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ON THE CATEGORIES OF PARAGRADED GROUPS AND MODULES OF TYPE Δ

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Abstract. In this paper we observe the categories of paragraded groups and R-modules with respect to the same set of grades Δ, where R is the paragraded ring with the set of grades Δ. We turn our attention to constructing new objects in those categories using the sets of morphisms of grade $\delta \in \Delta$. This process defines bifunctors which happen to be left exact. Thus we may construct the right derived functor and it turns out that it behaves the same as in the case of the category of abstract modules.

1. Introduction

Paragraded groups were introduced by M. Krasner and M. Vuković ([7]) in order to solve a problem of graded groups: they are not closed with respect to the direct product in sense that the homogeneous part of the direct product is not the direct product of the homogeneous parts of its components. Actually, from the homogeneous point of view, a multigroupoid does not have to be a homogroupoid ([7]). Paragraded groups are defined in [7] by the six-axiom system as follows.

Definition 1.1. *The map* $\pi : \Delta \to \mathrm{Sg}(G)$, $\pi(\delta) = G_\delta$ $(\delta \in \Delta)$, *of a partially ordered set* $(\Delta, <)$, *which is from bellow a complete semi-lattice and from above inductively ordered, to the set* $\mathrm{Sg}(G)$ *of subgroups of the group* G, *is called a* paragraduation *if it satisfies the following six-axiom system:*

i) $\pi(0) = G_0 = \{e\}$, *where* $0 = \inf \Delta$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

Remark 1.2. $H = \bigcup_{\delta \in \Delta} G_\delta$ is called *the homogeneous part* of G with respect to π, and elements from H are called the *homogeneous elements* of G.

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Remark 1.3. If $x \in H$, we say that $\delta(x) = \inf\{ \delta \in \Delta \mid x \in G_\delta \}$ is a grade of x. We have $\delta(x) = 0$ iff $x = e$. The elements $\delta(x)$, $x \in H$, are called the *principal grades* and they form a set which we will denote by Δ_p.

ii) $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} G_\delta = G_{\inf\theta}$;
iii) *If $x, y \in H$ and $yx = zxy$, then $z \in H$ and $\delta(z) \leq \inf(\delta(x), \delta(y))$;*
iv) *The homogeneous part H is a generating set of G;*
v) *Let $A \subseteq H$ be a subset such that for all $x, y \in A$ there exists an upper bound for $\delta(x), \delta(y)$. Then there exists an upper bound for all $\delta(x)$, $x \in A$;*
vi) *G is generated by H with the set of H-internal and left commutation relations (see* [7]*).*

The group is called paragraded *if it has a paragraduation.*

If we replace the sixth axiom with the following axiom:

vi') Let $\delta_1, \dots, \delta_s \in \Delta_p$ be pairwise incomparable and let $x_i, x_i' \in H$ $(i = 1, \dots, s)$ be such that $x_1 \cdot \dots \cdot x_s = x_1' \cdot \dots \cdot x_s'$ and $x_i, x_i' \in G_{\delta_i}$ for all $i = 1, \dots, s$. Then $\delta(x_i^{-1}x_i') < \delta_i$,

we get the notion of an *extragraded group.*

Theorem 1.4 ([7])**.** *Every extragraded group is a paragraded group.*

We shall consider maps between paragraded groups as well.
Let G_1 and G_2 be two paragraded groups with sets of grades Δ_1 and Δ_2, paragraduations π_1 and π_2, and homogeneous parts H_1 and H_2 respectively.

Definition 1.5. ([7]) *We say that a homomorphism $f : G_1 \to G_2$ is a* quasihomogeneous *if*

$$(\forall x \in H_1)\ f(x) \in H_2.$$

As we will see, we shall confine ourselves to the case $\Delta_1 = \Delta_2$.

