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B. Y. CHEN'S INEQUALITIES FOR BI-SLANT SUBMANIFOLDS IN COSYMPLECTIC SPACE FORMS

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ABSTRACT. In this paper we obtain B. Y. Chen's inequalities for a bi-slant submanifold M of a cosymplectic space form $\overline{M}$(c), when the structure vector field ξ of the ambient space is tangent to M.

1. Introduction

In the theory of Riemannian submanifolds it is quite interesting to establish a relationship between the intrinsic and extrinsic invariants. Basically, the Riemannian invariants are intrinsic characteristics of Riemannian manifolds. In 1993, B. Y. Chen [6] has obtained an inequality between sectional curvature K, the scalar curvature τ (intrinsic invariant) and the mean curvature function $||H||$ (extrinsic invariant) of a submanifold M of the real space form of constant curvature c. Moreover, Chen [4] also introduced a new type of Riemannian invariants of a Riemannian manifold.

Let M be a Riemannian manifold of dimension m and let $\{e_1, e_2, \ldots, e_m\}$ be any orthonormal basis of the tangent space T_pM at any point $p \in M$. then the scalar curvature τ at $p \in M$ is given by

$$\tau = \sum_{1 \le i < j \le m} K(e_i \wedge e_j) \tag{1.1}$$

for any point $p \in M$, we denote

$$(\inf K)(p) = \inf\{K(\pi) : \pi \subset T_pM, \dim \pi = 2\} \tag{1.2}$$

where $K(\pi)$ denotes the sectional curvature of M associated with a plane section $\pi \subset T_pM$ at $p \in M$.

The Chen invariant δ_M at any point $p \in M$ is defined as

$$\delta_M(p) = \tau(p) - (\inf K)(p). \tag{1.3}$$

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For a submanifold M of a real space form $\overline{M}(c)$, Chen has given a basic inequality in terms of the intrinsic invariant δ_M and the squared mean curvature of the immersion, as

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)}||H||^2 + \frac{1}{2}(m+1)(m-2)c. \tag{1.4}$$

The above inequality also holds good in case M is an anti-invariant submanifold of complex space form $\overline{M}(c)$[7]. In case of contact manifold, Defever, Mihai and Verstralen [11] obtained an inequality similar to that of (1.4), for C-totally real submanifold of a Sasakian space form with constant φ-sectional curvature c, given by

$$\delta_M \leq \frac{m^2(m-2)}{2(m-1)}||H||^2 - \frac{1}{2}(m+1)(m-2)\frac{c+3}{4}. \tag{1.5}$$

2. Preliminaries

A (2m+1)-dimensional Riemannian manifold $\overline{M}$ is said to be an almost contact metric manifold if there exists structure tensors (ϕ, ξ, η, g), where ϕ is a (1, 1) tensor field, ξ a vector field, η a 1-form and g the Riemannian metric on $\overline{M}$ satisfying [9]

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi)$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An almost contact metric manifold $\overline{M}$ is called a cosymplectic manifold if [13],

$$(\overline{\nabla}_X \phi)Y = 0 \quad \text{and} \quad \overline{\nabla}_X \xi = 0 \tag{2.2}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

The curvature tensor $\overline{R}$ of a cosymplectic space form $\overline{M}(c)$ is given by [14],

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + \eta(X)\eta(Z)Y \\ &- \eta(Y)\eta(Z)X + \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi \\ &- g(\phi X, Z)\phi Y + g(\phi Y, Z)\phi X + 2g(X, \phi Y)\phi Z\}\end{aligned} \tag{2.3}$$

for all $X, Y, Z \in T\overline{M}$.

Now, let$M$ be an m-dimensional isometrically immersed Riemannian submanifold of a cosymplectic manifold $\overline{M}$ with induced metric g. Denoting by

TM the tangent bundle of M and by $T^{\perp}M$ the set of all vector fields normal to M, we write

$$\phi X = PX + FX \tag{2.4}$$

for any $X \in TM$, where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX.

From now on we assume that the structure vector field ξ is tangent to M. We make the direct orthogonal decomposition $TM = D \oplus \xi$.

A submanifold M is said to be slant if for any non zero vector X tangent to M at p such that X is not proportional to ξ_p, the angle $\theta(X)$ between ϕX and T_pM is constant i. e., is independent of the choice of $p \in M$ and $X \in T_pM - \{\xi_p\}$. Sometime the angle $\theta(X)$ is termed as the wirtinger angle of the slant immersion.

Invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a proper slant immersion.

A submanifold M tangent to structure vector field ξ is said to be a bi-slant submanifold of a cosymplectic manifold $\overline{M}$, if there exist two orthogonal differentiable distributions D_1 and D_2 on M, such that

(i) TM possesses an orthogonal direct decomposition of D_1 and D_2 i. e. $TM = D_1 \oplus D_2 \oplus \xi$.
(ii) D_i is slant distribution with slant angle θ_i for any $i = 1, 2$.

If we take the $\dim D_1 = 2n_1$ and $\dim D_2 = 2n_2$, then it is obvious that in case either n_1 vanishes or n_2, the bi-slant submanifold reduces to a slant submanifold. Hence, the bi-slant submanifolds are generalized cases of slant submanifolds. moreover, slant submanifolds, invariant submanifolds and anti-invariant submanifolds are particular cases of bi-slant submanifolds.

Let R and $\overline{R}$ denote the curvature tensors of the submanifold M and cosymplectic space form $\overline{M}(c)$, respectively. Then the equation of Gauss is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,W),h(Y,Z)) + g(h(X,Z),h(Y,W)) \tag{2.5}$$

for all $X, Y, Z, W \in TM$.

We denote by h the second fundamental form of M and by A_N the Weingarten map associated with $N \in T^{\perp}M$. We put

$$h^r_{i,j} = g(h(e_i,e_j),e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{m} g(h(e_i,e_j),h(e_i,e_j)) \tag{2.6}$$

for any $e_i, e_j \in TM$ and $e_r \in T^{\perp}M$.

The mean curvature vector H is defined as $H = \frac{1}{m}(trace\, h)$. We say that the submanifold M is minimal, if the mean curvature vector H vanishes identically. It is well known that for a cosymplectic manifold

$$h(X, \xi) = 0. \tag{2.7}$$

For a given orthonormal frame $\{e_1, e_2, \ldots, e_m\}$ of a differentiable distribution D, we denote the squared norms of P and F respectively, by

$$\|P\|^2 = \sum_{i,j=1}^{m} g^2(e_i, Pe_j) \quad \text{and} \quad \|F\|^2 = \sum_{i=1}^{m} \|Fe_i\|^2. \tag{2.8}$$

It can be readily seen that $\|P\|^2$ and $\|F\|^2$ are independent of the choice of the above orthonormal frame.

For any $i = 1, 2, \ldots, m$ where $\{e_1, e_2, \ldots, e_m, \xi\}$ is a local orthonormal frame, we have

$$\sum_{j=1}^{m} g^2(e_i, \phi e_j) = cos^2\theta. \tag{2.9}$$

A plane section π in a cosymplectic manifold $\overline{M}$ is said to be a ϕ-section, if it is spanned by a unit tangent vector X orthonormal to ξ and ϕX, i. e.

$$K(\pi) = K(X, \phi X) = g(\overline{R}(X, \phi X)\phi X, X). \tag{2.10}$$

The sectional curvature of a ϕ-section is called ϕ-sectional curvature. A cosymplectic manifold $\overline{M}$ with constant ϕ-sectional curvature c is said to be a cosymplectic space form and is usually denoted by $\overline{M}(c)$.

For an orthonormal basis $\{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\}$ of the tangent space T_pM at $p \in M$, from (1.1), the scalar curvature τ at p of M assumes the form

$$2\tau = \sum_{i \neq j}^{m} K(e_i \wedge e_j) + 2\sum_{i=1}^{m} K(e_i \wedge \xi). \tag{2.11}$$

Now, we mention the following results for our subsequent use.

Corollary 2.1. [12] *Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(PX, PY) = cos^2\theta\{g(X, Y) - \eta(X)\eta(Y)\} \tag{2.12}$$

$$g(FX, FY) = sin^2\theta\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{2.13}$$

Lemma 2.1. [6] *Let $a_1, a_2, \ldots, a_k, c$ be $k + 1 \quad (k \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)\left(\sum_{i=1}^{k} a_i^2 + c\right).$$

Then $2a_1a_2 \geq c$ and the equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_k$.

3. Chen's inequality for bi-slant submanifolds in cosymplectic space forms

Theorem 3.1. *Let $\psi : M \to \overline{M}$ be an isometric immersion from a Riemannian $(m+1 = 2n_1 + 2n_2 + 1)$-dimensional bi-slant submanifold M into a cosymplectic space form $\overline{M}(c)$ of dimension $2m+1$. Then, we have*

$$\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2) + \frac{3c}{4}[(n_1-1)cos^2\theta_1 + n_2 cos^2\theta_2] \quad (3.1)$$

on D_1, and

$$\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2) + \frac{3c}{4}[n_1 cos^2\theta_1 + (n_2-1)cos^2\theta_2] \quad (3.2)$$

on D_2.

