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FRACTIONAL INTEGRAL INEQUALITIES INVOLVING CONVEXITY

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Dedicated to Professor Mustafa Kulenović on the occasion of his 60th birthday

Abstract. Here we present general integral inequalitites involving convex and increasing functions applied to products of functions. As specific applications we derive a wide range of fractional inequalities of Hardy type. These involve the left and right: Erdélyi-Kober fractional integrals, mixed Riemann-Liouville fractional multiple integrals. Next we produce multivariate Poincaré type fractional inequalitites involving left fractional radial derivatives of Canavati type, Riemann-Liouville and Caputo types. The exposed inequalities are of L_p type, $p \geq 1$, and exponential type.

1. Introduction

We start with some facts about fractional derivatives needed in the sequel, for more details see, for instance [1], [10].

Let $a < b$, $a, b \in \mathbb{R}$. By $C^N([a,b])$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order N, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^N([a,b])$, we denote the space of all functions g with $g^{(N-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$, we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$), and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N},\ n \geq \alpha\}$). By $L_1(a,b)$, we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with the definition of the Riemann-Liouville fractional integrals, see [13]. Let $[a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order $\alpha > 0$

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are defined by

$$\left(I_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}f(t)(x-t)^{\alpha-1}dt,\quad (x>a), \tag{1}$$

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}f(t)(t-x)^{\alpha-1}dt,\quad (x<b), \tag{2}$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and the right-sided fractional integrals. We mention some properties of the operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ of order $\alpha>0$, see also [16]. The first result yields that the fractional integral operators $I_{a+}^{\alpha}f$ and $I_{b-}^{\alpha}f$ are bounded in $L_p(a,b)$, $1\le p\le\infty$, that is

$$\left\|I_{a+}^{\alpha}f\right\|_p\le K\|f\|_p,\qquad \left\|I_{b-}^{\alpha}f\right\|_p\le K\|f\|_p, \tag{3}$$

where

$$K=\frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}. \tag{4}$$

Inequality (3), that is the result involving the left-sided fractional integral, was proved by H. G. Hardy in one of his first papers, see [11]. He did not write down the constant, but the calculation of the constant was hidden inside his proof.

Next we follow [12].

Let $(\Omega_1,\Sigma_1,\mu_1)$ and $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with positive σ-finite measures, and let $k:\Omega_1\times\Omega_2\to\mathbb{R}$ be a nonnegative measurable function, $k(x,\cdot)$ measurable on Ω_2 and

$$K(x)=\int_{\Omega_2}k(x,y)\,d\mu_2(y),\quad x\in\Omega_1. \tag{5}$$

We suppose that $K(x)>0$ a.e. on Ω_1, and by a weight function (shortly: a weight), we mean a nonnegative measurable function on the actual set. Let the measurable functions $g:\Omega_1\to\mathbb{R}$ with the representation

$$g(x)=\int_{\Omega_2}k(x,y)f(y)\,d\mu_2(y), \tag{6}$$

where $f:\Omega_2\to\mathbb{R}$ is a measurable function.

Theorem 1. ([12]) *Let u be a weight function on Ω_1, k a nonnegative measurable function on $\Omega_1\times\Omega_2$, and K be defined on Ω_1 by (5). Assume that the function $x\mapsto u(x)\frac{k(x,y)}{K(x)}$ is integrable on Ω_1 for each fixed $y\in\Omega_2$. Define ν on Ω_2 by*

$$\nu(y):=\int_{\Omega_1}u(x)\frac{k(x,y)}{K(x)}d\mu_1(x)<\infty. \tag{7}$$

If $\Phi : [0, \infty) \to \mathbb{R}$ is convex and increasing function, then the inequality

$$\int_{\Omega_1} u(x)\,\Phi\left(\left|\frac{g(x)}{K(x)}\right|\right) d\mu_1(x) \le \int_{\Omega_2} \nu(y)\,\Phi(|f(y)|)\, d\mu_2(y) \tag{8}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that:

(i) *$f, \Phi(|f|)$ are both $k(x,y)\,d\mu_2(y)$-integrable, μ_1 -a.e. in $x \in \Omega_1$,*

(ii) *$\nu\Phi(|f|)$ is μ_2-integrable,*

and for all corresponding functions g given by (6).

Important assumptions (i) and (ii) are missing from Theorem 2.1. of [12].

In this article we use and generalize Theorem 1 for products of several functions and we give wide applications to Fractional Calculus.

2. Main results

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, and let $k_i : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be nonnegative measurable functions, $k_i(x, \cdot)$ measurable on Ω_2, and

$$K_i(x) = \int_{\Omega_2} k_i(x,y)\, d\mu_2(y), \quad \text{for any } x \in \Omega_1, \tag{9}$$

$i = 1, \ldots, m$. We assume that $K_i(x) > 0$ a.e. on Ω_1, and the weight functions are nonnegative measurable functions on the related set.

We consider measurable functions $g_i : \Omega_1 \to \mathbb{R}$ with the representation

$$g_i(x) = \int_{\Omega_2} k_i(x,y)\, f_i(y)\, d\mu_2(y), \tag{10}$$

where $f_i : \Omega_2 \to \mathbb{R}$ are measurable functions, $i = 1, \ldots, m$.

Here u stands for a weight function on Ω_1.

The first introductory result is proved for $m = 2$.

Theorem 2. *Assume that the functions $(i = 1, 2)$ $x \mapsto \left(u(x)\frac{k_i(x,y)}{K_i(x)}\right)$ are integrable on Ω_1, for each fixed $y \in \Omega_2$. Define u_i on Ω_2 by*

$$u_i(y) := \int_{\Omega_1} u(x)\frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{11}$$

Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Let the functions $\Phi_1, \Phi_2 : \mathbb{R}_+ \to \mathbb{R}_+$, be convex and increasing. Then

$$\int_{\Omega_1} u(x)\,\Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)\Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x) \le$$

$$\left(\int_{\Omega_2} u_1(y)\,\Phi_1(|f_1(y)|)^p\, d\mu_2(y)\right)^{\frac{1}{p}}\left(\int_{\Omega_2} u_2(y)\,\Phi_2(|f_2(y)|)^q\, d\mu_2(y)\right)^{\frac{1}{q}}, \tag{12}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ $(i = 1, 2)$ *such that*

(i) f_1, $\Phi_1(|f_1|)^p$ *are both* $k_1(x, y)\, d\mu_2(y)$*-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(ii) $f_2, \Phi_2(|f_2|)^q$ *are both* $k_2(x, y)\, d\mu_2(y)$ *-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$,

(iii) $u_1\Phi_1(|f_1|)^p$, $u_2\Phi_2(|f_2|)^q$, *are both* μ_2*-integrable,*

and for all corresponding functions g_i $(i = 1, 2)$ *given by* (10).

Proof. Notice that Φ_1, Φ_2 are continuous functions. Here we use Hölder's inequality. We have

$$\int_{\Omega_1} u(x)\, \Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right) \Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x)$$

$$= \int_{\Omega_1} u(x)^{\frac{1}{p}}\, \Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right) u(x)^{\frac{1}{q}}\, \Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right) d\mu_1(x) \tag{13}$$

$$\leq \left(\int_{\Omega_1} u(x)\, \Phi_1\left(\left|\frac{g_1(x)}{K_1(x)}\right|\right)^p d\mu_1(x)\right)^{\frac{1}{p}}$$

$$\cdot \left(\int_{\Omega_1} u(x)\, \Phi_2\left(\left|\frac{g_2(x)}{K_2(x)}\right|\right)^q d\mu_1(x)\right)^{\frac{1}{q}}$$

(notice here that Φ_1^p, Φ_2^q are convex, increasing and continuous nonnegative functions, and by Theorem 1 we get)

$$\leq \left(\int_{\Omega_2} u_1(y)\, \Phi_1(|f_1(y)|)^p\, d\mu_2(y)\right)^{\frac{1}{p}} \left(\int_{\Omega_2} u_2(y)\, \Phi_2(|f_2(y)|)^q\, d\mu_2(y)\right)^{\frac{1}{q}}. \tag{14}$$

□

The general result follows

Theorem 3. *Assume that the functions* $(i = 1, 2, \ldots, m \in \mathbb{N})$ $x \mapsto \big(u(x) \cdot \frac{k_i(x,y)}{K_i(x)}\big)$ *are integrable on* Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x)\, \frac{k_i(x, y)}{K_i(x)} d\mu_1(x) < \infty. \tag{15}$$

Let $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$*. Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$*,* $i = 1, \ldots, m$*, be convex and increasing. Then*

$$\int_{\Omega_1} u(x) \prod_{i=1}^{m} \Phi_i\left(\left|\frac{g_i(x)}{K_i(x)}\right|\right) d\mu_1(x) \leq$$

$$\prod_{i=1}^{m} \left(\int_{\Omega_2} u_i(y)\, \Phi_i(|f_i(y)|)^{p_i}\, d\mu_2(y)\right)^{\frac{1}{p_i}}, \tag{16}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ $(i = 1, \ldots, m)$ *such that*

(i) f_i, $\Phi_i\left(|f_i|\right)^{p_i}$ *are both* $k_i\left(x,y\right)d\mu_2\left(y\right)$*-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$, $i=1,\ldots,m$,

(ii) $u_i\Phi_i\left(|f_i|\right)^{p_i}$ *is* μ_2*-integrable,* $i=1,\ldots,m$,

and for all corresponding functions g_i $(i=1,\ldots,m)$ *given by* (10).