Definition 1.6. ([7]) *The ring $(R, +, \cdot)$ is called paragraded if its additive group $(R, +)$ is a paragraded group, with paragraduation π and set of grades Δ, and if*

$$(\forall \xi, \eta \in \Delta)(\exists \zeta \in \Delta)\ R_\xi R_\eta \subseteq R_\zeta.$$

Definition 1.7. ([7]) *If R is a paragraded ring with paragraduation π, then the map $(\xi, \eta) \to \xi\eta$ from $\Delta \times \Delta$ to Δ is called* Δ-multiplication of grades *if the following holds:*

a) $R_\xi R_\eta \subseteq R_{\xi\eta}$;
b) $(\forall \xi, \xi', \eta, \eta' \in \Delta)\ \xi \leq \xi' \wedge \eta \leq \eta' \Rightarrow \xi\eta \leq \xi'\eta'$.

If R is a paragraded ring with paragraduation π, then there exists $\zeta = \sup(\ \delta(z) \mid z \in R_\xi R_\eta\ )$. If we put $\zeta = \xi\eta$, we will get Δ-multiplication and we call it *minimal multiplication* ([7]).

Definition 1.8. ([7]) *Let R_1 and R_2 be two paragraded rings and $f : R_1 \to R_2$ a homomorphism. We say that this homomorphism is a* quasihomogeneous *if it is a* quasihomogeneous *homomorphism from a paragraded group $(R_1, +)$ to a paragraded group $(R_2, +)$.*

Now, we will give the definition of a paragraded module.

Definition 1.9. ([7]) *Let R be a paragraded ring with paragraduation E and set of grades Δ, M a commutative paragraded group with paragraduation F and set of grades D and suppose M is an R-module. Denote $E(\delta)$ by R_δ and $F(d)$ by M_d, where $\delta \in \Delta$, $d \in D$. The R-module M is called* paragraded *if*

$$(\forall \delta \in \Delta)(\forall d \in D)(\exists t \in D)\ R_\delta M_d \subseteq M_t.$$

Definition 1.10. ([7]) *The map $\Delta \times D \to D : (\delta, d) \to \delta d$ is called (Δ, D)-*multiplication *if:*

1. $R_\delta M_d \subseteq M_{\delta d}$;
2. $(\forall \delta, \delta' \in \Delta)(\forall d, d' \in D)\ \delta \le \delta' \wedge d \le d' \Rightarrow \delta d \le \delta' d'$.

It is always possible to construct (Δ, D)-multiplication by putting $\delta d = \sup_{z \in R_\delta M_d} d(z)$ ([7]). This multiplication is called minimal multiplication.

The main feature of paragraded structures is described in the following theorem.

Theorem 1.11 ([7])**.** *The direct product of paragraded structures (groups, rings and modules) is also a paragraded structure and the homogeneous part of direct product is the direct product of the homogeneous parts of the components.*

2. The categories G^P_Δ and M^P_Δ

Let us observe the category of paragraded groups whose set of grades is Δ and denote it by G^P_Δ. We call it the *category of paragraded groups of type* Δ. Objects of such a category are paragraded groups, and morphisms are the elements of the set

$$\hom_{G^P_\Delta}(G, G') = \{\ f \in \hom(G, G') \mid f(G_\delta) \subseteq G'_\delta, \delta \in \Delta\ \},$$

where G, G' are paragraded groups.

Let us observe the paragraded R-modules M, where R is a paragraded ring with set of grades Δ and M is a commutative paragraded group of type Δ. These modules together with the set of morphisms

$$\{\ f \in \hom(M, M') \mid f(M_\delta) \subseteq M'_\delta, \delta \in \Delta\ \}$$

form a category which we will denote by M^P_Δ. We call it *the category of paragraded R-modules of type* Δ.

If $\Delta' \subseteq \Delta$ and if π is the paragraduation of G, then $\pi' = \pi|_{\Delta'}$ is also a paragraduation if Δ' is as ordered as Δ. So, if that is the case, we may observe the category $G^P_{\Delta'}$. Next, we examine the nature of the functor $F : G^P_\Delta \to G^P_{\Delta'}$.

