be a homeomorphism and $T_i : K \to H$ be (g, t_i) strongly monotone and (g, μ_i) Lipschitz continuous mappings satisfying $t_i\delta > \mu_i\sqrt{\delta^2 - 1}$, for $i = 1, 2, 3$. Suppose that $x^*, y^*, z^* \in K$ form a solution to (SNGVI), then the sequence $\{x_k, y_k, z_k\}$ generated by* (1.6) *strongly converges to (x^*, y^*, z^*), provided that the following conditions are satisfied:*

$$\frac{t_1}{\mu_1^2} - \Delta_1 \leq \rho \leq \min\left\{\frac{t_1}{\mu_1^2} + \Delta_1,\ \frac{s}{1 + \|T_1(y_n)\|},\ \frac{s}{1 + \|T_1(y^*)\|}\right\} ,$$
$$\frac{t_2}{\mu_2^2} - \Delta_2 \leq \eta \leq \min\left\{\frac{t_2}{\mu_2^2} + \Delta_2,\ \frac{s}{1 + \|T_2(z_n)\|},\ \frac{s}{1 + \|T_2(z^*)\|}\right\} ,$$
$$\frac{t_3}{\mu_3^2} - \Delta_3 \leq \sigma \leq \min\left\{\frac{t_3}{\mu_3^2} + \Delta_3,\ \frac{s}{1 + \|T_3(x_n)\|},\ \frac{s}{1 + \|T_3(x^*)\|}\right\} ,$$

where $\Delta_i = \frac{\sqrt{\delta^2 t_i^2 - \mu_i^2(\delta^2 - 1)}}{\delta\mu_i^2}$, *for* $i = 1, 2, 3$.

Proof. Since (x^*, y^*, z^*) is a solution of (SNGVI), from the conditions on the parameters ρ, η and σ, we have

$$g(x^*) = \mathrm{proj}_K(g(y^*) - \rho T_1(y^*)) ,$$
$$g(y^*) = \mathrm{proj}_K(g(z^*) - \eta T_2(z^*)) , \text{ and}$$
$$g(z^*) = \mathrm{proj}_K(g(x^*) - \sigma T_3(x^*)) .$$

Using (1.6), we can write

$$\begin{aligned}\|g(x_{k+1}) - g(x^*)\| &= \|\,\mathrm{proj}_K(g(y_k) - \rho T_1(y_k)) - \mathrm{proj}_K(g(y^*) - \rho T_1(y^*))\| \\ &\leq \delta \|(g(y_k) - \rho T_1(y_k)) - (g(y^*) - \rho T_1(y^*))\| . \qquad (2.1)\end{aligned}$$

Because of choice of ρ we have $g(y_k) - \rho T_1(y_k)$ and $g(y^*) - \rho T_1(y^*)$ belongs to K_s. Since T_1 is (g, t_1) strongly monotone and (g, μ_1) Lipschitzian, we have

$$\begin{aligned}
\|g(y_k) &- g(y^*) - \rho\,(T_1(y_k) - T_1(y^*))\|^2 \\
&= \|g(y_k) - g(y^*)\|^2 - 2\rho\,\langle T_1(y_k) - T_1(y^*), g(y_k) - g(y^*)\rangle \\
&\qquad + \rho^2\,\|T_1(y_k) - T_1(y^*)\|^2 \\
&\le \|g(y_k) - g(y^*)\|^2 - 2\rho t_1\,\|g(y_k) - g(y^*)\|^2 + \rho^2\mu_1^2\,\|g(y_k) - g(y^*)\|^2 \\
&= \left(1 - 2\rho t_1 + \rho^2\mu_1^2\right)\|g(y_k) - g(y^*)\|^2 \qquad (2.2)
\end{aligned}$$

By (2.1) and (2.2), we have

$$\|g\,(x_{k+1}) - g\,(x^*)\| \le \delta\theta_1\,\|g(y_k) - g(y^*)\|\ , \qquad (2.3)$$

where $\theta_1 = \sqrt{1 - 2\rho t_1 + \rho^2\mu_1^2}$.