The equality cases in (3.1) and (3.2) hold at a point $p \in M$ if and only if there exist an orthonormal basis $\{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\}$ of T_pM and an orthonormal basis $\{e_{m+2}, e_{m+3}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that the shape operators of M in $\overline{M}(c)$, at a point p take the following forms

$$A_{m+2} = \begin{pmatrix} a & 0 & 0 & . & . & . & 0 \\ 0 & b & 0 & . & . & . & 0 \\ 0 & 0 & & & \lambda I_{m-1} & & \end{pmatrix}, \quad a+b=\lambda \quad (3.3)$$

$$A_{e_r} = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & . & . & . & 0 \\ h_{12}^r & -h_{11}^r & 0 & . & . & . & 0 \\ 0 & 0 & & & 0_{m-1} & & \end{pmatrix}, \quad r \in \{m+3, \ldots, 2m+1\} \quad (3.4)$$

Proof. Using Gauss equation in the expression of the curvature tensor $\overline{R}$ of cosymplectic space form $\overline{M}(c)$ given by (2.3), we obtain

$$\begin{aligned} R(X,Y,Z,W) &= g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)) \\ &+\frac{c}{4}\Big\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + \eta(X)\eta(Z)g(Y,W) \\ &-\eta(Y)\eta(Z)g(X,W) + \eta(Y)\eta(W)g(X,Z) - \eta(X)\eta(W)g(Y,Z) \\ &-g(\phi X,Z)g(\phi Y,W) + g(\phi Y,Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)\Big\} \quad (3.5) \end{aligned}$$

for any $X, Y, Z, W \in TM$.

For an orthonormal basis $\{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\}$ of T_pM at $p \in M$, putting $X = W = e_i$ and $Y = Z = e_j, \forall i, j \in \{1, \ldots, m+1\}$, in (3.5), we get

$$\begin{aligned}\sum_{i,j=1}^{m+1} R(e_i, e_j, e_j, e_i) &= g(h(e_i, e_i), h(e_j, e_j)) - g(h(e_i, e_j), h(e_j, e_i)) \\ &+\frac{c}{4}\Big\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\Big\} + \frac{c}{4}\Big\{\eta(e_i)\eta(e_j)g(e_j, e_i) \\ &-\eta(e_j)\eta(e_j)g(e_i, e_i) + \eta(e_j)\eta(e_i)g(e_i, e_j) - \eta(e_i)\eta(e_i)g(e_j, e_j) \\ &-g(\phi e_i, e_j)g(\phi e_j, e_i) + g(\phi e_j, e_j)g(\phi e_i, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\Big\}\end{aligned}$$

or,

$$\sum_{i,j=1}^{m+1} R(e_i, e_j, e_j, e_i) = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4}\{(m+1)^2 - (m+1)\}$$

$$+\frac{c}{4}\Big\{1 - (m+1) + 1 - (m+1) + 3\sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j)\Big\}$$

or,

$$\sum_{i\neq j}^{m} R(e_i, e_j, e_j, e_i) + 2\sum_{i=1}^{m} R(e_i, \xi, \xi, e_i) = (m+1)^2\|H\|^2 - \|h\|^2$$

$$+\frac{c}{4}\{(m+1)^2 - (m+1)\} + \frac{c}{4}\Big\{-2m + 3\sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j)\Big\}.$$

Now using (2.11) in the above equation, we get

$$2\tau = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4}m(m+1) + \frac{c}{4}\Big\{-2m + 3\sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j)\Big\}$$

or,

$$2\tau = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4}m(m-1) + 3\frac{c}{4}\sum_{i,j=1}^{m+1} g^2(e_i, \phi e_j). \tag{3.6}$$

Since M^{m+1} is bi-slant submanifold of a cosymplectic space form $\overline{M}^{2m+1}(c)$, where $(m+1) = 2n_1 + 2n_2 + 1$, we may consider an adapted bi-slant orthonormal frame as follows:

$$e_1, e_2 = \sec\theta_1 Pe_1, \ldots, e_{2n_1-1}, e_{2n_1} = \sec\theta_1 Pe_{2n_1-1}$$

$$e_{2n_1+1}, e_{2n_1+2} = \sec\theta_2 Pe_{2n_1+1}, \ldots, e_{2n_1+2n_2-1}, e_{2n_1+2n_2}$$
$$= \sec\theta_2 Pe_{2n_1+2n_2-1} \ \text{ and } \ e_{2n_1+2n_2+1} = \xi.$$