Proof. Notice that $\Phi_i, i=1,\ldots,m$, are continuous functions. Here we use the generalized Hölder's inequality. We have

$$\int_{\Omega_1} u\left(x\right)\prod_{i=1}^{m}\Phi_i\left(\left|\frac{g_i\left(x\right)}{K_i\left(x\right)}\right|\right)d\mu_1\left(x\right)$$

$$=\int_{\Omega_1}\prod_{i=1}^{m}\left(u\left(x\right)^{\frac{1}{p_i}}\Phi_i\left(\left|\frac{g_i\left(x\right)}{K_i\left(x\right)}\right|\right)\right)d\mu_1\left(x\right) \tag{17}$$

$$\leq\prod_{i=1}^{m}\left(\int_{\Omega_1}u\left(x\right)\Phi_i\left(\left|\frac{g_i\left(x\right)}{K_i\left(x\right)}\right|\right)^{p_i}d\mu_1\left(x\right)\right)^{\frac{1}{p_i}}$$

(notice here that $\Phi_i^{p_i}$, $i=1,\ldots,m$, are convex, increasing and continuous, nonnegative functions, and by Theorem 1 we get)

$$\leq\prod_{i=1}^{m}\left(\int_{\Omega_2}u_i\left(y\right)\Phi_i\left(|f_i\left(y\right)|\right)^{p_i}d\mu_2\left(y\right)\right)^{\frac{1}{p_i}}. \tag{18}$$

proving the claim. □

When $k\left(x,y\right):=k_1\left(x,y\right)=k_2\left(x,y\right)=\cdots=k_m\left(x,y\right)$, then $K\left(x\right):=K_1\left(x\right)=K_2\left(x\right)=\cdots=K_m\left(x\right)$, we get by Theorems 2, 3 the following:

Corollary 4. *Assume that the function* $x\mapsto\left(u\left(x\right)\frac{k(x,y)}{K(x)}\right)$ *is integrable on* Ω_1*, for each fixed* $y\in\Omega_2$*. Define* U *on* Ω_2 *by*

$$U\left(y\right):=\int_{\Omega_1}u\left(x\right)\frac{k\left(x,y\right)}{K\left(x\right)}d\mu_1\left(x\right)<\infty. \tag{19}$$

Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*. Let the functions* $\Phi_1,\Phi_2:\mathbb{R}_+\to\mathbb{R}_+$*, be convex and increasing. Then*

$$\int_{\Omega_1}u\left(x\right)\Phi_1\left(\left|\frac{g_1\left(x\right)}{K\left(x\right)}\right|\right)\Phi_2\left(\left|\frac{g_2\left(x\right)}{K\left(x\right)}\right|\right)d\mu_1\left(x\right)$$

$$\leq\left(\int_{\Omega_2}U\left(y\right)\Phi_1\left(|f_1\left(y\right)|\right)^{p}d\mu_2\left(y\right)\right)^{\frac{1}{p}}\left(\int_{\Omega_2}U\left(y\right)\Phi_2\left(|f_2\left(y\right)|\right)^{q}d\mu_2\left(y\right)\right)^{\frac{1}{q}}, \tag{20}$$

for all measurable functions $f_i:\Omega_2\to\mathbb{R}$ $(i=1,2)$ *such that*

(i) f_1, f_2, $\Phi_1\left(|f_1|\right)^p$, $\Phi_2\left(|f_2|\right)^q$ *are all* $k\left(x,y\right)d\mu_2\left(y\right)$*-integrable,* μ_1 *-a.e. in* $x\in\Omega_1$,

(ii) $U\Phi_1(|f_1|)^p$, $U\Phi_2(|f_2|)^q$, *are both* μ_2*-integrable,*

and for all corresponding functions g_i $(i=1,2)$ *given by* (10).

Corollary 5. *Assume that the function* $x \mapsto \left(u(x)\frac{k(x,y)}{K(x)}\right)$ *is integrable on* Ω_1*, for each fixed* $y \in \Omega_2$*. Define* U *on* Ω_2 *by*

$$U(y) := \int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)} d\mu_1(x) < \infty. \tag{21}$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$*. Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\dots,m$*, be convex and increasing. Then*

$$\int_{\Omega_1} u(x)\prod_{i=1}^m \Phi_i\left(\left|\frac{g_i(x)}{K(x)}\right|\right) d\mu_1(x)$$
$$\leq \prod_{i=1}^m \left(\int_{\Omega_2} U(y)\,\Phi_i(|f_i(y)|)^{p_i} d\mu_2(y)\right)^{\frac{1}{p_i}}, \tag{22}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $i=1,\dots,m$*, such that*

(i) f_i, $\Phi_i(|f_i|)^{p_i}$ *are both* $k(x,y)\,d\mu_2(y)$*-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$*, for all* $i = 1,\dots,m$*,*

(ii) $U\Phi_i(|f_i|)^{p_i}$ *is* μ_2*-integrable,* $i = 1,\dots,m$*,*

and for all corresponding functions g_i $(i = 1,\dots,m)$ *given by* (10).

Next we give two applications of Theorem 3.

Theorem 6. *Assume that the functions* $(i = 1,2,\dots,m \in \mathbb{N})$ $x \mapsto \big(u(x) \cdot\frac{k_i(x,y)}{K_i(x)}\big)$ *are integrable on* Ω_1*, for each fixed* $y \in \Omega_2$*. Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x)\frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{23}$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$*;* $\alpha_i \geq 1$*,* $i = 1,\dots,m$*.*

Then

$$\int_{\Omega_1} u(x)\left(\prod_{i=1}^m \left|\frac{g_i(x)}{K_i(x)}\right|^{\alpha_i}\right) d\mu_1(x)$$
$$\leq \prod_{i=1}^m \left(\int_{\Omega_2} u_i(y)\,|f_i(y)|^{\alpha_i p_i} d\mu_2(y)\right)^{\frac{1}{p_i}}, \tag{24}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $i = 1,\dots,m$*, such that*

(i) f_i, $|f_i|^{\alpha_i p_i}$ *are* $k_i(x,y)\,d\mu_2(y)$*-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$*,* $i = 1,\dots,m$*,*

(ii) $u_i|f_i|^{\alpha_i p_i}$ *is* μ_2*-integrable,* $i = 1,\dots,m$*,* *and for all corresponding functions* g_i $(i = 1, \dots, m)$ *given by* (10).

Theorem 7. *Assume that the functions* $(i = 1, 2, \dots, m \in \mathbb{N})$ $x \mapsto \left(u(x) \cdot \frac{k_i(x,y)}{K_i(x)}\right)$ *are integrable on* Ω_1, *for each fixed* $y \in \Omega_2$. *Define* u_i *on* Ω_2 *by*

$$u_i(y) := \int_{\Omega_1} u(x) \frac{k_i(x,y)}{K_i(x)} d\mu_1(x) < \infty. \tag{25}$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. *Then*

$$\int_{\Omega_1} u(x) \left(e^{\sum_{i=1}^m \left| \frac{g_i(x)}{K_i(x)} \right|} \right) d\mu_1(x) \leq \prod_{i=1}^m \left(\int_{\Omega_2} u_i(y) e^{p_i |f_i(y)|} d\mu_2(y) \right)^{\frac{1}{p_i}}, \tag{26}$$

for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, \dots, m$, *such that*

(i) f_i, $e^{p_i|f_i|}$ *are* $k_i(x,y)\,d\mu_2(y)$*-integrable,* μ_1 *-a.e. in* $x \in \Omega_1$, $i = 1, \dots, m$,

(ii) $u_i e^{p_i|f_i|}$ *is* μ_2*–integrable,* $i = 1, \dots, m$,

and for all corresponding functions g_i $(i = 1, \dots, m)$ *given by* (10).

We need

Definition 8. ([16]) *Let* (a, b), $0 \leq a < b < \infty$; $\alpha, \sigma > 0$. *We consider the left- and right-sided fractional integrals of order* α *as follows:*

1) *for* $\eta > -1$, *we define*

$$\left(I^{\alpha}_{a+;\sigma,\eta} f\right)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)\,dt}{\left(x^\sigma - t^\sigma\right)^{1-\alpha}}, \tag{27}$$

2) *for* $\eta > 0$, *we define*

$$\left(I^{\alpha}_{b-;\sigma,\eta} f\right)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)\,dt}{\left(t^\sigma - x^\sigma\right)^{1-\alpha}}. \tag{28}$$

These are the Erdélyi-Kober type fractional integrals.

We remind the Beta function

$$B(x, y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt, \tag{29}$$

for $\Re(x), \Re(y) > 0$, and the Incomplete Beta function

$$B(x; \alpha, \beta) = \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt, \tag{30}$$

where $0 < x \leq 1$; $\alpha, \beta > 0$.

We make

Remark 9. Regarding (27) we have

$$k(x,y)=\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\chi_{(a,x]}(y)\frac{y^{\sigma\eta+\sigma-1}}{(x^{\sigma}-y^{\sigma})^{1-\alpha}}, \tag{31}$$

$x,y\in(a,b)$, χ stands for the characteristic function.