Proposition 2.1. *The functor* $F : G^P_\Delta \to G^P_{\Delta'}$ *has a right adjoint.*

Proof. Let $G' \in G^P_{\Delta'}$ and let G and G' be equal as abstract groups, $\pi(\delta) = \pi'(\delta)$ for $\delta \in \Delta'$ and $\pi(\delta) = \{e\}$ for $\delta \notin \Delta'$. Thus, we defined the functor U on objects. Now, we define it on morphisms. If $\varphi' \in \hom_{G^P_{\Delta'}}(G'_1, G'_2)$, then let $U(\varphi') = \varphi$ be a quasihomogeneous homomorphism defined via $\varphi(x) = \varphi'(x)$ if $x \in H'$ and $\varphi(x) = e$ otherwise, where $H' = \bigcup_{\delta\in\Delta'} G_\delta$. This functor is the right adjoint. Indeed, for $G \in G^P_\Delta$ and $G' \in G^P_{\Delta'}$, define a map

$$f : \hom_{G^P_{\Delta'}}(F(G), G') \to \hom_{G^P_\Delta}(G, U(G'))$$

by $f(\varphi)(x) = \varphi(x)$ if $x \in H'$, and $f(\varphi)(x) = e$ if $x \notin H'$, for any $\varphi \in \hom_{G^P_{\Delta'}}(F(G), G')$. One easily verifies that for all $g : (G, G') \to (G_1, G'_1)$ the following diagram commutes

$$\begin{array}{ccc}
\hom_{G^P_{\Delta'}}(F(G), G') & \longrightarrow & \hom_{G^P_\Delta}(G, U(G')) \\
\Big\downarrow & & \Big\downarrow \\
\hom_{G^P_{\Delta'}}(F(G_1), G'_1) & \longrightarrow & \hom_{G^P_\Delta}(G_1, U(G'_1))
\end{array}$$

and that f is a bijection, i.e. f is a natural isomorphism. □

Proposition 2.2. *The category* M^P_Δ *is abelian.*

Proof. We will only check whether the category M^P_Δ has products and coproducts, since all other axioms are trivial.

Let M and M' be paragraded modules of type Δ. We claim that $M\oplus M'$ is their coproduct in the category M^P_Δ. For $\delta \in \Delta$ define $\pi : \Delta \to \mathrm{Sg}(M \oplus M')$ by

$$\pi(\delta) = M_\delta \oplus M'_\delta.$$

Since in our case we have the same set of grades Δ of paragraduations, it is the special case of the Theorem 1.11 and so, π is the paragraduation of $M \oplus M'$. Now, observe the maps $\alpha : M \to M \oplus M'$ and $\beta : M' \to M \oplus M'$ defined by $\alpha(m) = (m, 0)$ and $\beta(m') = (0, m')$ for all $m \in M$ and $m' \in M'$. One can easily verify that the maps α and β belong to $\hom_{M^P_\Delta}(M, M \oplus M')$ and $\hom_{M^P_\Delta}(M', M \oplus M')$, respectively. Let $X \in M^P_\Delta$

and $f \in \hom_{M_\Delta^P}(M, X)$, $g \in \hom_{M_\Delta^P}(M', X)$. Define $\theta : M \oplus M' \to X$ by $\theta(m, m') = f(m) + g(m')$. If $\delta \in \Delta$, then we have

$$\theta(\pi(\delta)) = \theta(M_\delta \oplus M'_\delta) = f(M_\delta) + g(M'_\delta) \subseteq X_\delta + X_\delta = X_\delta.$$

Hence,

$$\theta \in \hom_{M_\Delta^P}(M \oplus M', X).$$

We have $\theta\alpha(m) = \theta(m, 0) = f(m)$ and $\theta\beta(m') = \theta(0, m') = g(m')$. If $\tau : M \oplus M' \to X$ is a morphism such that $\tau\alpha = f$ and $\tau\beta = g$, then $\tau(m, 0) = f(m)$ for all $m \in M$ and $\tau(0, m') = g(m')$ for all $m' \in M'$. One can easily prove that $\tau(m, m') = f(m) + g(m')$ and therefore, $\tau = \theta$. Analogously, we can prove that the category M_Δ^P has products. □

Corollary 2.3. *The category of abelian paragraded groups of type* Δ *is abelian.*

Let G, $G' \in G_\Delta^P$ and $M, M' \in M_\Delta^P$.