Then, we have

$$g(e_1, \phi e_2) = -g(\phi e_1, e_2) = -g(\phi_1, \sec\theta_1 P e_1)$$

or,

$$g(e_1, \phi e_2) = -\sec\theta_1\, g(Pe_1, Pe_1).$$

Now, using (2.12), we get

$$g(e_1, \phi e_2) = -cos\ \theta_1$$

or,

$$g^2(e_1, \phi e_2) = \cos^2\theta_1.$$

Similarly,

$$g^2(e_i, \phi e_{i+1}) = \begin{cases} \cos^2\theta_1, & \text{for } i = 1, \dots, 2n_1 - 1 \\ \cos^2\theta_2, & \text{for } i = 2n_1 + 1, \dots, 2n_1 + 2n_2 - 1. \end{cases}$$

Hence, we have

$$\sum_{i,\, j=1}^{m+1} g^2(e_i, \phi e_j) = 2\{n_1 \cos^2\theta_1 + n_2 \cos^2\theta_2\}.$$

Using this relation in (3.6), we obtain

$$2\tau = (m+1)^2\|H\|^2 - \|h\|^2 + \frac{c}{4}m(m-1) + \frac{3c}{4}[2(n_1\cos^2\theta_1 + n_2\cos^2\theta_2)]. \quad (3.7)$$

Putting

$$\epsilon = 2\,\tau - \frac{(m+1)^2(m-1)}{m}\|H\|^2 - \frac{c}{4}(m+1)(m-2) - \frac{3c}{2}[n_1\cos^2\theta_1 + n_2\cos^2\theta_2] \quad (3.8)$$

in (3.7), we get

$$\epsilon = \frac{(m+1)^2}{m}\|H\|^2 - \|h\|^2 + \frac{c}{2}$$

or,

$$(m+1)^2\|H\|^2 = m\|h\|^2 + m\Big\{\epsilon - \frac{c}{2}\Big\}. \quad (3.9)$$

Let $p \in M$, $\pi \subset T_pM$, $\dim \pi = 2$ and π is orthogonal to ξ.

Now, we consider the following two cases:

Case (i). Let π be tangent to the differentiable distribution D_1 and let it be spanned by orthonormal basis vectors e_1 and e_2. If we take e_{m+2} in the direction of mean curvature vector H i. e. $e_{m+2} = \frac{H}{\|H\|}$, then from (3.9), we get

$$\Big(\sum_{i=1}^{m+1} h_{i\,i}^{m+2}\Big)^2 = m\Big\{\sum_{i=1}^{m+1}\Big(h_{i\,i}^{m+2}\Big)^2 + \sum_{i\neq j}\Big(h_{i\,j}^{m+2}\Big)^2 + \sum_{r=m+3}^{2m+1}\sum_{i,\,j}\Big(h_{i\,j}^{r}\Big)^2 + \epsilon - \frac{c}{2}\Big\}. \quad (3.10)$$

Now using lemma (2.1) in (3.10), we get

$$2h_{11}^{m+2}h_{22}^{m+2} \geq \sum_{i\neq j}\left(h_{ij}^{m+2}\right)^2 + \sum_{r=m+3}^{2m+1}\sum_{i,\,j}\left(h_{ij}^{r}\right)^2 + \epsilon - \frac{c}{2}. \tag{3.11}$$

On the other hand, we have

$$K(\pi) = R(e_1, e_2, e_2, e_1) = g(h(e_1, e_1), h(e_2, e_2)) - g(h(e_1, e_2), h(e_1, e_2)) + \frac{c}{4} + \frac{3c}{4}\cos^2\theta_1$$

or,

$$K(\pi) = \sum_{r=m+2}^{2m+1}\Big\{g(h(e_1, e_1), e_r)\, g(h(e_2, e_2), e_r) - g(h(e_1, e_2), e_r)\, g(h(e_1, e_2), e_r) + \frac{c}{4} + \frac{3c}{4}\cos^2\theta_1\Big\}$$

or,

$$K(\pi) = \sum_{r=m+2}^{2m+1}\Big\{h_{11}^r h_{22}^r - (h_{12}^r)^2\Big\} + \frac{c}{4} + 3\frac{c}{4}\cos^2\theta_1 \tag{3.12}$$

or,

$$K(\pi) = h_{11}^{m+2}h_{22}^{m+2} + \sum_{r=m+3}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=m+2}^{2m+1}\left(h_{12}^r\right)^2 + \frac{c}{4} + \frac{3c}{4}\cos^2\theta_1.$$

