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A UNIFIED THEORY OF WEAKLY g-CLOSED SETS AND WEAKLY g-CONTINUOUS FUNCTIONS

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Abstract. We introduce the notion of weakly mng-closed sets as a unified form of weakly ω-closed sets [38], weakly rg-closed sets [23], weakly πg-closed sets [40] and weakly mg^*-closed sets [29]. Moreover, we introduce and study the notion of weakly mng-continuous functions to unify some modifications of weakly g-continuous functions.

1. Introduction

The concept of generalized closed (briefly g-closed) sets in a topological space was introduced by Levine [16]. These sets were further considered by Dunham and Levine [13]. In 1981, Munshi and Bassan [21] introduced the notion of g-continuous functions. The notion of g-continuity is also studied in [5], [8], [9], [10] and other papers. Various forms of g-continuity are studied in [12], [31], [38], [41], [43] and other papers. A unified form of g-closed sets is obtained in [26].

A weak form of g-closed sets is introduced in [42]. Some forms of weakly g-closed sets and weak g-continuity are introduced and studied in [29], [31], [38] and [40].

In [32] and [33], the present authors introduced and studied the notions of m-structures, m-spaces and m-continuity. A set with two minimal structures is used in Theorems 4.1 and 4.2 of [36], Theorems 4.2 and 4.3 of [37], and Theorems 7.4 and 7.5 of [27]. The notion of bi-m-spaces is introduced in [25]. A similar notion was recently introduced in [7].

In the present paper, we introduce the notion of weakly mng-closed sets as a unified form of weakly g-closed sets [42], weakly ω-closed sets [38], weakly rg-closed sets [23], weakly πg-closed sets [40] and weakly mg^*-closed sets [29]. Moreover, we introduce and study the notion of weakly mng-continuous functions to obtain a unified form of some modifications of weakly

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g-continuous functions. By using m-continuity, we obtain several characterizations and properties of weakly mng-continuous functions.

2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A is said to be *regular open* if $A = \mathrm{Int}(\mathrm{Cl}(A))$. We recall some generalized open sets in a topological space.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *α-open* [24] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,
(2) *semi-open* [15] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,
(3) *preopen* [18] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$,
(4) *β-open* [1] or *semi-preopen* [3] if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$,
(5) *b-open* [4] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,
(6) *π-open* [45] if A is the finite union of regular open sets.

The family of all α-open (resp. semi-open, preopen, b-open, β-open, π-open, regular open) sets in (X, τ) is denoted by $\alpha(X)$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, $\beta(X)$, $\pi(X)$, $\mathrm{RO}(X)$).

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *α-closed* [19] (resp. *semi-closed* [11], *preclosed* [18], *b-closed* [4], *β-closed* [1], *π-closed*) if the complement of A is α-open (resp. semi-open, preopen, b-open, β-open, π-open).

Definition 2.3. Let (X, τ) be a topological space and A a subset of X. The intersection of all α-closed (resp. semi-closed, preclosed, b-closed, β-closed, π-closed) sets of X containing A is called the *α-closure* [19] (resp. *semi-closure* [11], *preclosure* [14], *b-closure* [4], *β-closure* [2], *π-closure*) of A and is denoted by $\alpha\mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\mathrm{bCl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\pi\mathrm{Cl}(A)$).

Definition 2.4. Let (X, τ) be a topological space and A a subset of X. The union of all α-open (resp. semi-open, preopen, b-open, β-open, π-open) sets of X contained in A is called the *α-interior* [19] (resp. *semi-interior* [11], *preinterior* [14], *b-interior* [4], *β-interior* [2], *π-interior*) of A and is denoted by $\alpha\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\mathrm{bInt}(A)$, ${}_\beta\mathrm{Int}(A)$, $\pi\mathrm{Int}(A)$).

3. Minimal structures and m-continuity

Definition 3.1. Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subfamily m_X of $\mathcal{P}(X)$ is called a *minimal structure* (briefly *m-structure*) on X [32], [33] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with an m-structure m_X on X and call it an m-space. Each member of m_X is said to be *m_X-open* (briefly m-open) and the complement of an m_X-open set is said to be *m_X-closed* (briefly m-closed).