Here

$$K(x)=\int_a^b k(x,t)\,dt=\left(I^{\alpha}_{a+;\sigma;\eta}1\right)(x)=\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\int_a^x\frac{t^{\sigma\eta+\sigma-1}}{(x^{\sigma}-t^{\sigma})^{1-\alpha}}dt \tag{32}$$

(setting $z=\frac{t}{x}$)

$$=\frac{\sigma}{\Gamma(\alpha)}\int_{\frac{a}{x}}^1 z^{\sigma\left((\eta+1)-\frac{1}{\sigma}\right)}(1-z^{\sigma})^{\alpha-1}dz$$

(setting $\lambda=z^{\sigma}$)

$$=\frac{1}{\Gamma(\alpha)}\int_{\left(\frac{a}{x}\right)^{\sigma}}^1\lambda^{\eta}(1-\lambda)^{\alpha-1}d\lambda. \tag{33}$$

Hence

$$K(x)=\frac{1}{\Gamma(\alpha)}\int_{\left(\frac{a}{x}\right)^{\sigma}}^1\lambda^{\eta}(1-\lambda)^{\alpha-1}d\lambda. \tag{34}$$

Indeed it is

$$K(x)=\left(I^{\alpha}_{a+;\sigma;\eta}(1)\right)(x)=\frac{B(\eta+1,\alpha)-B\left(\left(\frac{a}{x}\right)^{\sigma};\eta+1,\alpha\right)}{\Gamma(\alpha)}. \tag{35}$$

We also make

Remark 10. Regarding (28) we have

$$k(x,y)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\chi_{[x,b)}(y)\frac{y^{\sigma(1-\eta-\alpha)-1}}{(y^{\sigma}-x^{\sigma})^{1-\alpha}}, \tag{36}$$

$x,y\in(a,b)$. Here

$$K(x)=\int_a^b k(x,t)\,dt=\left(I^{\alpha}_{b-;\sigma;\eta}1\right)(x)=\frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)}\int_x^b\frac{t^{\sigma(1-\eta-\alpha)-1}}{(t^{\sigma}-x^{\sigma})^{1-\alpha}}dt \tag{37}$$

(setting $z=\frac{t}{x}$)

$$=\frac{\sigma}{\Gamma(\alpha)}\int_1^{\left(\frac{b}{x}\right)}(z^{\sigma}-1)^{\alpha-1}z^{\sigma(1-\eta-\alpha)-1}dz$$

(setting $\lambda=z^{\sigma}$, $1\le\lambda<\left(\frac{b}{x}\right)^{\sigma}$)

$$=\frac{1}{\Gamma(\alpha)}\int_1^{\left(\frac{b}{x}\right)^{\sigma}}(\lambda-1)^{\alpha-1}\lambda^{-\eta-\alpha}d\lambda \tag{38}$$

$$= \frac{1}{\Gamma(\alpha)} \int_1^{\left(\frac{b}{x}\right)^\sigma} \frac{1}{\lambda^{\eta+1}} \left(1 - \frac{1}{\lambda}\right)^{\alpha-1} d\lambda$$

(setting $w := \frac{1}{\lambda}$, $0 < \left(\frac{x}{b}\right)^\sigma < w \leq 1$)

$$= \frac{1}{\Gamma(\alpha)} \int_{\left(\frac{x}{b}\right)^\sigma}^1 w^{\eta-1} (1-w)^{\alpha-1} dw = \frac{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma; \eta, \alpha\right)\right)}{\Gamma(\alpha)}. \quad (39)$$

That is

$$K(x) = \left(I_{b-;\sigma;\eta}^\alpha (1)\right)(x) = \frac{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma; \eta, \alpha\right)\right)}{\Gamma(\alpha)}. \quad (40)$$

We give

Theorem 11. *Assume that the function*

$$x \mapsto \left(u(x) \frac{\chi_{(a,x]}(y)\, \sigma x^{-\sigma(\alpha+\eta)} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha} \left[B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1, \alpha\right)\right]}\right) \quad (41)$$

is integrable on (a,b), for each $y \in (a,b)$. Here $\alpha, \sigma > 0$, $\eta > -1$, $0 \leq a < b < \infty$. Define u_1 on (a,b) by

$$u_1(y) := \sigma y^{\sigma\eta+\sigma-1} \int_y^b \frac{u(x)\, x^{-\sigma(\alpha+\eta)} (x^\sigma - y^\sigma)^{\alpha-1}}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1, \alpha\right)\right)} dx < \infty. \quad (42)$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. Let the functions $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, \ldots, m$, be convex and increasing.

Then

$$\int_a^b u(x) \prod_{i=1}^m \Phi_i \left(\frac{\left|I_{a+;\sigma;\eta}^\alpha f_i(x)\right| \Gamma(\alpha)}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1, \alpha\right)\right)}\right) dx$$

$$\leq \prod_{i=1}^m \left(\int_a^b u_1(y)\, \Phi_i(|f_i(y)|)^{p_i} dy\right)^{\frac{1}{p_i}}, \quad (43)$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, \ldots, m$, such that

(i) *f_i, $\Phi_i(|f_i|)^{p_i}$ are both $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \chi_{(a,x]}(y) \frac{y^{\sigma\eta+\sigma-1} dy}{(x^\sigma - y^\sigma)^{1-\alpha}}$ -integrable, a.e. in $x \in (a,b)$, for all $i = 1, \ldots, m$,*

(ii) *$u_1 \Phi_i(|f_i|)^{p_i}$ is Lebesgue integrable, $i = 1, \ldots, m$,*

Proof. By Corollary 5. □

Remark 12. In (42), if we choose

$$u(x) = x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\alpha; \eta+1, \alpha\right)\right), \ x \in (a,b), \quad (44)$$

then

$$u_1(y) = \sigma y^{\sigma\eta+\sigma-1} \int_y^b x^{\sigma-1} (x^\sigma - y^\sigma)^{\alpha-1} dx$$

(setting $w := x^\sigma$, $\frac{dw}{dx} = \sigma x^{\sigma-1}$, $dx = \frac{dw}{\sigma x^{\sigma-1}}$)

$$= y^{\sigma\eta+\sigma-1} \int_{y^\sigma}^{b^\sigma} (w - y^\sigma)^{\alpha-1} dw = y^{\sigma\eta+\sigma-1} \frac{(b^\sigma - y^\sigma)^\alpha}{\alpha}. \tag{45}$$

That is

$$u_1(y) = y^{\sigma\eta+\sigma-1} \frac{(b^\sigma - y^\sigma)^\alpha}{\alpha}, \quad y \in (a,b). \tag{46}$$

Based on the above, (43) becomes

$$\int_a^b x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1, \alpha \right) \right) \cdot$$

$$\prod_{i=1}^m \Phi_i \left(\frac{\left| I_{a+;\sigma;\eta}^\alpha f_i(x) \right| \Gamma(\alpha)}{\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1,\alpha \right) \right)} \right) dx$$

$$\leq \frac{1}{\alpha} \prod_{i=1}^m \left(\int_a^b y^{\sigma\eta+\sigma-1} (b^\sigma - y^\sigma)^\alpha \, \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}} \tag{47}$$

$$\leq \frac{(b^\sigma - a^\sigma)^\alpha}{\alpha} \prod_{i=1}^m \left(\int_a^b y^{\sigma(\eta+1)-1} \Phi_i(|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) following (43), and

(ii) * $y^{\sigma(\eta+1)-1} \Phi_i(|f_i(y)|)^{p_i}$ is Lebesgue integrable on (a,b), $i = 1, \ldots, m$.

Corollary 13. *Let $0 \leq a < b$; $\alpha, \sigma > 0$, $\eta > -1$; $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$; $\beta_i \geq 1$, $i = 1, \ldots, m$. Then*

$$\int_a^b x^{\sigma(\alpha+\eta+1)-1} \left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma; \eta+1,\alpha \right) \right)^{\left(1-\sum_{i=1}^m \beta_i\right)}$$

$$\cdot \left(\prod_{i=1}^m \left| I_{a+;\sigma;\eta}^\alpha f_i(x) \right|^{\beta_i} \right) dx \leq \frac{1}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^m \beta_i}}$$

$$\cdot \prod_{i=1}^m \left(\int_a^b y^{\sigma\eta+\sigma-1} (b^\sigma - y^\sigma)^\alpha |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}} \tag{48}$$

$$\leq \left(\frac{(b^\sigma - a^\sigma)^\alpha}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^m \beta_i}} \right) \prod_{i=1}^m \left(\int_a^b y^{\sigma(\eta+1)-1} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}},$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, \ldots, m$ such that

(i) $|f_i|^{\beta_i p_i}$ *is* $\left(\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\chi_{(a,x]}(y)\frac{y^{\sigma\eta+\sigma-1}dy}{(x^\sigma-y^\sigma)^{1-\alpha}}\right)$*–integrable, a.e. in* $x \in (a,b)$,

(ii) $y^{\sigma(\eta+1)-1}|f_i(y)|^{\beta_i p_i}$ *is Lebesgue integrable on* (a,b); $i = 1,\ldots,m$.

Proof. By Theorem 11 and (47). □

Corollary 14. *Let* $0 \le a < b$; $\alpha, \sigma > 0$, $\eta > -1$; $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. *Then*

$$\int_a^b x^{\sigma(\alpha+\eta+1)-1}\left(B(\eta+1,\alpha) - B\left(\left(\frac{a}{x}\right)^\sigma;\eta+1,\alpha\right)\right)$$

$$\cdot e^{\frac{\Gamma(\alpha)\left(\sum_{i=1}^m\left|I^\alpha_{a+;\sigma;\eta}f_i(x)\right|\right)}{\left(B(\eta+1,\alpha)-B\left(\left(\frac{a}{x}\right)^\sigma;\eta+1,\alpha\right)\right)}}dx$$

$$\le \frac{1}{\alpha}\prod_{i=1}^m\left(\int_a^b y^{\sigma(\eta+1)-1}(b^\sigma - y^\sigma)^\alpha e^{p_i|f_i(y)|}dy\right)^{\frac{1}{p_i}}$$

$$\le \frac{(b^\sigma - a^\sigma)^\alpha}{\alpha}\prod_{i=1}^m\left(\int_a^b y^{\sigma(\eta+1)-1}e^{p_i|f_i(y)|}dy\right)^{\frac{1}{p_i}}, \tag{49}$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1,\ldots,m$ *such that*

(i) f_i, $e^{p_i|f_i|}$ *are both* $\frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)}\chi_{(a,x]}(y)\frac{y^{\sigma\eta+\sigma-1}dy}{(x^\sigma-y^\sigma)^{1-\alpha}}$*–integrable, a.e. in* $x \in (a,b)$,

(ii) $y^{\sigma(\eta+1)-1}e^{p_i|f_i(y)|}$ *is Lebesgue integrable on* (a,b); $i = 1,\ldots,m$.