Definition 2.4. *For a homomorphism* $f : G \to G'$ *we say that it is* a morphism of grade δ *if*

$$(\forall \delta' \in \Delta)\ f(G_{\delta'}) \subseteq G'_\delta. \tag{2.1}$$

For a homomorphism $f : M \to M'$ *we say that it is* a morphism of grade δ *if*

$$(\forall \delta' \in \Delta)\ f(M_{\delta'}) \subseteq M'_{\delta'\delta}, \tag{2.2}$$

where $\delta'\delta$ *is minimal multiplication.*

Let us denote the set of all the morphisms of grade δ by $\hom(G, G')_\delta$ and by $\hom(M, M')_\delta$, respectively.

Lemma 2.5. *Let* G, G' *be commutative paragraded groups of type* Δ *and let* M, M' *be paragraded* R*-modules of type* Δ*. Then:*

a) *The set* $\hom(G, G')_\delta$ *is the subgroup of* $\hom(G, G')$;
b) *The set* $\hom(M, M')_\delta$ *is the subgroup of* $\hom(M, M')$.

Proof. **a)** Let $f, g \in \hom(G, G')_\delta$. Then,

$$(\forall \delta' \in \Delta)\ f(G_{\delta'}) \subseteq G'_\delta \wedge g(G_{\delta'}) \subseteq G'_\delta,$$

and hence, $f \cdot g(G_{\delta'}) = f(G_{\delta'})g(G_{\delta'}) \subseteq G'_\delta G'_\delta = G'_\delta$ and $f^{-1}(G_{\delta'}) = (f(G_{\delta'}))^{-1} \subseteq (G'_\delta)^{-1} = G'_\delta$. So, fg and f^{-1} belong to $\hom(G, G')_\delta$, hence, $\hom(G, G')_\delta < \hom(G, G')$.

b) Let $f, g \in \hom(M, M')_\delta$. Then,

$$(\forall \delta' \in \Delta)\ f(M_{\delta'}) \subseteq M'_{\delta'\delta} \wedge g(M_{\delta'}) \subseteq M'_{\delta'\delta},$$

and hence, $f+g(M_{\delta'}) = f(M_{\delta'}) + g(M_{\delta'}) \subseteq M'_{\delta'\delta} + M'_{\delta'\delta} = M'_{\delta'\delta}$ and $-f(M_{\delta'}) \subseteq (-M'_{\delta'\delta}) = M'_{\delta'\delta}$. So, $f+g$ and $-f$ belong to $\hom(M,M')_\delta$, hence, $\hom(M,M')_\delta < \hom(M,M')$. □

Now, consider the sets

$$\mathrm{HOM}(G,G') = \Big\langle \bigcup_{\delta\in\Delta} \hom(G,G')_\delta \Big\rangle \qquad \text{and}$$

$$\mathrm{HOM}(M,M') = \Big\langle \bigcup_{\delta\in\Delta} \hom(M,M')_\delta \Big\rangle.$$

Theorem 2.6. *Let G and M, M' be a commutative paragraded group and paragraded R-modules of type Δ, respectively, and let G' be a commutative extragraded group of type Δ. Then*

a) $\mathrm{HOM}(G,G')$ *is the commutative paragraded group of type* Δ;
b) $\mathrm{HOM}(M,M')$ *is the commutative paragraded group of type* Δ.