Remark 3.1. Let (X, τ) be a topological space. Then the family $\alpha(X)$ is a topology which is finer than τ. The families $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\mathrm{BO}(X)$, $\beta(X)$, $\pi(X)$ and $\mathrm{RO}(X)$ are all m-structures on X.

Definition 3.2. Let X be a nonempty set and m_X an m-structure on X. For a subset A of X, the *m_X-closure* of A and the *m_X-interior* of A are defined in [17] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X - F \in m_X\}$,
(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$, $\mathrm{BO}(X)$, $\pi(X)$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, ${}_\beta\mathrm{Cl}(A)$, $\mathrm{bCl}(A)$, $\pi\mathrm{Cl}(A)$),
(2) $\mathrm{mInt}(A)=\mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, ${}_\beta\mathrm{Int}(A)$, $\mathrm{bInt}(A)$, $\pi\mathrm{Int}(A)$).

Lemma 3.1 (Maki et al. [17])**.** *Let X be a nonempty set and m_X a minimal structure on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X - A) = X - \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X - A) = X - \mathrm{mCl}(A)$,
(2) *If* $(X - A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,
(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,
(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,
(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,
(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property $\mathcal{B}$* [17] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 3.3. If (X, τ) is a topological space, then $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$ and $\mathrm{BO}(X)$ have property $\mathcal{B}$.

Lemma 3.2. (Popa and Noiri [35])**.** *Let X be a nonempty set and m_X an m-structure on X satisfying property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,
(2) *A is m_X-closed if and only if* $\mathrm{mCl}(A) = A$,
(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Definition 3.4. Let (Y, σ) be a topological space. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *m-continuous* [33] at $x \in X$ if for each open set V containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$. The function f is said to be *m-continuous* if it has this property at each point $x \in X$.

Theorem 3.1. (Popa and Noiri [33]). *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is m-continuous;*
(2) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ *for every open set V of Y;*
(3) $f^{-1}(F) = \mathrm{mCl}(f^{-1}(F))$ *for every closed set F of Y;*
(4) $\mathrm{mCl}(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}(B))$ *for every subset B of Y;*
(5) $f(\mathrm{mCl}(A)) \subset \mathrm{Cl}(f(A))$ *for every subset A of X;*
(6) $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{mInt}(f^{-1}(B))$ *for every subset B of Y.*

Corollary 3.1. (Popa and Noiri [33]). *For a function $f : (X, m_X) \to (Y, \sigma)$, where m_X has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is m-continuous;*
(2) *$f^{-1}(V)$ is m_X-open in X for every open set V of Y;*
(3) *$f^{-1}(F)$ is m_X-closed in X for every closed set F of Y.*

Definition 3.5. A function $f : (X, m_X) \to (Y, \sigma)$ is said to be *m^*-continuous* [20] if $f^{-1}(V)$ is m_X-open in X for each open set V of Y.

Remark 3.4.

(1) If $f : (X, m_X) \to (Y, \sigma)$ is m^*-continuous, then it is m-continuous. By Example 3.4 of [20], every m-continuous function may not be m^*-continuous.
(2) Let m_X have property $\mathcal{B}$, then it follows from Corollary 3.1 that f is m-continuous if and only if f is m^*-continuous.

For a function $f : (X, m_X) \to (Y, \sigma)$, we define $D_m(f)$ as follows:

$$D_m(f) = \{x \in X : f \text{ is not } m\text{-continuous at } x\}.$$

Theorem 3.2. (Popa and Noiri [34]). *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties hold:*

$$\begin{aligned} D_m(f) &= \textstyle\bigcup_{G\in\sigma}\{f^{-1}(G) - \mathrm{mInt}(f^{-1}(G))\} \\ &= \textstyle\bigcup_{B\in\ \mathcal{P}(Y)}\ \{f^{-1}(\mathrm{Int}(B)) - \mathrm{mInt}(f^{-1}(B))\} \\ &= \textstyle\bigcup_{B\in\ \mathcal{P}(Y)}\ \{\mathrm{mCl}(f^{-1}(B)) - f^{-1}(\mathrm{Cl}(B))\} \\ &= \textstyle\bigcup_{A\in\ \mathcal{P}(X)}\ \{\mathrm{mCl}(A) - f^{-1}(\mathrm{Cl}(f(A)))\} \\ &= \textstyle\bigcup_{F\in\ \mathcal{F}}\ \{\mathrm{mCl}(f^{-1}(F)) - f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ is the family of closed sets of (Y, σ).