Proof. By Theorem 11 and (47). □

We present

Theorem 15. *Assume that the function*

$$x \mapsto \left(u(x)\frac{\sigma x^{\sigma\eta}\chi_{[x,b)}(y)\,y^{\sigma(1-\eta-\alpha)-1}}{(y^\sigma - x^\sigma)^{1-\alpha}\left[B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma;\eta,\alpha\right)\right]}\right)$$

is integrable on (a,b), *for each* $y \in (a,b)$. *Here* $\alpha, \sigma, \eta > 0$, $0 \le a < b < \infty$. *Define* u_2 *on* (a,b) *by*

$$u_2(y) := \sigma y^{\sigma(1-\eta-\alpha)-1}\int_a^y \frac{u(x)\,x^{\sigma\eta}(y^\sigma - x^\sigma)^{\alpha-1}dx}{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma;\eta,\alpha\right)\right)} < \infty. \tag{50}$$

Let $p_i > 1 : \sum_{i=1}^m \frac{1}{p_i} = 1$. *Let the functions* $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1,\ldots,m$, *be convex and increasing. Then*

$$\int_a^b u(x)\prod_{i=1}^m \Phi_i\left(\frac{\left|I^\alpha_{b-;\sigma;\eta}f_i(x)\right|\Gamma(\alpha)}{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^\sigma;\eta,\alpha\right)\right)}\right)dx$$

$$\leq \prod_{i=1}^{m} \left(\int_a^b u_2(y) \Phi_i (|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}}, \tag{51}$$

for all measurable functions $f_i : (a,b) \to \mathbb{R}$, $i = 1, \ldots, m$, *such that*

(i) f_i, $\Phi_i (|f_i|)^{p_i}$ *are both* $\left(\frac{\sigma x^{\sigma\eta} \chi_{[x,b)}(y) y^{\sigma(1-\eta-\alpha)-1} dy}{\Gamma(\alpha)(y^\sigma - x^\sigma)^{1-\alpha}} \right)$*–integrable, a.e. in* $x \in (a,b)$, *for all* $i = 1, \ldots, m$,

(ii) $u_2 \Phi_i (|f_i|)^{p_i}$ *is Lebesgue integrable on* (a,b), $i = 1, \ldots, m$.

Proof. By Corollary 5. □

Remark 16. Here $0 < a < b < \infty$; $\alpha, \sigma, \eta > 0$.

In (50), if we choose

$$u(x) = x^{\sigma(1-\eta)-1} \left(B(\eta, \alpha) - B\left(\left(\frac{x}{b} \right)^{\alpha}; \eta, \alpha \right) \right), \quad x \in (a,b), \tag{52}$$

then

$$u_2(y) = \sigma y^{\sigma(1-\eta-\alpha)-1} \int_a^y x^{\sigma-1} (y^\sigma - x^\sigma)^{\alpha-1} dx$$

(setting $w := x^\sigma$, $dx = \frac{dw}{\sigma x^{\sigma-1}}$)

$$= y^{\sigma(1-\eta-\alpha)-1} \int_{a^\sigma}^{y^\sigma} (y^\sigma - w)^{\alpha-1} dw = y^{\sigma(1-\eta-\alpha)-1} \frac{(y^\sigma - a^\sigma)^\alpha}{\alpha}. \tag{53}$$

That is

$$u_2(y) = y^{\sigma(1-\eta-\alpha)-1} \frac{(y^\sigma - a^\sigma)^\alpha}{\alpha}, \quad y \in (a,b). \tag{54}$$

Based on the above, (51) becomes

$$\int_a^b x^{\sigma(1-\eta)-1} \left(B(\eta, \alpha) - B\left(\left(\frac{x}{b} \right)^{\sigma}; \eta, \alpha \right) \right)$$

$$\cdot \prod_{i=1}^{m} \Phi_i \left(\frac{\left| I^\alpha_{b-;\sigma;\eta} f_i(x) \right| \Gamma(\alpha)}{\left(B(\eta, \alpha) - B\left(\left(\frac{x}{b} \right)^{\sigma}; \eta, \alpha \right) \right)} \right) dx$$

$$\leq \frac{1}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} (y^\sigma - a^\sigma)^\alpha \Phi_i (|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}} \tag{55}$$

$$\leq \frac{(b^\sigma - a^\sigma)^\alpha}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\alpha(1-\eta-\alpha)-1} \Phi_i (|f_i(y)|)^{p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) following (51), and

(ii)* $y^{\sigma(1-\eta-\alpha)-1} \Phi_i (|f_i(y)|)^{p_i}$ is Lebesgue integrable on (a,b), $i = 1, \ldots, m$.

Corollary 17. *Let $0 < a < b < \infty$; $\alpha, \sigma, \eta > 0$; $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$; $\beta_i \geq 1$, $i = 1, \ldots, m$. Then*

$$\int_a^b x^{\sigma(1-\eta)-1} \left(B(\eta, \alpha) - B\left(\left(\frac{x}{b} \right)^{\sigma}; \eta, \alpha \right) \right)^{\left(1 - \sum_{i=1}^{m} \beta_i\right)}$$

$$\cdot \left(\prod_{i=1}^{m} \left| I_{b-;\sigma;\eta}^{\alpha} f_i(x) \right|^{\beta_i} \right) dx \leq \frac{1}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^{m} \beta_i}}$$

$$\cdot \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} (y^{\sigma} - a^{\sigma})^{\alpha} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}} \tag{56}$$

$$\leq \frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha (\Gamma(\alpha))^{\sum_{i=1}^{m} \beta_i}} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} |f_i(y)|^{\beta_i p_i} dy \right)^{\frac{1}{p_i}},$$

under the assumptions:

(i) $|f_i|^{\beta_i p_i}$ *is* $\left(\frac{\sigma x^{\sigma\eta} \chi_{[x,b)}(y) y^{\sigma(1-\eta-\alpha)-1} dy}{\Gamma(\alpha)(y^{\sigma} - x^{\sigma})^{1-\alpha}} \right)$*–integrable, a.e. in* $x \in (a, b)$*, for all* $i = 1, \ldots, m$*,*

(ii) $y^{\sigma(1-\eta-\alpha)-1} |f_i(y)|^{\beta_i p_i}$ *is Lebesgue integrable on* (a, b), $i = 1, \ldots, m$.

Proof. By Theorem 15 and (55). □

Corollary 18. *Let $0 < a < b < \infty$; $\alpha, \sigma, \eta > 0$; $p_i > 1 : \sum_{i=1}^{m} \frac{1}{p_i} = 1$. Then*

$$\int_a^b x^{\sigma(1-\eta)-1} \left(B(\eta, \alpha) - B\left(\left(\frac{x}{b} \right)^{\sigma}; \eta, \alpha \right) \right) \cdot e^{\frac{\Gamma(\alpha)\left(\sum_{i=1}^{m} \left| I_{b-;\sigma;\eta}^{\alpha} f_i(x) \right|\right)}{\left(B(\eta,\alpha) - B\left(\left(\frac{x}{b}\right)^{\sigma};\eta,\alpha\right)\right)}} dx$$

$$\leq \frac{1}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} (y^{\sigma} - a^{\sigma})^{\alpha} e^{p_i |f_i(y)|} dy \right)^{\frac{1}{p_i}}$$

$$\leq \frac{(b^{\sigma} - a^{\sigma})^{\alpha}}{\alpha} \prod_{i=1}^{m} \left(\int_a^b y^{\sigma(1-\eta-\alpha)-1} e^{p_i |f_i(y)|} dy \right)^{\frac{1}{p_i}}, \tag{57}$$

under the assumptions:

(i) f_i, $e^{p_i |f_i|}$ *are both* $\left(\frac{\sigma x^{\sigma\eta} \chi_{[x,b)}(y) y^{\sigma(1-\eta-\alpha)-1} dy}{\Gamma(\alpha)(y^{\sigma} - x^{\sigma})^{1-\alpha}} \right)$*–integrable, a.e. in* $x \in (a, b)$, $i = 1, \ldots, m$*,*

(ii) $y^{\sigma(1-\eta-\alpha)-1} e^{p_i |f_i(y)|}$ *is Lebesgue integrable on* (a, b); $i = 1, \ldots, m$.

Proof. By Theorem 15 and (55). □

We make

Remark 19. Let $\prod_{i=1}^{N}(a_i,b_i) \subset \mathbb{R}^N$, $N>1$, $a_i<b_i$, $a_i,b_i \in \mathbb{R}$. Let $\alpha_i>0$, $i=1,\dots,N$; $f \in L_1\left(\prod_{i=1}^{N}(a_i,b_i)\right)$, and set $a=(a_1,\dots,a_N)$, $b=(b_1,\dots,b_N)$, $\alpha=(\alpha_1,\dots,\alpha_N)$, $x=(x_1,\dots,x_N)$, $t=(t_1,\dots,t_N)$.

We define the left mixed Riemann-Liouville fractional multiple integral of order α (see also [14]):

$$\left(I_{a+}^{\alpha}f\right)(x) := \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{a_1}^{x_1}\cdots\int_{a_N}^{x_N} \prod_{i=1}^{N}(x_i-t_i)^{\alpha_i-1} f(t_1,\dots,t_N)\,dt_1\dots dt_N, \tag{58}$$

with $x_i>a_i$, $i=1,\dots,N$.

We also define the right mixed Riemann-Liouville fractional multiple integral of order α (see also [12]):

$$\left(I_{b-}^{\alpha}f\right)(x) := \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{x_1}^{b_1}\cdots\int_{x_N}^{b_N} \prod_{i=1}^{N}(t_i-x_i)^{\alpha_i-1} f(t_1,\dots,t_N)\,dt_1\dots dt_N, \tag{59}$$

with $x_i<b_i$, $i=1,\dots,N$.

Notice $I_{a+}^{\alpha}(|f|)$, $I_{b-}^{\alpha}(|f|)$ are finite if $f \in L_\infty\left(\prod_{i=1}^{N}(a_i,b_i)\right)$.

