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INVERSE PROBLEMS FOR STURM-LIOUVILLE-TYPE DIFFERENTIAL EQUATION WITH THE FIXED DELAY UNDER DISCONTINUITY CONDITIONS

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ABSTRACT. In this manuscript, we consider boundary value problems for the equation with the fixed delay and transmission conditions on $\frac{\pi}{2}$. We study the case when all parameters in transmission conditions are known. We will prove theorem of uniqueness from two spectra, first with Dirichlet-Dirichlet boundary conditions and second with Dirichlet-Neumann boundary condition. Also, we implicitly give the algorithm for determining the potential from these two spectra.

1. INTRODUCTION

Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. These problems are often in connection with discontinuous material properties.

Inverse spectral problems are problems in which we study operator from some spectral characteristics of this operator. We will use spectra of the operators for reconstruction of parameters of the operators.

The Sturm-Liouville-type operators are generated by second order differential expression and boundary conditions (see [1] and references therein). In this paper we study Sturm-Liouville operators with fixed delay under Dirichlet-Dirichlet and Dirichlet-Neumann boundary conditions. Some result of the inverse problems for operators with constant delay can be found in [2]-[8]. In the papers [10]-[11] authors study inverse problem for operators of the Sturm-Liouville-type with discontinuity in interior points. Latest result in the inverse problems of operators with constant delay, with discontinuity in interior points, is given in [9]. In the paper [9] authors study these problems for unknown delay from $\left(\frac{\pi}{2}, \pi\right)$ under Dirichlet-Dirichlet and Dirichlet-Neumann boundary conditions. We study the problem when $\tau = \frac{\pi}{2}$, which is defined as follows:

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$$-y''(x)+q(x)y\left(x-\frac{\pi}{2}\right)=\lambda y(x),\, x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right), \tag{1.1}$$
$$y(0)=0, \tag{1.2}$$
$$y(\pi)=0, \tag{1.3}$$
$$y'(\pi)=0, \tag{1.4}$$

with jump conditions

$$y\left(\frac{\pi}{2}+0\right)=by\left(\frac{\pi}{2}-0\right), y'\left(\frac{\pi}{2}+0\right)=b^{-1}y'\left(\frac{\pi}{2}-0\right)+cy\left(\frac{\pi}{2}-0\right). \tag{1.5}$$

Let L_0 is boundary value problem defined with (1.1, 1.2, 1.3, 1.5) and L_1 is boundary value problem defined with (1.1, 1.2, 1.4, 1.5). The following notation is used: λ is the spectral parameter, $q(x)$ is a complex-valued function which we call potential, such that $q\in L^2(0,\pi)$, and $q(x)\equiv 0$ for $x\in[0,\tau]$ and b,c are real parameters which we assume that are known.
The spectra of L_0 and L_1 are countable. We will prove that the potential q is uniquely determined from the spectra of L_0 and L_1. Let $(\lambda_n)_{n=1}^{\infty}$ be the eigenvalues of L_0 and $(\mu_n)_{n=1}^{\infty}$ be the eigenvalues of L_1.
The inverse problem is to prove that $q(x)$ is uniquely determined from $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, and find $q(x)$ from $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$.

2. Preliminaries

Let the function $Y_1(x)$ be the solution of the differential equation (1.1) on the interval $\left(0,\frac{\pi}{2}\right)$ which satisfying initial conditions $Y_1(0)=0, Y_1'(0)=1$. Since the differential equation (1.1) has the form $y''(x)+\lambda y(x)=0$ on the interval $\left(0,\frac{\pi}{2}\right)$, we have $Y_1(x)=\frac{sinzx}{z}$ where $\lambda=z^2$.
The differential equation (1.1) on the interval $\left(\frac{\pi}{2},\pi\right)$ has the form

$$-y''(x)+q(x)\frac{\sin z(x-\frac{\pi}{2})}{z}=z^2y(x).$$

Using method of variation of parameters we have general solution of the differential equation (1.1) on the interval $\left(\frac{\pi}{2},\pi\right)$

$$Y_2(x)=\frac{1}{z^2}\int_{\frac{\pi}{2}}^{x}q(t)sinz\left(t-\frac{\pi}{2}\right)sinz(x-t)dt+C_1sinzx+C_2coszx.$$