4. Weakly mng-closed sets

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be *g-closed* [16] (resp. ω-closed [41], *rg-closed* [30], *πg-closed* [12]) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open (resp. semi-open, regular open, π-open) in X.

Remark 4.1. An ω-closed set is said to be *sg^**-closed [22], $\hat{g}$-closed [43] or semi-star generalized closed [39].

Definition 4.2. Let (X, τ) be a topological space and m_X an m-structure on X. A subset A of X is said to be *mg^*-closed* [26] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Remark 4.2. Let (X, τ) be a topological space and m_X an m-structure on X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{RO}(X)$, $\pi(X)$), then we obtain Definition 4.1.

Definition 4.3. Let (X, τ) be a topological space. A subset A of X is said to be *weakly g-closed* [42] (resp. weakly ω-closed [38], *weakly rg-closed* [23], *weakly πg-closed* [40]) if $\mathrm{Cl}(\mathrm{Int}(A)) \subset U$ whenever $A \subset U$ and U is open (resp. semi-open, regular open, π-open) in X.

The following definition is a generalization of Definition 4.3.

Definition 4.4. Let (X, τ) be a topological space and m_X an m-structure on X. A subset A is said to be *weakly mg^*-closed* (briefly *wmg^**-closed) [29] if $\mathrm{Cl}(\mathrm{Int}(A)) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Remark 4.3. Let (X, τ) be a topological space and m_X an m-structure on X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{RO}(X)$, $\pi(X)$), then we obtain Definition 4.3.

Recently, a new generalization of weakly g-closed sets is introduced as follows:

Definition 4.5. Let (X, m_X) be an m-space. A subset A of X is said to be *m-weakly g-closed* [31] if $\mathrm{mCl}(\mathrm{mInt}(A)) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Definition 4.6. Let X be a nonempty set and m_X, n_X minimal structures on X. A set X with two minimal structures is called a *bi-m-space* [25] or a biminimal structure space [7] and is denoted by (X, m_X, n_X).

A subset A of a bi-m-space (X, m_X, n_X) is said to be *mng-closed* [25] if $\mathrm{nCl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$. Now, we introduce the notion of weakly g-closed sets in a bi-m-space (X, m_X, n_X) as follows:

Definition 4.7. Let (X, m_X, n_X) be a bi-m-space. A subset A of X is said to be *weakly mng-closed* (briefly *$wmng$*-closed) if $\mathrm{nCl}(\mathrm{nInt}(A)) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Remark 4.4. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X.

(1) If $n_X = \tau$ and $m_X = \tau$ (resp. SO(X), RO(X), $\pi(X)$), then we obtain the definition of weakly g-closed sets [42] (resp. weakly ω-closed sets [38], weakly rg-closed sets [23], weakly πg-closed sets [40]).

(2) If $n_X = \tau$, then we obtain the definition of weakly mg^*-closed sets [29].

(3) If $n_X = m_X$, then we obtain the definition of m-weakly g-closed sets [31].

Theorem 4.1. *Let (X, m_X, n_X) be a bi-m-space and A a subset of X. If A is mng-closed, then A is wmng-closed.*

Proof. Since A is mng-closed, we have $\mathrm{nCl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$ and hence $\mathrm{nCl}(\mathrm{nInt}(A)) \subset \mathrm{nCl}(A) \subset U$. Therefore, A is $wmng$-closed. □

Remark 4.5. The converse of Theorem 4.1 is not true as shown by Example 3.3 of [31], Example 3.5 of [40] and Example 3.6 of [38].

A subset A of an m-space (X, m_X) is said to be m_X*-regular closed* [6] if $A = \mathrm{mCl}(\mathrm{mInt}(A))$.

Theorem 4.2. *Let (X, m_X, n_X) be a bi-m-space. Then every n_X-regular closed set is wmng-closed.*

Proof. Let A be n_X-regular closed, $A \subset U$ and $U \in m_X$. Then $\mathrm{nCl}(\mathrm{nInt}(A)) = A \subset U$. Therefore, A is $wmng$-closed. □

Remark 4.6. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X.