Proof. **a)** According to the previous Lemma, it is easy to establish that $\mathrm{HOM}(G,G')$ is a group. Now we set off to prove that it is paragraded. Define the map

$$\pi : \Delta \to \mathrm{Sg}(\mathrm{HOM}(G,G'))$$

by

$$\pi(\delta) = \hom(G,G')_\delta.$$

We first need to consider $\pi(0)$ i.e. the set $\hom(G,G')_0$. It is the set of all morphisms $f : G \to G'$ such that $f(G_{\delta'}) \subseteq G'_0$ for all $\delta' \in \Delta$. We know (see *i)*) that $G'_0 = \{e\}$, and hence $f(G_{\delta'}) = \{e\}$ for all $\delta' \in \Delta$, i.e. $\hom(G,G')_0 = \{f_0\}$, where by f_0 we denoted the map $g \to e$ $(g \in G)$. Now, let $\delta_1 < \delta_2$. Then, $\pi(\delta_1) = \hom(G,G')_{\delta_1}$ and $\pi(\delta_2) = \hom(G,G')_{\delta_2}$. Take $f \in \hom(G,G')_{\delta_1}$. Then, for all $\delta' \in \Delta$, we have $f(G_{\delta'}) \subseteq G'_{\delta_1}$. From $\delta_1 < \delta_2$ it follows that $G'_{\delta_1} \subseteq G'_{\delta_2}$, according to *i)*. Hence, $f(G_{\delta'}) \subseteq G'_{\delta_2}$ for all $\delta' \in \Delta$, so $f \in \hom(G,G')_{\delta_2}$ and $\hom(G,G')_{\delta_1} \subseteq \hom(G,G')_{\delta_2}$.

Let us now consider the subset $\theta \subset \Delta$. We wonder what $\bigcap_{\delta\in\theta} \hom(G,G')_\delta$ is. If $f \in \bigcap_{\delta\in\theta} \hom(G,G')_\delta$, then $f(G_{\delta'}) \subseteq G'_\delta$ for all $\delta' \in \Delta$ and for all $\delta \in \theta$. Hence, $f(G_{\delta'}) \subseteq \bigcap_{\delta\in\theta} G'_\delta$ for all $\delta' \in \Delta$ i.e. $f(G_{\delta'}) \subseteq G'_{\inf\theta}$ for all $\delta' \in \Delta$ according to *ii)*. Thus, $\bigcap_{\delta\in\theta} \hom(G,G')_\delta = \hom(G,G')_{\inf\theta}$.

Denote by H the set $\bigcup_{\delta\in\Delta} \hom(G,G')_\delta$. Take two elements $f, g \in H$. Then there are δ_1 and δ_2 from Δ such that $f \in \hom(G,G')_{\delta_1}$ and $g \in \hom(G,G')_{\delta_2}$. Let h be $gfg^{-1}f^{-1}$. We wish to prove that $h \in H$. For all $\delta' \in \Delta$ we have

$$\begin{aligned} h(G_{\delta'}) &= g(G_{\delta'})f(G_{\delta'})g^{-1}(G_{\delta'})f^{-1}(G_{\delta'}) \\ &\subseteq G'_{\delta_2}G'_{\delta_1}(G'_{\delta_2})^{-1}(G'_{\delta_1})^{-1}. \end{aligned}$$

We will now show how we can avoid the commutativity of G' in this case. Since G'_{δ_1} and G'_{δ_2} are normal subgroups (see [7]), it follows that

$$h(G_{\delta'}) \subseteq G'_{\delta_1} \cap G'_{\delta_2} \overset{ii)}{=} G'_{\inf(\delta_1,\delta_2)} \ (\delta' \in \Delta). \tag{2.3}$$

By (2.3), $h \in \hom(G, G')_{\inf(\delta_1,\delta_2)} \subseteq H$, as we claimed. Now, we prove that $\delta(h) \leq \inf(\delta(f), \delta(g))$. But this follows from the fact that $\delta(h) \leq \inf(\delta_1, \delta_2)$.