Using (3.11) in the above equation, we obtain

$$k(\pi) \geq \frac{1}{2}\sum_{i\neq j}\left(h_{ij}^{m+2}\right)^2 + \frac{1}{2}\sum_{r=m+3}^{2m+1}\sum_{ij=1}^{2m+1}\left(h_{ij}^r\right)^2 + \frac{\epsilon}{2} - \frac{c}{4} + \sum_{r=m+3}^{2m+1} h_{11}^r h_{22}^r - \sum_{r=m+2}^{2m+1}\left(h_{12}^r\right)^2 + \frac{c}{4} + \frac{3c}{4}\cos^2\theta_1$$

or,

$$K(\pi) \geq \frac{\epsilon}{2} + 3\frac{c}{4}\cos^2\theta_1. \tag{3.13}$$

Now using (3.8) in (3.13), we obtain

$$\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2) + \frac{3c}{4}[(n_1 - 1)cos^2\theta_1 + n_2 cos^2\theta_2].$$

Case (ii). If π is tangent to D_2, we obtain, as in Case (i)

$$\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2)$$
$$+\frac{3c}{4}[n_1 cos^2\theta_1 + (n_2-1)cos^2\theta_2].$$

These are the desired inequalities.

If at any point $p \in M$, equality in (3.1) and (3.2) hold, then the inequalities in (3.11) and (3.13) become equalities. Hence, we have

$$h_{1j}^{m+2} = h_{2j}^{m+2} = h_{ij}^{m+2} = 0, \quad i \neq j > 2$$
$$h_{ij}^r = 0, \quad \forall\, i \neq j, \quad i,\, j = 3, \ldots, 2m+1,\; r = m+3, \ldots, 2m+1$$
$$h_{11}^r + h_{22}^r = 0, \quad \forall\, r = m+3, \ldots, 2m+1$$
$$h_{11}^{m+2} + h_{22}^{m+2} = h_{33}^{m+2} = \cdots = h_{m+1\, m+1}^{m+2}.$$

Now, if we take $e_1,\ e_2$ such that $h_{12}^{m+2} = 0$ and letting $a = h_{11}^r$, $b = h_{22}^r$, $\lambda = h_{33}^{m+2} = \cdots = h_{m+1\,m+1}^{m+2}$, it follows that the shape operators assume the desired form. □

Corollary 3.1. *Let M be an $m+1$-dimensional contact CR-submanifold with in a $2m+1$-dimensional cosymplectic space form $\overline{M}(c)$. Then, we have*

$$\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2) + \frac{3c}{4}(n_1-1)$$

on D_1, and

$$\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2) + \frac{3c}{4}n_1$$

on D_2.

Now, we have the following result.

Theorem 3.2. *Let M be an $(m+1)$-dimensional θ-slant submanifold with $\theta_1 = \theta_2 = \theta$ in a $(2m+1)$-dimensional cosymplectic space form $\overline{M}(c)$. Then, we have*

$$\delta_M \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2) + \frac{3c}{8}(m-2)\cos^2\theta.$$

The equality holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_m, e_{m+1} = \xi\}$ of T_pM and an orthonormal basis $\{e_{m+2}, e_{m+3}, \ldots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M

in cosymplectic space form $\overline{M}(c)$ take the following forms

$$A_{m+2} = \begin{pmatrix} a & 0 & 0 & . & . & . & 0 \\ 0 & b & 0 & . & . & . & 0 \\ 0 & 0 & & & \lambda I_{m-1} & & \end{pmatrix}, \quad a+b=\lambda$$

$$A_{e_r} = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & . & . & . & 0 \\ h^r_{12} & -h^r_{11} & 0 & . & . & . & 0 \\ 0 & 0 & & & 0_{m-1} & & \end{pmatrix}, \quad r = m+3, \ldots, 2m+1.$$