(1) If $n_X = \tau$ and $m_X = $ SO(X) (resp. $\pi(X)$), then by Theorem 4.2 we obtain Proposition 3.3 of [38] (resp. Theorem 3.6 of [40]).

(2) The converse of Theorem 4.2 is not true as shown by Example 3.7 of [40] and Example 3.6 of [38].

Theorem 4.3. *Let (X, m_X, n_X) be a bi-m-space. If A is an n_X-closed set, then A is wmng-closed.*

Proof. Let A be an n_X-closed set. Then, by Lemma 3.1, $A = \mathrm{nCl}(A)$. Let $A \subset U$ and $U \in m_X$, then $\mathrm{nCl}(\mathrm{nInt}(A)) \subset \mathrm{nCl}(A) = A \subset U$. Hence A is $wmng$-closed. □

Remark 4.7. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X.

(1) If $n_X = \tau$ and $m_X = $ SO(X) (resp. $\pi(X)$), then by Theorem 4.3 we obtain Corollary 3.4 of [38] (resp. Theorem 3.4 of [40]).

(2) If $n_X = m_X$, then by Theorem 4.3 we obtain Lemma 3.4 of [31].

(3) The converse of Theorem 4.3 is not true as shown by Example 3.3 of [40] and Example 3.6 of [38].

Theorem 4.4. *Let (X, m_X, n_X) be a bi-m-space. If A is a wmng-closed set and $A \subset B \subset \mathrm{nCl}(\mathrm{nInt}(A))$, then B is wmng-closed.*

Proof. Let $B \subset U$ and $U \in m_X$. Since A is *wmng*-closed and $A \subset U$, $\mathrm{nCl}(\mathrm{nInt}(A)) \subset U$. By Lemma 3.1, $\mathrm{nCl}(\mathrm{nInt}(B)) \subset \mathrm{nCl}(\mathrm{nInt}(\mathrm{nCl}(\mathrm{nInt}(A))))$ $\subset \mathrm{nCl}(\mathrm{nInt}(A)) \subset U$. Hence B is *wmng*-closed. □

Remark 4.8. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X. By Theorem 4.4, the following hold:

(1) If $n_X = \tau$ and $m_X = \mathrm{SO}(X)$ (resp. $\pi(X)$), then we obtain Theorem 3.15 of [38] (resp. Theorem 3.23 of [40]).
(2) If $n_X = \tau$, then we obtain Theorem 5.2 of [29].
(3) If $m_X = n_X$, then we obtain Lemma 3.4(iii) of [31].

Theorem 4.5. *Let (X, m_X, n_X) be a bi-m-space and n_X have property $\mathcal{B}$. If A is wmng-closed, n_X-open and m_X-open, then A is n_X-closed.*

Proof. Since A is *wmng*-closed and m_X-open, $\mathrm{nCl}(\mathrm{nInt}(A)) \subset A$. Since A is n_X-open, $\mathrm{nCl}(A) \subset A$ and hence by Lemma 3.1 $\mathrm{nCl}(A) = A$. Since n_X has property $\mathcal{B}$, A is n_X-closed. □

Remark 4.9. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X.

(1) If $n_X = \tau$ and $m_X = \pi(X)$, then by Theorem 4.5 we obtain Theorem 3.17 of [40].
(2) If $m_X = n_X$, then by Theorem 4.5 we obtain Lemma 3.4(ii) of [31].

Theorem 4.6. *Let (X, m_X, n_X) be a bi-m-space. A subset A of X is wmng-closed if and only if* $\mathrm{nCl}(\mathrm{nInt}(A)) \cap F = \emptyset$ *whenever $A \cap F = \emptyset$ and F is m_X-closed.*

Proof. Necessity. Suppose that A is *wmng*-closed. Let $A \cap F = \emptyset$ for any m_X-closed set F. Then $A \subset X - F \in m_X$ and $\mathrm{nCl}(\mathrm{nInt}(A)) \subset X - F$. Therefore, $\mathrm{nCl}(\mathrm{nInt}(A)) \cap F = \emptyset$.