One can rewrite (58) and (59) as follows:

$$\left(I_{a+}^{\alpha}f\right)(x) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{\prod_{i=1}^{N}(a_i,b_i)} \chi_{\prod_{i=1}^{N}(a_i,x_i]}(t) \prod_{i=1}^{N}(x_i-t_i)^{\alpha_i-1} f(t)\,dt, \tag{60}$$

with $x_i>a_i$, $i=1,\dots,N$, and

$$\left(I_{b-}^{\alpha}f\right)(x) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{\prod_{i=1}^{N}(a_i,b_i)} \chi_{\prod_{i=1}^{N}[x_i,b_i)}(t) \prod_{i=1}^{N}(t_i-x_i)^{\alpha_i-1} f(t)\,dt, \tag{61}$$

with $x_i<b_i$, $i=1,\dots,N$.

The corresponding $k(x,y)$ for I_{a+}^{α}, I_{b-}^{α} are

$$k_{a+}(x,y) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}, \tag{62}$$

$\forall\, x,y \in \prod_{i=1}^{N}(a_i,b_i)$, and

$$k_{b-}(x,y) = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}, \tag{63}$$

$\forall\ x, y \in \prod_{i=1}^{N} (a_i, b_i)$.

The corresponding $K(x)$ for I_{a+}^{α} is:

$$K_{a+}(x) = \int_{\prod_{i=1}^{N}(a_i,b_i)} k_{a+}(x,y)\, dy = \left(I_{a+}^{\alpha}1\right)(x)$$

$$= \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{a_1}^{x_1} \cdots \int_{a_N}^{x_N} \prod_{i=1}^{N} (x_i - t_i)^{\alpha_i - 1}\, dt_1 \dots dt_N$$

$$= \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \int_{a_i}^{x_i} (x_i - t_i)^{\alpha_i - 1}\, dt_i = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \frac{(x_i - a_i)^{\alpha_i}}{\alpha_i}$$

$$= \prod_{i=1}^{N} \left(\frac{(x_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)} \right),$$

that is

$$K_{a+}(x) = \prod_{i=1}^{N} \frac{(x_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{64}$$

$\forall\ x \in \prod_{i=1}^{N} (a_i, b_i)$.

Similarly the corresponding $K(x)$ for I_{b-}^{α} is:

$$K_{b-}(x) = \int_{\prod_{i=1}^{N}(a_i,b_i)} k_{b-}(x,y)\, dy = \left(I_{b-}^{\alpha}1\right)(x) =$$

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \int_{x_1}^{b_1} \cdots \int_{x_N}^{b_N} \prod_{i=1}^{N} (t_i - x_i)^{\alpha_i - 1}\, dt_1 \dots dt_N =$$

$$\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \int_{x_i}^{b_i} (t_i - x_i)^{\alpha_i - 1}\, dt_i = \frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)} \prod_{i=1}^{N} \frac{(b_i - x_i)^{\alpha_i}}{\alpha_i}$$

$$= \prod_{i=1}^{N} \frac{(b_i - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)},$$

that is

$$K_{b-}(x) = \prod_{i=1}^{N} \frac{(b_i - x_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}, \tag{65}$$

$\forall\ x \in \prod_{i=1}^{N} (a_i, b_i)$.

Next we form

$$\frac{k_{a+}(x,y)}{K_{a+}(x)} = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \prod_{i=1}^{N} (x_i - y_i)^{\alpha_i - 1} \prod_{i=1}^{N} \frac{\Gamma(\alpha_i + 1)}{(x_i - a_i)^{\alpha_i}}$$

$$= \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \frac{(x_i - y_i)^{\alpha_i - 1}}{(x_i - a_i)^{\alpha_i}} \right),$$

that is

$$\frac{k_{a+}(x,y)}{K_{a+}(x)} = \chi_{\prod_{i=1}^{N}(a_i,x_i]}(y) \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \frac{(x_i - y_i)^{\alpha_i - 1}}{(x_i - a_i)^{\alpha_i}} \right), \tag{66}$$

$\forall\ x, y \in \prod_{i=1}^{N} (a_i, b_i)$.

Similarly we form

$$\frac{k_{b-}(x,y)}{K_{b-}(x)} = \frac{1}{\prod_{i=1}^{N} \Gamma(\alpha_i)} \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \prod_{i=1}^{N} (y_i - x_i)^{\alpha_i - 1} \prod_{i=1}^{N} \frac{\Gamma(\alpha_i + 1)}{(b_i - x_i)^{\alpha_i}}$$

$$= \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \frac{(y_i - x_i)^{\alpha_i - 1}}{(b_i - x_i)^{\alpha_i}} \right),$$

that is

$$\frac{k_{b-}(x,y)}{K_{b-}(x)} = \chi_{\prod_{i=1}^{N}[x_i,b_i)}(y) \left(\prod_{i=1}^{N} \alpha_i \right) \left(\prod_{i=1}^{N} \frac{(y_i - x_i)^{\alpha_i - 1}}{(b_i - x_i)^{\alpha_i}} \right), \tag{67}$$

$\forall\ x, y \in \prod_{i=1}^{N} (a_i, b_i)$.

We choose the weight function $u_1(x)$ on $\prod_{i=1}^{N} (a_i, b_i)$ such that the function $x \mapsto \left(u_1(x) \frac{k_{a+}(x,y)}{K_{a+}(x)} \right)$ is integrable on $\prod_{i=1}^{N} (a_i, b_i)$, for each fixed $y \in \prod_{i=1}^{N} (a_i, b_i)$. We define w_1 on $\prod_{i=1}^{N} (a_i, b_i)$ by

$$w_1(y) := \int_{\prod_{i=1}^{N}(a_i,b_i)} u_1(x) \frac{k_{a+}(x,y)}{K_{a+}(x)} dx < \infty. \tag{68}$$

We have that

$$w_1(y) = \left(\prod_{i=1}^{N}\alpha_i\right)\int_{y_1}^{b_1}\cdots\int_{y_N}^{b_N} u_1(x_1,\ldots,x_N)\left(\prod_{i=1}^{N}\frac{(x_i-y_i)^{\alpha_i-1}}{(x_i-a_i)^{\alpha_i}}\right)dx_1\ldots dx_N, \tag{69}$$

$\forall\ y\in\prod_{i=1}^{N}(a_i,b_i)$.

We also choose the weight function $u_2(x)$ on $\prod_{i=1}^{N}(a_i,b_i)$ such that the function $x\mapsto\left(u_2(x)\frac{k_{b-}(x,y)}{K_{b-}(x)}\right)$ is integrable on $\prod_{i=1}^{N}(a_i,b_i)$, for each fixed $y\in\prod_{i=1}^{N}(a_i,b_i)$. We define w_2 on $\prod_{i=1}^{N}(a_i,b_i)$ by

$$w_2(y) := \int_{\prod_{i=1}^{N}(a_i,b_i)} u_2(x)\frac{k_{b-}(x,y)}{K_{b-}(x)}dx<\infty. \tag{70}$$

We have that

$$w_2(y) = \left(\prod_{i=1}^{N}\alpha_i\right)\int_{a_1}^{y_1}\cdots\int_{a_N}^{y_N} u_2(x_1,\ldots,x_N)\left(\prod_{i=1}^{N}\frac{(y_i-x_i)^{\alpha_i-1}}{(b_i-x_i)^{\alpha_i}}\right)dx_1\ldots dx_N, \tag{71}$$

$\forall\ y\in\prod_{i=1}^{N}(a_i,b_i)$.

If we choose as

$$u_1(x)=u_1^*(x):=\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}, \tag{72}$$

then

$$\begin{aligned} w_1^*(y) := w_1(y) &= \left(\prod_{i=1}^{N}\alpha_i\right)\int_{y_1}^{b_1}\cdots\int_{y_N}^{b_N}\left(\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}\right)dx_1\ldots dx_N \\ &= \left(\prod_{i=1}^{N}\alpha_i\right)\left(\prod_{i=1}^{N}\int_{y_i}^{b_i}(x_i-y_i)^{\alpha_i-1}dx_i\right) \\ &= \left(\prod_{i=1}^{N}\alpha_i\right)\left(\prod_{i=1}^{N}\frac{(b_i-y_i)^{\alpha_i}}{\alpha_i}\right)=\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}. \end{aligned}$$

that is

$$w_1^*(y)=\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}, \quad \forall y\in\prod_{i=1}^{N}(a_i,b_i). \tag{73}$$

If we choose as

$$u_2(x)=u_2^*(x):=\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}, \tag{74}$$

then

$$w_2^*(y) := w_2(y) = \left(\prod_{i=1}^N \alpha_i\right) \int_{a_1}^{y_1} \cdots \int_{a_N}^{y_N} \left(\prod_{i=1}^N (y_i - x_i)^{\alpha_i - 1}\right) dx_1 \ldots dx_N$$

$$= \left(\prod_{i=1}^N \alpha_i\right) \left(\prod_{i=1}^N \int_{a_i}^{y_i} (y_i - x_i)^{\alpha_i - 1} dx_i\right)$$

$$= \left(\prod_{i=1}^N \alpha_i\right) \left(\prod_{i=1}^N \frac{(y_i - a_i)^{\alpha_i}}{\alpha_i}\right) = \prod_{i=1}^N (y_i - a_i)^{\alpha_i}.$$

That is

$$w_2^*(y) = \prod_{i=1}^N (y_i - a_i)^{\alpha_i}, \quad \forall y \in \prod_{i=1}^N (a_i, b_i). \tag{75}$$

Here we choose $f_j : \prod_{i=1}^N (a_i, b_i) \to \mathbb{R}$, $j = 1, \ldots, m$, that are Lebesgue measurable and $I_{a+}^{\alpha}(|f_j|)$, $I_{b-}^{\alpha}(|f_j|)$ are finite a.e., one or the other, or both.