Since

$$Y_1\left(\frac{\pi}{2}-0\right)=\frac{sin\frac{z\pi}{2}}{z}, Y_1'\left(\frac{\pi}{2}-0\right)=cos\frac{z\pi}{2},$$
$$Y_2\left(\frac{\pi}{2}+0\right)=C_1sin\frac{z\pi}{2}+C_2cos\frac{z\pi}{2}, Y_2'\left(\frac{\pi}{2}+0\right)=zC_1cos\frac{z\pi}{2}-zC_2sin\frac{z\pi}{2},$$

from jump conditions (1.5) we have

$$C_1 sin\frac{z\pi}{2} + C_2 cos\frac{z\pi}{2} = b\frac{sin\frac{z\pi}{2}}{z},$$
$$zC_1 cos\frac{z\pi}{2} - zC_2 sin\frac{z\pi}{2} = b^{-1}cos\frac{z\pi}{2} + c\frac{sin\frac{z\pi}{2}}{z}.$$

This system of linear equations has unique solution

$$C_1 = \frac{b}{z}sin^2\frac{z\pi}{2} + \frac{1}{bz}cos^2\frac{z\pi}{2} + \frac{c}{2z^2}sinz\pi,$$
$$C_2 = -\frac{1}{2bz}sinz\pi - \frac{c}{z^2}sin^2\frac{z\pi}{2} + \frac{b}{2z}sinz\pi.$$

From (1.3) we have characteristic function of boundary value problem L_0

$$\Delta_0(\lambda) = Y_2(\pi) = \frac{1}{z^2}\int_{\frac{\pi}{2}}^{\pi} q(t)sinz\left(t-\frac{\pi}{2}\right)sinz(\pi-t)dt+ \tag{2.1}$$

$$+\left(\frac{b-b^{-1}}{z}sin^2\frac{z\pi}{2} + \frac{1}{bz} + \frac{c}{2z^2}sinz\pi\right)sinz\pi + \left(\frac{b-b^{-1}}{2z}sinz\pi - \frac{c}{z^2}sin^2\frac{z\pi}{2}\right)cosz\pi,$$

and from (1.4) we have characteristic function of boundary value problem L_1

$$\Delta_1(\lambda) = Y_2'(\pi) = \frac{1}{z}\int_{\frac{\pi}{2}}^{\pi} q(t)sinz\left(t-\frac{\pi}{2}\right)cosz(\pi-t)dt+ \tag{2.2}$$

$$+\left((b-b^{-1})sin^2\frac{z\pi}{2} + \frac{1}{b} + \frac{c}{2z}sinz\pi\right)cosz\pi + \left(\frac{-b+b^{-1}}{2}sinz\pi + \frac{c}{z}sin^2\frac{z\pi}{2}\right)sinz\pi$$

After elementary transformation we obtain

$$\Delta_0(\lambda) = \frac{1}{z^2}\int_{\frac{\pi}{2}}^{\pi} q(t)sinz\left(t-\frac{\pi}{2}\right)sinz(\pi-t)dt + \frac{b+b^{-1}}{2z}sinz\pi + \frac{c}{2z^2}(1-cosz\pi), \tag{2.3}$$

$$\Delta_1(\lambda) = \frac{1}{z}\int_{\frac{\pi}{2}}^{\pi} q(t)sinz\left(t-\frac{\pi}{2}\right)cosz(\pi-t)dt - \frac{b-b^{-1}}{2} + \frac{b+b^{-1}}{2}cosz\pi + \frac{c}{2z}sinz\pi \tag{2.4}$$

The functions $\Delta_0(\lambda), \Delta_1(\lambda)$ are entire in λ of order $1/2$, (see [9]). It is clear that the set of zeros of functions $\Delta_0(\lambda), \Delta_1(\lambda)$ is equivalent to the spectrum of boundary spectral problems L_0, L_1, respectively (see [1]). By using Hadamard's factorization theorem we conclude that spectra uniquely determine functions $\Delta_0(\lambda), \Delta_1(\lambda)$, (see [9], *Lemma* 2.1). Asymptotic formula for the eigenvalues and characteristic function $\Delta_0(\lambda)$ are known (see [9]).

3. Main results

We consider the case with fixed delay $\tau = \frac{\pi}{2}$. Since characteristic function $\Delta_0(\lambda)$ have the same form like characteristic function $\Delta_0(\lambda)$ in [9], from *Lemma* 3.2 in [9] we have that the integral $\int_\tau^\pi q(t)dt$ is determined by spectrum $(\lambda_n)_{n=1}^\infty$. By using $q(x) \equiv 0$ for $x \in [0,\tau]$, we obtain $\int_0^\pi q(t)dt = \int_\tau^\pi q(t)dt$. In this way, the first Fourier coefficient of the potential q on the $[0,\pi]$ is uniquely determined from the spectrum.