The set $\mathrm{HOM}(G, G')$ is generated by H as it can be seen from the construction of the set $\mathrm{HOM}(G, G')$.

Assume now that we have a subset $A \subseteq H$ such that for all $f, g \in A$ there exists $\delta \in \Delta$ such that $fg \in \hom(G, G')_\delta$. We wish to prove that there exists $\delta_1 \in \Delta$ such that $A \subseteq \hom(G, G')_{\delta_1}$. For any $\delta' \in \Delta$ choose $x \in G_{\delta'}$. Then $f(x)g(x) \in H_{G'}$ and so by $v)$, $f(x) \in G'_{\delta_1}$ for some $\delta_1 \in \Delta$. The map f was arbitrary, so $A \subseteq \hom(G, G')_{\delta_1}$ as we wished to prove.

Let $\delta_1, \ldots, \delta_s \in \Delta_p$ be mutually incomparable and $f_1 \ldots f_s = f'_1 \ldots f'_s$, where $f_i, f'_i \in \hom(G, G')_{\delta_i}$ $(i = \overline{1, s})$. That means that for arbitrarily chosen $x \in H_G$ one has

$$f_1(x) \ldots f_s(x) = f'_1(x) \ldots f'_s(x)$$

and $f_i(x), f'_i(x) \in G'_{\delta_i}$ $(i = \overline{1, s})$. Since $f_i(x), f'_i(x) \in G'_{\delta_i} \subseteq G'$ and G' is extragraded, by $vi)$ it follows that $\delta(f_i(x)^{-1} f'_i(x)) < \delta_i$ and hence $f_i^{-1} f'_i \in \hom(G, G')_{\delta_i}$ $(i = \overline{1, s})$.

We have proven so far that six axioms of extragraduation are satisfied, hence $\mathrm{HOM}(G, G')$ is an extragraded group, but since every extragraduation is a paragraduation, as is stated in Theorem 1.4, $\mathrm{HOM}(G, G')$ is a paragraded group.

b) It is easy to establish that $\mathrm{HOM}(M, M')$ is a commutative group. This group is a paragraded group, since $\mathrm{HOM}(M, M')$ is a homogeneous subgroup of $\hom(M, M')$, and $\hom(M, M')$ is a paragraded group according to [7]. □

Remark 2.7. From the last proof we notice that $\mathrm{HOM}(G, G')$ is a post-paragraded group if $G, G' \in G^P_\Delta$.

Remark 2.8. In the same way that we defined the categories G^P_Δ and M^P_Δ, we define the categories of extragraded groups and modules with the set of grades Δ and denote it by G^E_Δ and M^E_Δ, respectively.

In what follows, all objects are assumed to be commutative.

Let us now observe the map $\mathrm{HOM}(G_1, -) : G^E_\Delta \to G^P_\Delta$ which sends each $G_2 \in G^E_\Delta$ to $\mathrm{HOM}(G_1, G_2) \in G^P_\Delta$ and each $g : G_2 \to G'_2$ from $\hom_{G^P_\Delta}(G_2, G'_2)$, where $G_2, G'_2 \in G^E_\Delta$, to a morphism

$$\mathrm{HOM}(1_{G_1}, g) : \mathrm{HOM}(G_1, G_2) \to \mathrm{HOM}(G_1, G'_2)$$

defined as follows. For each $x \in \mathrm{HOM}(G_1, G_2)$ we have

$$x = \langle f_\delta \mid \delta \in \Delta \rangle,$$

where $f_\delta \in \mathrm{hom}(G_1, G_2)_\delta$. So, let

$$\mathrm{HOM}(1_{G_1}, g)(x) = \langle g \circ f_\delta \mid \delta \in \Delta \rangle.$$

Then $\mathrm{HOM}(G_1, -)$ represents a covariant functor. First,

$$\mathrm{HOM}(1_{G_1}, 1_{G_2})(x) = \langle f_\delta \mid \delta \in \Delta \rangle = x.$$