Sufficiency. Let $A \subset U$ and $U \in m_X$. Then $A \cap (X - U) = \emptyset$ and $X - U$ is m_X-closed. By hypothesis $\mathrm{nCl}(\mathrm{nInt}(A)) \cap (X - U) = \emptyset$ and hence $\mathrm{nCl}(\mathrm{nInt}(A)) \subset U$. Therefore, A is *wmng*-closed. □

Remark 4.10. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X. If $n_X = \tau$ (resp. $m_X = n_X$), then by Theorem 4.6 we obtain Theorem 5.3 of [29] (resp. Theorem 3.9 of [31]).

Theorem 4.7. *Let (X, m_X, n_X) be a bi-m-space. If a subset A of X is wmng-closed, then* $\mathrm{nCl}(\mathrm{nInt}(A)) - A$ *does not contain any nonempty m_X-closed set. Moreover, the converse holds if $n_X \subset m_X$ and both n_X and m_X have property $\mathcal{B}$.*

Proof. Suppose that A is a *wmng*-closed set. Let F be an m_X-closed set and $F \subset \mathrm{nCl}(\mathrm{nInt}(A)) - A$. Then $A \subset X - F$ and $X - F \in m_X$ and hence $\mathrm{nCl}(\mathrm{nInt}(A)) \subset X - F$. Therefore, we have $F \subset X - \mathrm{nCl}(\mathrm{nInt}(A))$. However, $F \subset \mathrm{nCl}(\mathrm{nInt}(A))$ and $F \subset \mathrm{nCl}(\mathrm{nInt}(A)) \cap (X - \mathrm{nCl}(\mathrm{nInt}(A))) = \emptyset$.

Conversely, suppose that A is not *wmng*-closed. Then $\emptyset \neq \mathrm{nCl}(\mathrm{nInt}(A)) - U$ for some $U \in m_X$ containing A. Since $n_X \subset m_X$ and both n_X and m_X have property $\mathcal{B}$, $\mathrm{nCl}(\mathrm{nInt}(A)) - U$ is m_X-closed. Moreover, we have $\emptyset \neq \mathrm{nCl}(\mathrm{nInt}(A)) - U \subset \mathrm{nCl}(\mathrm{nInt}(A)) - A$. Therefore, $\mathrm{nCl}(\mathrm{nInt}(A)) - A$ contains a nonempty m_X-closed set. □

Remark 4.11. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X. Then, by Theorem 4.7, the following hold:

(1) If $n_X = \tau$ and $m_X = \mathrm{SO}(X)$ (resp. $\pi(X)$), then we obtain Theorem 3.12 of [38] (resp. Theorem 3.19 of [40]).
(2) If $n_X = \tau$, then we obtain Theorem 5.5 of [29].
(3) If $n_X = m_X$, then we obtain Theorem 2.8(ii) of [31].

Definition 4.8. Let (X, m_X, n_X) be a bi-m-space. A subset A of X is said to be *weakly mng-open* (briefly *wmng*-open) if $X - A$ is weakly *mng*-closed. The family of all *wmng*-open sets in (X, m_X, n_X) is denoted by $\mathrm{wmnGO}(X)$.

Remark 4.12. Let (X, m_X, n_X) be a bi-m-space and τ a topology for X.

(1) If $n_X = \tau$ and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\pi(X)$, $\mathrm{RO}(X)$), then a *wmng*-open set is weakly g-open (resp. weakly ω-open [38], weakly πg-open [40], weakly rg-open [23]).
(2) If $n_X = \tau$, then a *wmng*-open set is wmg^*-open [29].
(3) If $n_X = m_X$, then a *wmng*-open set is *mwg*-open [31].

The family of weakly g-open (resp. weakly ω-open, weakly πg-open, weakly rg-open, wmg^*-open) sets is denoted by $\mathrm{wGO}(X)$ (resp. $\mathrm{w}\omega(X)$, $\mathrm{w}\pi\mathrm{GO}(X)$, $\mathrm{wrGO}(X)$, $\mathrm{wmG^*O}(X)$). These families are obviously minimal structures on X and they are called *wmng-structures* on X. In general, *wmng*-structures do not have property $\mathcal{B}$.

Definition 4.9. Let (X, m_X, n_X) be a bi-m-space and $\mathrm{wmnGO}(X)$ a *wmng*-structure on X. For a subset A of X, the *wmng-closure* and the *wmng-interior* of A are defined as follows:

(1) $wmn\mathrm{Cl}_g(A) = \cap\{F : A \subset F, X - F \in \mathrm{wmnGO}(X)\}$,
(2) $wmn\mathrm{Int}_g(A) = \cup\{U : U \subset A, U \in \mathrm{wmnGO}(X)\}$.