Let $p_j > 1 : \sum_{j=1}^m \frac{1}{p_j} = 1$ and the functions $\Phi_j : \mathbb{R}_+ \to \mathbb{R}_+$, $j = 1, \ldots, m$, to be convex and increasing.

Then by (22) we obtain

$$\int_{\prod_{i=1}^N (a_i, b_i)} u_1(x) \prod_{j=1}^m \Phi_j \left(\frac{\left|I_{a+}^{\alpha}(f_j)(x)\right| \prod_{i=1}^N \Gamma(\alpha_i + 1)}{\prod_{i=1}^N (x_i - a_i)^{\alpha_i}} \right) dx$$

$$\leq \prod_{j=1}^m \left(\int_{\prod_{i=1}^N (a_i, b_i)} w_1(y) \Phi_j(|f_j(y)|)^{p_j} dy \right)^{\frac{1}{p_j}}, \tag{76}$$

under the assumptions:

(i) f_j, $\Phi_j(|f_j|)^{p_j}$ are both $\frac{1}{\prod_{i=1}^N \Gamma(\alpha_i)} \chi_{\prod_{i=1}^N (a_i, x_i]}(y) \prod_{i=1}^N (x_i - y_i)^{\alpha_i - 1} dy$ -integrable, a.e. in $x \in \prod_{i=1}^N (a_i, b_i)$, for all $j = 1, \ldots, m$,

(ii) $w_1 \Phi_j(|f_j|)^{p_j}$ is Lebesgue integrable, $j = 1, \ldots, m$.

Similarly, by (22), we obtain

$$\int_{\prod_{i=1}^N (a_i, b_i)} u_2(x) \prod_{j=1}^m \Phi_j \left(\frac{\left|I_{b-}^{\alpha}(f_j)(x)\right| \prod_{i=1}^N \Gamma(\alpha_i + 1)}{\prod_{i=1}^N (b_i - x_i)^{\alpha_i}} \right) dx$$

$$\leq \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} w_2(y)\,\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}}, \tag{77}$$

under the assumptions:

(i) f_j, $\Phi_j\left(|f_j|\right)^{p_j}$ are both $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i-x_i)^{\alpha_i-1}dy$ -integrable, a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$, for all $j=1,\dots,m$,

(ii) $w_2\Phi_j\left(|f_j|\right)^{p_j}$ is Lebesgue integrable, $j=1,\dots,m$.

Using (72) and (73) we rewrite (76), as follows

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|I_{a+}^{\alpha}(f_j)(x)\right|\prod_{i=1}^{N}\Gamma(\alpha_i+1)}{\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}}\right)dx$$

$$\leq \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}\right)\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}} \leq \tag{78}$$

$$\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) following (76) and

(ii)* $\Phi_j\left(|f_j|\right)^{p_j}$ is Lebesgue integrable, $j=1,\dots,m$.

Similarly, using (74) and (75) we rewrite (77),

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}\right)\prod_{j=1}^{m}\Phi_j\left(\frac{\left|I_{b-}^{\alpha}(f_j)(x)\right|\prod_{i=1}^{N}\Gamma(\alpha_i+1)}{\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}}\right)dx$$

$$\leq \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(y_i-a_i)^{\alpha_i}\right)\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}} \tag{79}$$

$$\leq \left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\Phi_j\left(|f_j(y)|\right)^{p_j}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) following (77), and

(ii)* $\Phi_j\left(|f_j|\right)^{p_j}$ is Lebesgue integrable, $j=1,\ldots,m$.

Let now $\beta_j \geq 1$, $j=1,\ldots,m$.

Then, by (78), we obtain

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)} \left(\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}\right) dx$$

$$\leq \left(\frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right)$$

$$\cdot \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}\right)|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}} \tag{80}$$

$$\leq \left(\frac{\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}}.$$

But it holds

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)} \left(\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}\right) dx$$

$$\geq \left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}dx\right). \tag{81}$$

So by (80) and (81) we derive

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\prod_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|^{\beta_j}dx \tag{82}$$

$$\leq \left(\prod_{i=1}^{N}\frac{(b_i-a_i)^{\alpha_i}}{\Gamma(\alpha_i+1)}\right)^{\sum_{j=1}^{m}\beta_j}\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}|f_j(y)|^{\beta_j p_j}\,dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) $|f_j|^{p_j\beta_j}$ is $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}\,dy$–integrable, a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$, for all $j=1,\ldots,m$,

(ii) $|f_j|^{p_j\beta_j}$ is Lebesgue integrable, $j=1,\ldots,m$.

We also have, by (78), that

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(x_i-a_i)^{\alpha_i}\right)e^{\left(\sum_{j=1}^{m}\left|I_{a+}^{\alpha}(f_j)(x)\right|\right)\left(\prod_{i=1}^{N}\frac{\Gamma(\alpha_i+1)}{(x_i-a_i)^{\alpha_i}}\right)}dx$$

$$\leq\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-y_i)^{\alpha_i}\right)e^{p_j|f_j(y)|}dy\right)^{\frac{1}{p_j}} \tag{83}$$

$$\leq\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}e^{p_j|f_j(y)|}dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) f_j, $e^{p_j|f_j|}$ are both $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}(a_i,x_i]}(y)\prod_{i=1}^{N}(x_i-y_i)^{\alpha_i-1}dy$ – integrable, a.e. in $x\in\prod_{i=1}^{N}(a_i,b_i)$, for all $j=1,\ldots,m$,

(ii) $e^{p_j|f_j|}$ is Lebesgue integrable, $j=1,\ldots,m$.

From (79) we get

$$\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(b_i-x_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)}\left(\prod_{j=1}^{m}\left|I_{b-}^{\alpha}(f_j)(x)\right|^{\beta_j}\right)dx$$

$$\leq\left(\frac{1}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right)$$

$$\cdot\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}\left(\prod_{i=1}^{N}(y_i-a_i)^{\alpha_i}\right)|f_j(y)|^{\beta_jp_j}dy\right)^{\frac{1}{p_j}} \tag{84}$$

$$\leq\left(\frac{\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}}{\left(\prod_{i=1}^{N}\Gamma(\alpha_i+1)\right)^{\sum_{j=1}^{m}\beta_j}}\right)\prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)}|f_j(y)|^{\beta_jp_j}dy\right)^{\frac{1}{p_j}}.$$

But it holds

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{i=1}^{N}(b_i - x_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)} \left(\prod_{j=1}^{m}\left|I_{b-}^{\alpha}(f_j)(x)\right|^{\beta_j}\right) dx$$

$$\geq \left(\prod_{i=1}^{N}(b_i - a_i)^{\alpha_i}\right)^{\left(1-\sum_{j=1}^{m}\beta_j\right)} \left(\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{j=1}^{m}\left|I_{b-}^{\alpha}(f_j)(x)\right|^{\beta_j}\right) dx\right). \tag{85}$$

So by (84) and (85) we obtain

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{j=1}^{m}\left|I_{b-}^{\alpha}(f_j)(x)\right|^{\beta_j}\right) dx$$

$$\leq \left(\prod_{i=1}^{N}\frac{(b_i - a_i)^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right)^{\sum_{j=1}^{m}\beta_j} \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} |f_j(y)|^{\beta_j p_j}\, dy\right)^{\frac{1}{p_j}}, \tag{86}$$

under the assumptions:

(i) $|f_j|^{p_j\beta_j}$ is $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i - x_i)^{\alpha_i - 1}\, dy$ –integrable, a.e. in $x \in \prod_{i=1}^{N}(a_i,b_i)$, for all $j = 1,\ldots,m$,

(ii) $|f_j|^{p_j\beta_j}$ is Lebesgue integrable, $j = 1,\ldots,m$.

We also have, by (79), that

$$\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{i=1}^{N}(b_i - x_i)^{\alpha_i}\right) e^{\left(\sum_{j=1}^{m}\left|I_{b-}^{\alpha}(f_j)(x)\right|\right)\left(\prod_{i=1}^{N}\frac{\Gamma(\alpha_i+1)}{(b_i - x_i)^{\alpha_i}}\right)} dx$$

$$\leq \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} \left(\prod_{i=1}^{N}(y_i - a_i)^{\alpha_i}\right) e^{p_j|f_j(y)|} dy\right)^{\frac{1}{p_j}} \tag{87}$$

$$\leq \left(\prod_{i=1}^{N}(b_i - a_i)^{\alpha_i}\right) \prod_{j=1}^{m}\left(\int_{\prod_{i=1}^{N}(a_i,b_i)} e^{p_j|f_j(y)|} dy\right)^{\frac{1}{p_j}},$$

under the assumptions:

(i) f_j, $e^{p_j|f_j|}$ are both $\frac{1}{\prod_{i=1}^{N}\Gamma(\alpha_i)}\chi_{\prod_{i=1}^{N}[x_i,b_i)}(y)\prod_{i=1}^{N}(y_i - x_i)^{\alpha_i - 1}\, dy$ –integrable, a.e. in $x \in \prod_{i=1}^{N}(a_i,b_i)$, for all $j = 1,\ldots,m$,

(ii) $e^{p_j|f_j|}$ is Lebesgue integrable, $j = 1,\ldots,m$.

Background 20. *In order to apply Theorem* 1 *to the case of a spherical shell we need:*

Let $N \geq 2$, $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$ *the unit sphere on* $\mathbb{R}^N$, *where* $|\cdot|$ *stands for the Euclidean norm in* $\mathbb{R}^N$. *Also denote the ball* $B(0, R) := \{x \in \mathbb{R}^N : |x| < R\} \subseteq \mathbb{R}^N$, $R > 0$, *and the spherical shell*

$$A := B(0, R_2) - \overline{B(0, R_1)}, \quad 0 < R_1 < R_2. \tag{88}$$

For the following see [15, pp. 149-150], *and* [17, pp. 87-88].