Now we will prove that the other Fourier coefficients of the potential q are uniquely determined. We introduce notation $a_n = \int_0^\pi q(t)\cos 2nt dt$ and $b_n = \int_0^\pi q(t)\sin 2nt dt$. So, we finally come to our main result.

Theorem 3.1. *Let $(\lambda_n)_{n=1}^\infty$ and $(\mu_n)_{n=1}^\infty$ be the spectra of boundary spectral problems $L_k, k = 0,1$ respectively, then potential q is uniquely determined by $(\lambda_n)_{n=1}^\infty$ and $(\mu_n)_{n=1}^\infty$.*

Proof. We introduce notation

$$F_0(z) = z^2\left[\Delta_0(\lambda) - \left(\frac{b+b^{-1}}{2z} sinz\pi + \frac{c}{2z^2}(1-cosz\pi)\right)\right]$$

$$F_1(z) = z\left[\Delta_1(\lambda) - \left(-\frac{b-b^{-1}}{2} + \frac{b+b^{-1}}{2} cosz\pi + \frac{c}{2z} sinz\pi\right)\right]$$

Since functions $\Delta_0(\lambda), \Delta_1(\lambda)$ are ordered from spectra $(\lambda_n)_{n=1}^\infty$ and $(\mu_n)_{n=1}^\infty$ we conclude that functions $F_0(z)$ and $F_1(z)$ are also ordered from these spectra. From (2.3) and (2.4) we have

$$F_0(z) = \int_{\frac{\pi}{2}}^{\pi} q(t) sinz\left(t - \frac{\pi}{2}\right) sinz(\pi - t)dt,$$

$$F_1(z) = \int_{\frac{\pi}{2}}^{\pi} q(t) sinz\left(t - \frac{\pi}{2}\right) cosz(\pi - t)dt.$$

Using trigonometric transformations: product of trigonometric functions to sum and addition formulas we have

$$F_0(z) = -\frac{\cos\frac{z\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt + \frac{\cos\frac{3z\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)cos2ztdt + \frac{\sin\frac{3z\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)sin2ztdt,$$

and

$$F_1(z) = \frac{\sin\frac{z\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt + \frac{\cos\frac{3z\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)sin2ztdt - \frac{\sin\frac{3z\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)cos2ztdt.$$

Now we put $z = n, n \in N$ and using $a_n = \int_0^\pi q(t)\cos 2nt dt$, $b_n = \int_0^\pi q(t)\sin 2nt dt$ we have

$$F_0(n) + \frac{\cos\frac{n\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt = \frac{\cos\frac{3n\pi}{2}}{2}a_n + \frac{\sin\frac{3n\pi}{2}}{2}b_n,$$

$$F_1(n) - \frac{\sin\frac{n\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt = -\frac{\sin\frac{3n\pi}{2}}{2}a_n + \frac{\cos\frac{3n\pi}{2}}{2}b_n.$$

This is linear system of two variables a_n and b_n with determinant $D = \frac{1}{4} \neq 0$, and we have unique solution

$$a_n = 2\cos\frac{3n\pi}{2}\left(F_0(n) + \frac{\cos\frac{n\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt\right) - 2\sin\frac{3n\pi}{2}\left(F_1(n) - \frac{\sin\frac{n\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt\right),$$

$$b_n = 2\cos\frac{3n\pi}{2}\left(F_1(n) - \frac{\sin\frac{n\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt\right) + 2\sin\frac{3n\pi}{2}\left(F_0(n) + \frac{\cos\frac{n\pi}{2}}{2}\int_{\frac{\pi}{2}}^{\pi} q(t)dt\right).$$

Since $\int_{\frac{\pi}{2}}^{\pi} q(t)dt$ and functions F_0, F_1 are ordered from the spectra $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$, Fourier coefficients a_n and b_n are also ordered from these spectra. Since $q \in L^2[0,\pi]$ is complex valued function, we have

$$q(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{2inx}$$

where is $c_n = \frac{1}{\pi}a_n - \frac{i}{\pi}b_n$. We finally conclude that potential q is uniquely determined from the spectra $(\lambda_n)_{n=1}^{\infty}$ and $(\mu_n)_{n=1}^{\infty}$. □

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