5. Weakly *mng*-continuous functions

Definition 5.1. Let (X, m_X, n_X) be a bi-m-space and $\mathrm{wmnGO}(X)$ a *wmng*-structure on X. A function $f : (X, m_X, n_X) \to (Y, \sigma)$ is said to be

(1) *wmng-continuous* at $x \in X$ if $f : (X, \text{wmnGO}(X)) \to (Y, \sigma)$ is m-continuous at x, equivalently if for each open set V containing $f(x)$ there exists a *wmng*-open set U containing x such that $f(U) \subset V$. The function f is said to be *wmng-continuous* if it has this property at each point $x \in X$.

(2) $w(mn)^*g$*-continuous* if $f : (X, \text{wmnGO}(X)) \to (Y, \sigma)$ is m^*-continuous, equivalently if $f^{-1}(K)$ is *wmng*-closed in X for each closed set K of Y.

Remark 5.1. Let (X, τ) be a topological space and m_X, n_X m-structures on X.

(1) If $n_X = \tau$ and $m_X = \text{SO}(X)$ (resp. $\pi(X)$), then a $w(mn)^*g$-continuous function is weakly ω-continuous [38] (resp. weakly πg-continuous [40]).

(2) If $n_X = \tau$, then a $w(mn)^*g$-continuous function is wm^*g^*-continuous [29].

Let (X, m_X, n_X) be a bi-m-space and $\text{wmnGO}(X)$ a *wmng*-structure on X. For a function $f : (X, m_X, n_X) \to (Y, \sigma)$, we denote the set of all points of X at which the function f is not *wmng*-continuous by $D_{wmng}(f)$.

Theorem 5.1. *Let* (X, m_X, n_X) *be a bi-m-space and* $\text{wmnGO}(X)$ *a wmng-structure on X. For a function* $f : (X, m_X, n_X) \to (Y, \sigma)$*, the following properties hold:*

$$\begin{aligned} D_{wmng}(f) &= \bigcup_{G\in\sigma}\{f^{-1}(G) - wmn\text{Int}_g(f^{-1}(G))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)}\ \{f^{-1}(\text{Int}(B)) - wmn\text{Int}_g(f^{-1}(B))\} \\ &= \bigcup_{B\in\ \mathcal{P}(Y)}\ \{wmn\text{Cl}_g(f^{-1}(B)) - f^{-1}(\text{Cl}(B))\} \\ &= \bigcup_{A\in\ \mathcal{P}(X)}\ \{wmn\text{Cl}_g(A) - f^{-1}(\text{Cl}(f(A)))\} \\ &= \bigcup_{F\in\ \mathcal{F}}\ \{wmn\text{Cl}_g(f^{-1}(F)) - f^{-1}(F)\}, \end{aligned}$$

where $\mathcal{F}$ *is the family of closed sets of* (Y, σ)*.*

Proof. The proof follows immediately from Theorem 3.2. □

Theorem 5.2. *Let* (X, m_X, n_X) *be a bi-m-space. Then for a function* $f : (X, m_X, n_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) *f is wmng-continuous;*
(2) $f^{-1}(V) = wmn\text{Int}_g(f^{-1}(V))$ *for every open set V of Y;*
(3) $f^{-1}(F) = wmn\text{Cl}_g(f^{-1}(F))$ *for every closed set F of Y;*
(4) $wmn\text{Cl}_g(f^{-1}(B)) \subset f^{-1}(\text{Cl}(B))$ *for every subset B of Y;*
(5) $f(wmn\text{Cl}_g(A)) \subset \text{Cl}(f(A))$ *for every subset A of X;*
(6) $f^{-1}(\text{Int}(B)) \subset wmn\text{Int}_g(f^{-1}(B))$ *for every subset B of Y.*

Proof. The proof follows immediately from Theorem 3.1. □

Corollary 5.1. *Let (X, m_X, n_X) be a bi-m-space and* wmnGO*(X) a wmng-structure on X with property $\mathcal{B}$. For a function $f : (X, m_X, n_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is wmng-continuous;*
(2) *$f^{-1}(V)$ is wmng-open in X for every open set V of Y;*
(3) *$f^{-1}(F)$ is wmng-closed in X for every closed set F of Y.*

Proof. The proof follows from Corollary 3.1. □

Definition 5.2. Let (X, m_X) be an m-space and A a subset of X. The m_X-*frontier* of A, mFr(A), [33] is defined by $\mathrm{mFr}(A) = \mathrm{mCl}(A) \cap \mathrm{mCl}(X - A) = \mathrm{mCl}(A) - \mathrm{mInt}(A)$.