Now, let $g_1, g_2 \in \mathrm{hom}_{G^P_\Delta}(G_2, G_2')$ and let g_1, g_2 be composable. Then we have

$$\begin{aligned} \mathrm{HOM}(1_{G_1}, g_1) \circ \mathrm{HOM}(1_{G_1}, g_2)(x) &= \mathrm{HOM}(1_{G_1}, g_1)\langle g_2 \circ f_\delta \mid \delta \in \Delta \rangle \\ &= \langle g_1 \circ g_2 \circ f_\delta \mid \delta \in \Delta \rangle \\ &= \mathrm{HOM}(1_{G_1}, g_1 \circ g_2)(x), \end{aligned}$$

for all $x \in \mathrm{HOM}(G_1, G_2)$.

Similarly, observe the map $\mathrm{HOM}(-, G_2) : {G^P_\Delta}^{\mathrm{op}} \to G^P_\Delta$ which sends each $G_1 \in {G^P_\Delta}^{\mathrm{op}}$ to $\mathrm{HOM}(G_1, G_2) \in G^P_\Delta$ and each $f : G_1 \to G_1'$ from $\mathrm{hom}_{G^P_\Delta}(G_1, G_1')$ to a morphism

$$\mathrm{HOM}(f, 1_{G_2}) : \mathrm{HOM}(G_1, G_2) \to \mathrm{HOM}(G_1', G_2)$$

defined by

$$\mathrm{HOM}(f, 1_{G_2})(x) = \langle f_\delta \circ f \mid \delta \in \Delta \rangle,$$

for all $x = \langle f_\delta \mid \delta \in \Delta \rangle \in \mathrm{HOM}(G_1, G_2)$. The proof that $\mathrm{HOM}(-, G_2)$ is a contravariant functor is similar to the one that $\mathrm{HOM}(G_1, -)$ is a covariant functor and we shall omit it. Also, it is easy to verify that the following diagram commutes

$$\begin{tikzcd} \mathrm{HOM}(G_1, G_2) \arrow[r, "\mathrm{HOM}(1_{G_1}, g)"] \arrow[d, "\mathrm{HOM}(f, 1_{G_2})"'] \arrow[dr, "\mathrm{HOM}(f, g)"] & \mathrm{HOM}(G_1, G_2') \arrow[d, "\mathrm{HOM}(f, 1_{G_2'})"] \\ \mathrm{HOM}(G_1', G_2) \arrow[r, "\mathrm{HOM}(1_{G_1'}, g)"'] & \mathrm{HOM}(G_1', G_2') \end{tikzcd}$$

Thus, $\mathrm{HOM}(-, -) : {G^P_\Delta}^{\mathrm{op}} \times G^E_\Delta \to G^P_\Delta$ is a bifunctor. Let us fix $G_1 \in G^P_\Delta$ and observe the functor $\mathrm{HOM}(G_1, -) : G^E_\Delta \to G^P_\Delta$. Suppose that the following sequence

$$0 \longrightarrow G_2 \xrightarrow{g} G_2' \xrightarrow{g'} G_2'' \longrightarrow 0 \qquad (2.4)$$

is exact, where G_2, G_2', $G_2'' \in G_\Delta^E$. If we apply the functor $\mathrm{HOM}(G_1, -)$ to the sequence (2.4), we wish to prove that the sequence

$$0 \longrightarrow \mathrm{HOM}(G_1, G_2) \xrightarrow{\mathrm{HOM}(1_{G_1}, g)} \mathrm{HOM}(G_1, G_2') \xrightarrow{\mathrm{HOM}(1_{G_1}, g')} \mathrm{HOM}(G_1, G_2'')$$