For $x \in \mathbb{R}^N - \{0\}$ *we can write uniquely* $x = r\omega$, *where* $r = |x| > 0$, *and* $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$.

Clearly here

$$\mathbb{R}^N - \{0\} = (0, \infty) \times S^{N-1}, \tag{89}$$

and

$$\overline{A} = [R_1, R_2] \times S^{N-1}. \tag{90}$$

We will be using

Theorem 21. [1, p. 322] *Let* $f : A \to \mathbb{R}$ *be a Lebesgue integrable function. Then*

$$\int_A f(x)\, dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(r\omega)\, r^{N-1} dr \right) d\omega. \tag{91}$$

So we are able to write an integral on the shell in polar form using the polar coordinates (r, ω).

We need

Definition 22. [1, p. 458] *Let* $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$, $f \in C^n\left(\overline{A}\right)$, *and* A *is a spherical shell. Assume that there exists function* $\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} \in C\left(\overline{A}\right)$, *given by*

$$\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial r} \left(\int_{R_1}^{r} (r - t)^{-\alpha} \frac{\partial^n f(t\omega)}{\partial r^n} dt \right), \tag{92}$$

where $x \in \overline{A}$; *that is* $x = r\omega$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$.

We call $\frac{\partial^\nu_{R_1} f}{\partial r^\nu}$ *the left radial Canavati-type fractional derivative of* f *of order* ν. *If* $\nu = 0$, *then set* $\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} := f(x)$.

Based on [1, p. 288], and [5] we have

Lemma 23. *Let* $\gamma \geq 0$, $m := [\gamma]$, $\nu > 0$, $n := [\nu]$, *with* $0 \leq \gamma < \nu$. *Let* $f \in C^n\left(\overline{A}\right)$ *and there exists* $\frac{\partial^\nu_{R_1} f(x)}{\partial r^\nu} \in C\left(\overline{A}\right)$, $x \in \overline{A}$, A *a sperical shell.*

Further assume that $\frac{\partial^j f(R_1\omega)}{\partial r^j} = 0$, $j = m, m+1, \ldots, n-1$, $\forall\ \omega \in S^{N-1}$. *Then there exists* $\frac{\partial^\gamma_{R_1} f(x)}{\partial r^\gamma} \in C\left(\overline{A}\right)$ *such that*

$$\frac{\partial^\gamma_{R_1} f(x)}{\partial r^\gamma} = \frac{\partial^\gamma_{R_1} f(r\omega)}{\partial r^\gamma} = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \frac{\partial^\nu_{R_1} f(t\omega)}{\partial r^\nu} dt, \tag{93}$$

$\forall\ \omega \in S^{N-1}$; *all* $R_1 \le r \le R_2$, *indeed* $f(r\omega) \in C^\gamma_{R_1}([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$.

We make

Remark 24. In the settings and assumptions of Theorem 1 and Lemma 23 we have

$$k(r,t) = \frac{1}{\Gamma(\nu-\gamma)} \chi_{[R_1,r]}(t)(r-t)^{\nu-\gamma-1}, \tag{94}$$

and

$$K(r) = \frac{(r-R_1)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}, \tag{95}$$

$r, t \in [R_1, R_2]$.

Furthermore we get

$$\frac{k(r,t)}{K(r)} = (\nu-\gamma)\chi_{[R_1,r]}(t) \frac{(r-t)^{\nu-\gamma-1}}{(r-R_1)^{\nu-\gamma}}, \tag{96}$$

and by choosing

$$u(r) := (r-R_1)^{\nu-\gamma}, \quad r \in [R_1, R_2], \tag{97}$$

we find

$$U(t) = (\nu-\gamma) \int_t^{R_2} (r-t)^{\nu-\gamma-1} dr = (R_2-t)^{\nu-\gamma}, \tag{98}$$

$t \in [R_1, R_2]$.

Then by (8) for $p \ge 1$ we find

$$\int_{R_1}^{R_2} (r-R_1)^{\nu-\gamma} \left| \frac{\partial^\gamma_{R_1} f(r\omega)}{\partial r^\gamma} \right|^p \frac{(\Gamma(\nu-\gamma+1))^p}{(r-R_1)^{(\nu-\gamma)p}} dr$$

$$\le \int_{R_1}^{R_2} (R_2-r)^{\nu-\gamma} \left| \frac{\partial^\nu_{R_1} f(r\omega)}{\partial r^\nu} \right|^p dr, \tag{99}$$

and

$$\int_{R_1}^{R_2} (r-R_1)^{(\nu-\gamma)(1-p)} \left| \frac{\partial^\gamma_{R_1} f(r\omega)}{\partial r^\gamma} \right|^p dr$$

$$\le \frac{1}{(\Gamma(\nu-\gamma+1))^p} \int_{R_1}^{R_2} (R_2-r)^{\nu-\gamma} \left| \frac{\partial^\nu_{R_1} f(r\omega)}{\partial r^\nu} \right|^p dr$$

$$\leq \frac{(R_2 - R_1)^{\nu-\gamma}}{(\Gamma(\nu-\gamma+1))^p} \int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr. \tag{100}$$

But it holds

$$\int_{R_1}^{R_2} (r - R_1)^{(\nu-\gamma)(1-p)} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr$$

$$\geq (R_2 - R_1)^{(\nu-\gamma)(1-p)} \int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr. \tag{101}$$

Consequently we derive

$$\int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \leq \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr, \tag{102}$$

$\forall\ \omega \in S^{N-1}$.

Here we have $R_1 \leq r \leq R_2$, and $R_1^{N-1} \leq r^{N-1} \leq R_2^{N-1}$, and $R_2^{1-N} \leq r^{1-N} \leq R_1^{1-N}$.

From (102) we have

$$R_2^{1-N} \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \leq \int_{R_1}^{R_2} r^{1-N} r^{N-1} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr$$

$$\leq \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} r^{1-N} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr$$

$$\leq R_1^{1-N} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr. \tag{103}$$

So we get

$$\int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr$$

$$\leq \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr, \tag{104}$$

$\forall\ \omega \in S^{N-1}$.

Hence

$$\int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\gamma} f(r\omega)}{\partial r^{\gamma}} \right|^p dr \right) d\omega$$

$$\leq \left(\frac{R_2}{R_1} \right)^{N-1} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right)^p \int_{S^{N-1}} \left(\int_{R_1}^{R_2} r^{N-1} \left| \frac{\partial_{R_1}^{\nu} f(r\omega)}{\partial r^{\nu}} \right|^p dr \right) d\omega. \tag{105}$$

By Theorem 21, equality (91), we obtain

$$\int_A \left| \frac{\partial_{R_1}^{\gamma} f(x)}{\partial r^{\gamma}} \right|^p dx \leq \left(\frac{R_2}{R_1}\right)^{N-1} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu - \gamma + 1)}\right)^p \int_A \left| \frac{\partial_{R_1}^{\nu} f(x)}{\partial r^{\nu}} \right|^p dx. \tag{106}$$

We have proved the following fractional Poincaré type inequalities on the shell.

Theorem 25. *Here all as in Lemma* 23*,* $p \geq 1$*.*

It holds

1)

$$\left\| \frac{\partial_{R_1}^{\gamma} f}{\partial r^{\gamma}} \right\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu - \gamma + 1)}\right) \left\| \frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}} \right\|_{p,A}, \tag{107}$$

2) *When* $\gamma = 0$*, we have*

$$\|f\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{\nu}}{\Gamma(\nu + 1)}\right) \left\| \frac{\partial_{R_1}^{\nu} f}{\partial r^{\nu}} \right\|_{p,A}. \tag{108}$$

See the related, and proof, results in [1, pp. 458-459] with different constants and proof in the corresponding inequalities.

Similar results can be produced for the right radial Canavati type fractional derivative. We choose to omit it.

We make

Remark 26. (from [1], p. 460) Here we denote $\lambda_{\mathbb{R}^N}(x) \equiv dx$ the Lebesgue measure on $\mathbb{R}^N$, $N \geq 2$, and by $\lambda_{S^{N-1}}(\omega) = d\omega$ the surface measure on S^{N-1}, where $\mathcal{B}_X$ stands for the Borel class on space X. Define the measure R_N on $\left((0, \infty), \mathcal{B}_{(0,\infty)}\right)$ by

$$R_N(B) = \int_B r^{N-1} dr, \text{ any } B \in \mathcal{B}_{(0,\infty)}.$$

Now let $F \in L_1(A) = L_1\left([R_1, R_2] \times S^{N-1}\right)$.

Call

$$K(F) := \{\omega \in S^{N-1} : F(\cdot\omega) \notin L_1\left([R_1, R_2], \mathcal{B}_{[R_1,R_2]}, R_N\right)\}. \tag{109}$$

We get, by Fubini's theorem and [17, pp. 87-88] that

$$\lambda_{S^{N-1}}(K(F)) = 0.$$

Of course

$$\theta(F) := [R_1, R_2] \times K(F) \subset A,$$

and

$$\lambda_{\mathbb{R}^N}(\theta(F)) = 0.$$

Above $\lambda_{S^{N-1}}$ is defined as follows: let $A \subset S^{N-1}$ be a Borel set, and let

$$\widetilde{A} := \{ru : 0 < r < 1, u \in A\} \subset \mathbb{R}^N;$$

we define

$$\lambda_{S^{N-1}}(A) := N\lambda_{\mathbb{R}^N}\left(\widetilde{A}\right).$$

We have that

$$\lambda_{S^{N-1}}\left(S^{N-1}\right) = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)},$$

the surface area of S^{N-1}.

See also [15, pp. 149-150], [17, pp. 87-88] and [1, p. 320].

Following [1, p. 466] we define the left Riemann-Liouville radial fractional derivative next.