If (X, m_X, n_X) is a bi-m-space and wmnGO(X) a *wmng*-structure, then $wmn\mathrm{Fr}_g(A) = wmn\mathrm{Cl}_g(A) \cap wmn\mathrm{Cl}_g(X - A) = wmn\mathrm{Cl}_g(A) - wmn\mathrm{Int}_g(A)$.

Lemma 5.1. (Popa and Noiri [33]). *The set of all points of X at which a function $f : (X, m_X) \to (Y, \sigma)$ is not m-continuous is identical with the union of the m-frontiers of the inverse images of open sets containing f(x).*

Theorem 5.3. *Let (X, m_X, n_X) is a bi-m-space and* wmnGO(X) *a wmng-structure. Then, the set of all points of X at which a function $f : (X, m_X, n_X) \to (Y, \sigma)$ is not wmng-continuous is identical with the union of the wmng-frontiers of the inverse images of open sets containing $f(x)$.*

Proof. The proof follows from Lemma 5.1. □

Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ for every open set V containing x. The set of all θ-cluster points of A is called the θ-*closure* of A and is denoted by $\mathrm{Cl}_\theta(A)$ [44]. If $A = \mathrm{Cl}_\theta(A)$, then A is said to be θ-*closed*. The complement of a θ-closed set is said to be θ-*open*.

Lemma 5.2. (Noiri and Popa [28]). *Let (Y, σ) be a regular space. For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is m-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B)) = \mathrm{mCl}(f^{-1}(\mathrm{Cl}_\theta(B)))$ *for every subset B of Y;*
(3) $f^{-1}(K) = \mathrm{mCl}(f^{-1}(K))$ *for every θ-closed set K of Y;*
(4) $f^{-1}(V) = \mathrm{mInt}(f^{-1}(V))$ *for every θ-open set V of Y.*

Corollary 5.2. (Noiri and Popa [28]). *Let (Y, σ) be a regular space and m_X an m-structure on X with property $\mathcal{B}$. For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is m-continuous;*
(2) *$f^{-1}(\mathrm{Cl}_\theta(B))$ is m-closed for every subset B of Y;*
(3) *$f^{-1}(K)$ is m-closed in X for every θ-closed set K of Y;*

(4) $f^{-1}(V)$ *is m-open in X for every θ-open set V of Y.*

Theorem 5.4. *Let* (Y, σ) *be a regular space,* (X, m_X, n_X) *a bi-m-space and* wmnGO(X) *a wmng-structure. For a function* $f : (X, m_X, n_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) *f is wmng-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B)) = wmn\mathrm{Cl}_g(f^{-1}(\mathrm{Cl}_\theta(B)))$ *for every subset B of Y;*
(3) $f^{-1}(K) = wmn\mathrm{Cl}_g(f^{-1}(K))$ *for every θ-closed set K of Y;*
(4) $f^{-1}(V) = wmn\mathrm{Int}_g(f^{-1}(V))$ *for every θ-open set V of Y.*

Proof. The proof follows from Lemma 5.2. □

Corollary 5.3. *Let* (Y, σ) *be a regular space,* (X, m_X, n_X) *a bi-m-space and* wmnGO(X) *a wmng-structure with property* $\mathcal{B}$ *on X. For a function* $f : (X, m_X, n_X) \to (Y, \sigma)$*, the following properties are equivalent:*

(1) *f is wmng-continuous;*
(2) $f^{-1}(\mathrm{Cl}_\theta(B))$ *is wmng-closed in X for every subset B of Y;*
(3) $f^{-1}(K)$ *is wmng-closed in X for every θ-closed set K of Y;*
(4) $f^{-1}(V)$ *is wmng-open in X for every θ-open set V of Y.*

Proof. The proof follows from Corollary 5.2. □

Remark 5.2. Let (X, τ) be a topological space. If $n_X = \tau$ and m_X is an m-structure on X, then by Theorem 5.2 and Corollary 5.3 we obtain Theorem 5.7 and Corollary 5.3 in [29], respectively.