is exact. Let $x = \langle f_\delta \mid \delta \in \Delta\rangle \in \ker \mathrm{HOM}(1_{G_1}, g)$. Then $\mathrm{HOM}(1_{G_1}, g)(x) = \langle g \circ f_\delta \mid \delta \in \Delta\rangle$. From $x \in \ker \mathrm{HOM}(1_{G_1}, g)$ and the exactness of the sequence (2.4), it follows that $x = f_0$ which proves the injectivity of $\mathrm{HOM}(G_1, -)$. If $y \in \mathrm{im}\ \mathrm{HOM}(1_{G_1}, g)$, then $y = \langle g \circ f_\delta \mid \delta \in \Delta\rangle$ for some $\langle f_\delta \mid \delta \in \Delta\rangle \in \mathrm{HOM}(G_1, G_2)$. By exactness of (2.4), $\mathrm{im}\ g \subseteq \ker g'$, so

$$\mathrm{HOM}(1_{G_1}, g')(y) = \langle g' \circ g \circ f_\delta \mid \delta \in \Delta\rangle = 0.$$

Hence, $y \in \ker \mathrm{HOM}(1_{G_1}, g')$. Now, if $y \in \ker \mathrm{HOM}(1_{G_1}, g')$, then if $a \in G_1$, we have $y(a) = \langle f_\delta'(a) \mid \delta \in \Delta\rangle$, where $f_\delta' \in \mathrm{hom}(G_1, G_2')_\delta$. Note that $f_\delta'(a) \in \ker g' = \mathrm{im}\ g$, for all $\delta \in \Delta$, so $f_\delta'(a) = g(b)$ for some $b = f_\delta(a) \in G_2$, where $f_\delta \in \mathrm{hom}(G_1, G_2)_\delta$. This proves that $y \in \mathrm{im}\ \mathrm{HOM}(1_{G_1}, g)$. Thus, $\mathrm{HOM}(G_1, -) : G_\Delta^P \to G_\Delta^P$ is a left exact functor. Similarly, one can prove that $\mathrm{HOM}(-, G_2) : G_\Delta^{P\,\mathrm{op}} \to G_\Delta^P$ is also a left exact functor. This means that the following proposition holds.

Proposition 2.9. *A bifunctor* $\mathrm{HOM}(-,-) : G_\Delta^{P\,\mathrm{op}} \times AbG_\Delta^E \to AbG_\Delta^P$ *is left exact.*

Analogously, one can prove that the following proposition holds as well.

Proposition 2.10. *A bifunctor* $\mathrm{HOM}(-,-) : M_\Delta^{P\,\mathrm{op}} \times M_\Delta^P \to AbG_\Delta^P$ *is left exact.*

Lemma 2.11. *Let M be a paragraded R-module with the set of grades Δ. Then there exists a free paragraded R-module.*

Proof. Let M' be a free R-module with base M. If we define the map $\pi' : \Delta \to \mathrm{Sg}(M', +)$ via $\pi'(\delta) = M_\delta'$, where M_δ' denotes the subgroup generated by the set M_δ, then it can be proven that π' is paragraduation of M'. □

Corollary 2.12. *The category M_Δ^P has enough injective and projective objects.*

The proofs of the following propositions are similar to the proofs in the case of abstract modules.

Proposition 2.13. *Let M be a paragraded left R-module of type Δ. Then:*

a) *M is projective iff the functor* $\mathrm{HOM}(M,-)$ *is exact;*
b) *M is injective iff the functor* $\mathrm{HOM}(-,M)$ *is exact.*

Proposition 2.14. *Let M' be a projective paragraded left R-module of type Δ and $\overline{M}$ an injective paragraded left R-module of type Δ. Then:*

a) $\mathrm{EXT}^n(M',M) = \{0\}$, *for all $n \geq 1$ and for every paragraded left R-module M of type Δ;*
b) $\mathrm{EXT}^n(M,\overline{M}) = \{0\}$, *for all $n \geq 1$ and for every paragraded left R-module M of type Δ;*

where EXT *designates the right derived functor of* HOM.

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