Definition 27. *Let $\beta > 0$, $m := [\beta] + 1$, $F \in L_1(A)$, and A is the spherical shell. We define*

$$\frac{\overline{\partial}_{R_1}^{\beta} F(x)}{\partial r^{\beta}} := \begin{cases} \frac{1}{\Gamma(m-\beta)} \left(\frac{\partial}{\partial r}\right)^m \int_{R_1}^{r} (r-t)^{m-\beta-1} F(t\omega)\, dt, \\ \qquad\qquad \text{for } \omega \in S^{N-1} - K(F), \\ 0, \quad \text{for } \omega \in K(F), \end{cases} \tag{110}$$

where $x = r\omega \in A$, $r \in [R_1, R_2]$, $\omega \in S^{N-1}$; $K(F)$ as in (109).

If $\beta = 0$, define

$$\frac{\overline{\partial}_{R_1}^{\beta} F(x)}{\partial r^{\beta}} := F(x).$$

We need the following important representation result for left Riemann-Liouville radial fractional derivatives, by [1, p. 466].

Theorem 28. *Let $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := [\nu]$, $m := [\gamma]$, $F : \overline{A} \to \mathbb{R}$ with $F \in L_1(A)$. Assume that $F(\cdot\omega) \in AC^n([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$, and that $\frac{\overline{\partial}_{R_1}^{\nu} F(\cdot\omega)}{\partial r^{\nu}}$ is measurable on $[R_1, R_2]$, $\forall\ \omega \in S^{N-1}$. Also assume $\exists$ $\frac{\overline{\partial}_{R_1}^{\nu} F(r\omega)}{\partial r^{\nu}} \in \mathbb{R}$, $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$, and $\frac{\overline{\partial}_{R_1}^{\nu} F(x)}{\partial r^{\nu}}$ is measurable on $\overline{A}$. Suppose $\exists\ M_1 > 0$:*

$$\left| \frac{\overline{\partial}_{R_1}^{\nu} F(r\omega)}{\partial r^{\nu}} \right| \leq M_1, \ \ \forall\ \ (r, \omega) \in [R_1, R_2] \times S^{N-1}. \tag{111}$$

We suppose that $\frac{\overline{\partial}^{j} F(R_1\omega)}{\partial r^{j}} = 0$, $j = m, m+1, \ldots, n-1$; $\forall\ \omega \in S^{N-1}$.

Then

$$\frac{\overline{\partial}_{R_1}^{\gamma} F(x)}{\partial r^{\gamma}} = \overline{D}_{R_1}^{\gamma} F(r\omega) = \frac{1}{\Gamma(\nu - \gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \left(\overline{D}_{R_1}^{\nu} F\right)(t\omega)\, dt, \tag{112}$$

valid $\forall\ x \in \overline{A}$*; that is, true* $\forall\ r \in [R_1, R_2]$ *and* $\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Here

$$\overline{D}_{R_1}^{\gamma} F(\cdot\omega) \in AC([R_1, R_2]), \tag{113}$$

$\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Furthermore

$$\frac{\overline{\partial}_{R_1}^{\gamma} F(x)}{\partial r^{\gamma}} \in L_{\infty}(A), \ \gamma > 0. \tag{114}$$

In particular, it holds

$$F(x) = F(r\omega) = \frac{1}{\Gamma(\nu)} \int_{R_1}^{r} (r-t)^{\nu-1} \left(\overline{D}_{R_1}^{\nu} F\right)(t\omega)\, dt, \tag{115}$$

true $\forall\ x \in \overline{A}$*; that is, true* $\forall\ r \in [R_1, R_2]$ *and* $\forall\ \omega \in S^{N-1}$*, and*

$$F(\cdot\omega) \in AC([R_1, R_2]), \quad \forall\ \omega \in S^{N-1}. \tag{116}$$

We give also the following fractional Poincaré type inequalities on the spherical shell.

Theorem 29. *Here all as in Theorem* 28*,* $p \geq 1$*. Then*

1)

$$\left\| \frac{\overline{\partial}_{R_1}^{\gamma} F}{\partial r^{\gamma}} \right\|_{p,A} \leq \left(\frac{R_2}{R_1} \right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu - \gamma + 1)} \right) \left\| \frac{\overline{\partial}_{R_1}^{\nu} F}{\partial r^{\nu}} \right\|_{p,A}, \tag{117}$$

2) *When* $\gamma = 0$*, we have*

$$\|F\|_{p,A} \leq \left(\frac{R_2}{R_1} \right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{\nu}}{\Gamma(\nu + 1)} \right) \left\| \frac{\overline{\partial}_{R_1}^{\nu} F}{\partial r^{\nu}} \right\|_{p,A}. \tag{118}$$

Proof. As in Theorem 25, based on Theorem 28. □

See also similar results in [1, p. 468].

We also need (see [1], p. 421).

Definition 30. *Let* $F : \overline{A} \to \mathbb{R}$*,* $\nu \geq 0$*,* $n := [\nu]$ *such that* $F(\cdot\omega) \in AC^n([R_1, R_2])$*, for all* $\omega \in S^{N-1}$*.*

We call the left Caputo radial fractional derivative the following function

$$\frac{\partial_{*R_1}^{\nu} F(x)}{\partial r^{\nu}} := \frac{1}{\Gamma(n-\nu)} \int_{R_1}^{r} (r-t)^{n-\nu-1} \frac{\partial^n F(t\omega)}{\partial r^n} dt, \tag{119}$$

where $x \in \overline{A}$*, i.e.* $x = r\omega$*,* $r \in [R_1, R_2]$*,* $\omega \in S^{N-1}$*.*

Clearly

$$\frac{\partial_{*R_1}^{0} F(x)}{\partial r^{0}} = F(x), \tag{120}$$

$$\frac{\partial^{\nu}_{*R_1} F(x)}{\partial r^{\nu}} = \frac{\partial^{\nu} F(x)}{\partial r^{\nu}}, \quad \textit{if } \nu \in \mathbb{N}.$$

Above function (119) exists almost everywhere for $x \in \overline{A}$, (see [1], p. 422).

We mention the following fundamental representation result (see [1], p. 422-423 and [5]).

Theorem 31. *Let $\nu \geq \gamma + 1$, $\gamma \geq 0$, $n := [\nu]$, $m := [\gamma]$, $F : \overline{A} \to \mathbb{R}$ with $F \in L_1(A)$. Assume that $F(\cdot\omega) \in AC^n([R_1, R_2])$, for all $\omega \in S^{N-1}$, and that $\frac{\partial^{\nu}_{*R_1} F(\cdot\omega)}{\partial r^{\nu}} \in L_{\infty}(R_1, R_2)$ for all $\omega \in S^{N-1}$.*

Further assume that $\frac{\partial^{\nu}_{*R_1} F(x)}{\partial r^{\nu}} \in L_{\infty}(A)$. More precisely, for these $r \in [R_1, R_2]$, for each $\omega \in S^{N-1}$, for which $D^{\nu}_{*R_1} F(r\omega)$ takes real values, there exists $M_1 > 0$ such that $\left| D^{\nu}_{*R_1} F(r\omega) \right| \leq M_1$.

We suppose that $\frac{\partial^{j} F(R_1\omega)}{\partial r^{j}} = 0$, $j = m, m+1, \ldots, n-1$; for every $\omega \in S^{N-1}$. Then

$$\frac{\partial^{\gamma}_{*R_1} F(x)}{\partial r^{\gamma}} = D^{\gamma}_{*R_1} F(r\omega) = \frac{1}{\Gamma(\nu-\gamma)} \int_{R_1}^{r} (r-t)^{\nu-\gamma-1} \left(D^{\nu}_{*R_1} F \right)(t\omega)\, dt, \tag{121}$$

valid $\forall\ x \in \overline{A}$; i.e. true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Here

$$D^{\gamma}_{*R_1} F(\cdot\omega) \in AC([R_1, R_2]), \tag{122}$$

$\forall\ \omega \in S^{N-1}$; $\gamma > 0$.

Furthermore

$$\frac{\partial^{\gamma}_{*R_1} F(x)}{\partial r^{\gamma}} \in L_{\infty}(A), \ \gamma > 0. \tag{123}$$

In particular, it holds

$$F(x) = F(r\omega) = \frac{1}{\Gamma(\nu)} \int_{R_1}^{r} (r-t)^{\nu-1} \left(D^{\nu}_{*R_1} F \right)(t\omega)\, dt, \tag{124}$$

true $\forall\ x \in \overline{A}$; i.e. true $\forall\ r \in [R_1, R_2]$ and $\forall\ \omega \in S^{N-1}$, and

$$F(\cdot\omega) \in AC([R_1, R_2]), \quad \forall\ \omega \in S^{N-1}. \tag{125}$$

We finish with the following Poincaré type inequalities involving left Caputo radial fractional derivatives.

Theorem 32. *Here all as in Theorem 31, $p \geq 1$. Then*

1)

$$\left\| \frac{\partial^{\gamma}_{*R_1} F}{\partial r^{\gamma}} \right\|_{p,A} \leq \left(\frac{R_2}{R_1} \right)^{\left(\frac{N-1}{p} \right)} \left(\frac{(R_2 - R_1)^{(\nu-\gamma)}}{\Gamma(\nu-\gamma+1)} \right) \left\| \frac{\partial^{\nu}_{*R_1} F}{\partial r^{\nu}} \right\|_{p,A}, \tag{126}$$

2) *When* $\gamma = 0$, *we have*

$$\|F\|_{p,A} \leq \left(\frac{R_2}{R_1}\right)^{\left(\frac{N-1}{p}\right)} \left(\frac{(R_2 - R_1)^{\nu}}{\Gamma(\nu+1)}\right) \left\| \frac{\partial^{\nu}_{*R_1} F}{\partial r^{\nu}} \right\|_{p,A}. \tag{127}$$

Proof. As in Theorem 25, based on Theorem 31. □

See also similar results in [1, p. 464].

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