Corollary 3.2. *Let M be an $(m+1)$-dimensional invariant submanifold of a $(2m+1)$-dimensional cosymplectic space form $\overline{M}(c)$. Then, we have*

$$\delta_M \leq \frac{c(m^2+2m-8)}{8}.$$

Corollary 3.3. *Let M be an $(m+1)$-dimensional anti-invariant submanifold of a $(2m+1)$-dimensional cosymplectic space form $\overline{M}(c)$. Then, we have*

$$\delta_M \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{c}{8}(m+1)(m-2).$$

4. Examples of bi-slant submanifolds of cosymplectic manifolds

Example 4.1. For any $\theta_1,\ \theta_2 \in [0,\ \pi/2]$

$$x\,(u,v,w,s,z) = (u,0,w,0,v\cos\theta_1, v\sin\theta_1, s\cos\theta_2, s\sin\theta_2, z)$$

defines a 5-dimensional bi-slant submanifold M, with slant angles θ_1 and θ_2 in R^9 with its usual cosymplectic structure (ϕ_0,ξ,η,g), given by:

$$\eta = dz, \qquad \xi = \frac{\partial}{\partial z}$$

$$g = \eta\otimes\eta + \Big\{\sum_{i=1}^{4}(dx^i\otimes dx^i + dy^i\otimes dy^i)\Big\}$$

and

$$\phi_0\Big\{\sum_{i=1}^{4}\Big(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\Big) + Z\frac{\partial}{\partial z}\Big\} = \sum_{i=1}^{4}\Big(-Y_i\frac{\partial}{\partial x^i} + X_i\frac{\partial}{\partial y^i}\Big).$$

Furthermore it is easy to see that:

$$e_1 = \frac{\partial}{\partial x^1}, \quad e_2 = \cos\theta_1\frac{\partial}{\partial y^1} + \sin\theta_1\frac{\partial}{\partial y^2}, \quad e_3 = \frac{\partial}{\partial x^3}$$

$$e_4 = \cos\theta_2\frac{\partial}{\partial y^3} + \sin\theta_1\frac{\partial}{\partial y^4} \quad \text{and} \quad e_5 = \frac{\partial}{\partial z} = \xi$$

form a local orthonormal frame of TM. If, we define $D_1 = \{e_1, e_2\}$ and $D_2 = \{e_3, e_4\}$, then a simple computation yields, $g(\phi_0 e_1, e_2) = \cos\theta_1$ and $g(\phi_0 e_3, e_4) = \cos\theta_2$ proving that the distribution D_1 is θ_1-slant and the distribution D_2 is θ_2-slant.

Example 4.2. For any $\theta_1, \theta_2 \in [0, \pi/2]$

$$x(u,v,w,s,z) = (\cos\alpha_1 \cos\alpha_2\, u - \sin\alpha_1\, s, \sin\alpha_1 \cos\alpha_2\, u + \cos\alpha_1\, s\, , \cos\alpha_1 \sin\alpha_2\, u, \sin\alpha_1 \sin\alpha_2\, u,\ w,\ -\sin\alpha_2\, v,\ 0\, , \cos\alpha_2\, v,\ z)$$

defines a 5-dimensional bi-slant submanifold M, with slant angles $\theta_1 = \pi/2$ and $\cos^2\theta_2 = \sin^2\alpha_1$ in R^9 with its usual cosymplectic structure.

We can choose orthonormal frame on TM, given by

$$e_1 = (\cos\alpha_1 \cos\alpha_2,\ \sin\alpha_1 \cos\alpha_2,\ \cos\alpha_1\ \sin\alpha_2,\ \sin\alpha_1\ \sin\alpha_2,\ 0,\ 0,\ 0,\ 0,\ 0)$$

$$e_2 = -\sin\alpha_2 \frac{\partial}{\partial y^2} + \cos\alpha_2 \frac{\partial}{\partial y^4}, \quad e_3 = \frac{\partial}{\partial y^1}$$

$$e_4 = -\sin\alpha_1 \frac{\partial}{\partial x^1} + \cos\alpha_1 \frac{\partial}{\partial x^2} \quad \text{and} \quad e_5 = \frac{\partial}{\partial z} = \xi$$

where, distributions are defined by $D_1 = \{e_1, e_2\}$ and $D_2 = \{e_3, e_4\}$. Then it can be easily seen that $g(e_1, \phi_0 e_2) = 0$ and $g(e_3, \phi_0 e_4) = \sin\alpha_1$, that is, distribution D_1 is θ_1-slant with $\theta_1 = \pi/2$ and the distribution D_2 is θ_2-slant with $\cos^2\theta_2 = \sin^2\alpha_1$.

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