6. Some properties of *wmng*-continuous functions

Definition 6.1. A function $f : (X, m_X) \to (Y, \sigma)$ is said to have a *strongly m-closed graph* (resp. *m-closed graph*) [33] if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in m_X$ containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).

Definition 6.2. Let (X, m_X, n_X) be a bi-m-space and wmnGO(X) a *wmng*-structure on X. A function $f : (X, m_X, n_X) \to (Y, \sigma)$ is said to have a *strongly wmng-closed graph* (resp. *wmng-closed graph*) if a function $f : (X, \mathrm{wmnGO}(X)) \to (Y, \sigma)$ has a strongly m-closed (resp. m-closed) graph, equivalently if for each $(x, y) \in (X \times Y) - \mathrm{G}(f)$, there exist $U \in \mathrm{wmnGO}(X)$ containing x and an open set V of Y containing y such that $[U \times \mathrm{Cl}(V)] \cap \mathrm{G}(f) = \emptyset$ (resp. $[U \times V] \cap \mathrm{G}(f) = \emptyset$).

Remark 6.1. Let (X, τ) be a topological space and m_X, n_X minimal structures on X. If $n_X = \tau$, then by Definition 6.2 we obtain the definition of strongly *wmg**-closed graphs and *wmg**-closed graphs in [29].

Lemma 6.1. (Popa and Noiri [33]). *A function $f : (X, m_X) \to (Y, \sigma)$ is m-continuous and (Y, σ) is a Hausdorff space, then f has a strongly m-closed graph.*

Theorem 6.1. *Let (X, m_X, n_X) be a bi-m-space and* wmnGO(X) *a wmng-structure on X. If a function $f : (X, m_X, n_X) \to (Y, \sigma)$ is wmng-continuous and (Y, σ) is a Hausdorff space, then f has a strongly wmng-closed graph.*

Proof. The proof follows from Lemma 6.1. □

Remark 6.2. Let (X, τ) be a topological space, $n_X = \tau$ and m_X an m-structure on X. Then, by Theorem 6.1 we obtain Theorem 6.4 in [29].

Lemma 6.2. (Popa and Noiri [33]). *Let (X, m_X) be an m-space and (Y, σ) a topological space. If $f : (X, m_X) \to (Y, \sigma)$ is a surjective function with a strongly m-closed graph, then (Y, σ) is Hausdorff.*

Theorem 6.2. *Let (X, m_X, n_X) be a bi-m-space and* wmnGO(X) *a wmng-structure on X. If $f : (X, m_X, n_X) \to (Y, \sigma)$ is a surjective function with a strongly wmng-closed graph, then (Y, σ) is Hausdorff.*

Proof. The proof follows from Lemma 6.2. □

Remark 6.3. Let (X, τ) be a topological space, $n_X = \tau$ and m_X an m-structure on X. Then, by Theorem 6.2 we obtain Theorem 6.5 in [29].

Lemma 6.3. (Popa and Noiri [33]). *Let (X, m_X) be an m-space, where m_X has property $\mathcal{B}$. If $f : (X, m_X) \to (Y, \sigma)$ is an m-continuous injection with an m-closed graph, then X is m-T_2.*

Theorem 6.3. *Let (X, m_X, n_X) be a bi-m-space and* wmnGO(X) *a wmng-structure on X satisfying property $\mathcal{B}$. If $f : (X, m_X, n_X) \to (Y, \sigma)$ is a wmng-continuous injection with a wmng-closed graph, then X is wmng-T_2.*

Proof. The proof follows from Lemma 6.3. □

Remark 6.4. Let (X, τ) be a topological space, $n_X = \tau$ and m_X an m-structure on X. Then, by Theorem 6.3 we obtain Theorem 6.6 in [29].

Remark 6.5. By using the results in [33] and [28], we obtain Theorem 6.1 of [29], Theorem 4.14 of [38], Theorem 4.14 of [40], Theorem 6.2 of [29], Theorem 6.3 of [38] and Theorem 4.15 of [40].

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