Since T_2 is (g, t_2) strongly monotone and (g, μ_2) Lipschitzian, we have

$$\|g\,(y_k) - g\,(y^*)\| \le \delta\theta_2\,\|g(z_k) - g(z^*)\|\ , \qquad (2.4)$$

where $\theta_2 = \sqrt{1 - 2\eta t_2 + \eta^2\mu_2^2}$.

Similarly, we have

$$\|g\,(z_k) - g\,(z^*)\| \le \delta\theta_3\,\|g(x_k) - g(x^*)\|\ , \qquad (2.5)$$

where $\theta_3 = \sqrt{1 - 2\sigma t_3 + \sigma^2\mu_3^2}$ and $\delta\theta_3 \le 1$.

Substituting (2.5) into (2.4), we have

$$\|g\,(y_k) - g\,(y^*)\| \le \delta\theta_2\delta\theta_3\,\|g(x_k) - g(x^*)\|\ . \qquad (2.6)$$

Combining (2.3), and (2.6), we get

$$\|g\,(x_{k+1}) - g\,(x^*)\| \le \varrho\,\|g(x_k) - g(x^*)\| \ \le \cdots \le \varrho^k\,\|g(x_1) - g(x^*)\|\ . \quad (2.7)$$

where $\varrho = \delta\theta_1\delta\theta_2\delta\theta_3 < 1$, since $\delta\theta_i < 1$, $i = 1, 2, 3$. It follows from (2.7) that

$$\lim_{k\to\infty}\|g\,(x_k) - g\,(x^*)\| = 0\,. \qquad (2.8)$$

Combining (2.5) and (2.8), we have

$$\lim_{k\to\infty}\|g\,(z_k) - g\,(z^*)\| = 0\,. \qquad (2.9)$$

Also, by combining (2.6) and (2.8), we have

$$\lim_{k\to\infty}\|g\,(y_k) - g\,(y^*)\| = 0\,. \qquad (2.10)$$

Since g is invertible, it follows form (2.8), (2.9) and (2.10) that

$$\lim_{k\to\infty} x_k = x^*\,, \quad \lim_{k\to\infty} y_k = y^*\,, \lim_{k\to\infty} z_k = z^*\,,$$

satisfying the (SNGVI).

This completes the proof. □

For $z^* = x^*$ and $\eta = \sigma$ and $T_2 = T_3$, the (SGVI) reduces to the following system of variational inequality problem:

Find $x^*, y^* \in K$ such that $g(x^*), g(y^*) \in K$ and

$$\begin{aligned} &\langle \rho T_1(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0\,, \quad \text{for all } x \in K,\\ &\langle \eta T_2(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0\,, \quad \text{for all } x \in K, \end{aligned} \tag{2.11}$$

where $g : H \to H$ be a given mapping, $T_1, T_2 : K \to H$ are nonlinear operators and ρ, η are nonnegative real numbers.

System (2.11) appears to be new one. If we take $T_1 = T_2$ and $g =$ identity mapping in the system (2.11), then we have following system of variational inequality: Find $x^*, y^* \in H$ such that

$$\begin{aligned} &\langle \rho T(y^*) + x^* - y^*, x - x^*\rangle \geq 0\,, \quad \text{for all } x \in K\,,\\ &\langle \eta T(x^*) + y^* - x^*, x - y^*\rangle \geq 0\,, \quad \text{for all } x \in K\,, \end{aligned} \tag{2.12}$$

where $T : K \to H$ be a nonlinear operators and ρ, η are nonnegative real numbers.

System (2.12) was studied by Moudafi [3]. Theorem 2.1 of [3] is a special case of Theorem 2.2. Now consider the particular case where $r = +\infty$, we have $\delta = 1$ and we can recover Theorem 3.1 of Verma [12] and Theorem 2.1 of Verma [11] from Theorem 2.2.

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