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BASINS OF ATTRACTION OF AN ANTI-COMPETITIVE DISCRETE RATIONAL SYSTEM

M. NURKANOVIĆ AND Z. NURKANOVIĆ

Dedicated to Professor Mustafa Kulenović on the occasion of his 60th birthday

Abstract. We investigate the global asymptotic behavior of solutions of the following anti-competitive system of difference equations

$$x_{n+1} = \frac{\gamma_1 y_n}{A_1 + x_n}, \quad y_{n+1} = \frac{\beta_2 x_n + \gamma_2 y_n}{y_n}, \quad n = 0, 1, \ldots,$$

where the parameters $\gamma_1, \gamma_2, \beta_2, A_1$ are positive numbers and the initial conditions $x_0 \geq 0, y_0 > 0$. We find the basins of attraction of all attractors of the system, which are the equilibrium point and period-two solutions.

1. Introduction and main result

Consider the following system of difference equations

$$\left.\begin{aligned} x_{n+1} &= \frac{\gamma_1 y_n}{A_1 + x_n}, \\ y_{n+1} &= \frac{\beta_2 x_n + \gamma_2 y_n}{y_n}, \end{aligned}\right\} \quad n = 0, 1, \ldots, \tag{1}$$

where the parameters A_1, β_2, γ_1 and γ_2 are positive numbers and the initial conditions $x_0 \geq 0, y_0 > 0$.

System (1) is the first order system of difference equations of the form

$$\left\{\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned}\right. , \quad n = 0, 1, \ldots, (x_0, y_0) \in \mathcal{R}, \tag{2}$$

$\mathcal{R} \subset \mathbb{R}^2, (f, g) : \mathcal{R} \to \mathcal{R}, f, g$ are continuous functions. System (2) is *competitive* if $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. If the functions f and g have

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monotonic character opposite of the monotonic character in competitive system, then System (2) will be called *anti-competitive.* It is clear that System (1) is anti-competitive.

In the classification of all linear fractional systems in [2], System (1) was mentioned as system (16,26). By the change variables $v_n = y_n - \gamma_2$ we transform System (1) into system which is classified as (31,16) and is dual of system (16, 31). Thus dynamics of System (1) is identical to dynamics of system (31,16). Also, System (1) has a similar dynamics as system (16,16), see [19].

Competitive system of the form (2) were studied by many authors and there is an extensive literature, see [1, 3, 6, 8, 11, 12, 17, 18, 21, 22].

The study of anti-competitive system started in [9] and has advanced since then (see [7, 10, 19]). The principal tool of study of anti-competitive systems is the fact that the second iterate of the map associated with anti-competitive system is a competitive map.

The main result of this paper, i.e. the main result on the global behavior of System (1) is the following theorem.

Theorem 1.1. a) *System* (1) *has a unique positive equilibrium point* $E = (\overline{x}, \overline{y})$ *for all values of the parameters.*

b) *If* $A_1\gamma_2 > \beta_2\gamma_1$*, then* E *is globally asymptotically stable, i.e. the basin of attraction of this equilibrium is* $\mathcal{B}(E) = \{(x, y) : x \geq 0, y > 0\}$.

c) *If* $A_1\gamma_2 < \beta_2\gamma_1$*, then* E *is a saddle point and then there exists a set* $\mathcal{C} \subset \mathcal{R}$ *which is an invariant subset of the basin of attraction of* E*. The set* $\mathcal{C}$ *is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates* $\mathcal{R}$ *into two connected and invariant components, namely*

$$\mathcal{W}_- := \{\mathbf{x} \in \mathcal{R} \backslash \mathcal{C} : \exists \mathbf{y} \in \mathcal{C} \text{ with } \mathbf{x} \preceq_{se} \mathbf{y}\},$$
$$\mathcal{W}_+ := \{\mathbf{x} \in \mathcal{R} \backslash \mathcal{C} : \exists \mathbf{y} \in \mathcal{C} \text{ with } \mathbf{y} \preceq_{se} \mathbf{x}\}.$$

which satisfy:

i) *If* $(x_0, y_0) \in \mathcal{W}_+$*, then*

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (\infty, \gamma_2) \quad \text{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (0, \infty).$$

ii) *If* $(x_0, y_0) \in \mathcal{W}_-$*, then*

$$\lim_{n\to\infty} (x_{2n}, y_{2n}) = (0, \infty) \quad \text{and} \quad \lim_{n\to\infty} (x_{2n+1}, y_{2n+1}) = (\infty, \gamma_2).$$

d) *Assume that* $A_1\gamma_2 = \beta_2\gamma_1$*. Then:*

i) *System* (1) *has an infinite number of minimal period-two solutions belong to a curve*

$$xy^2\gamma_1 + y(A_1 + x)(xA_1 - \gamma_1\gamma_2) - x\gamma_2 A_1 (A_1 + x) = 0.$$

ii) *The equilibrium point E and all minimal period-two solutions are stable but not asymptotically stable.*

iii) *There exists a family of strictly increasing curves $\mathcal{C}, \mathcal{C}_x$ and $\mathcal{C}^x$ for $x > 0$, that emanate from E, A_x and B_x, respectively, given by* (10) *and* (11)*, such that the curves are pairwise disjoint, the union of all the curves equals $\mathbb{R}^2_+$, and solutions with initial point in $\mathcal{C}$ converging to E, solutions with initial point in $\mathcal{C}_x$ have even-indexed terms converging to A_x and odd-terms converging to B_x, and, solution with initial point in $\mathcal{C}^x$ have even-indexed terms converging to B_x and odd-terms converging to A_x.*

2. Preliminaries

We now give some basic notions about competitive systems and maps in the plane of the form of (2) where f and g are continuous functions and $f(x, y)$ is non-decreasing in x and non-increasing in y and $g(x, y)$ is non-increasing in x and non-decreasing in y in some domain A with non-empty interior.

Consider a map $T = (f, g)$ on a set $\mathcal{R} \subset \mathbb{R}^2$, and let $E \in \mathcal{R}$. The point $E \in \mathcal{R}$ is called a *fixed point* if $T(E) = E$. An *isolated* fixed point is a fixed point that has a neighborhood with no other fixed points in it. A fixed point $E \in \mathcal{R}$ is an *attractor* if there exists a neighborhood $\mathcal{U}$ of E such that $T^n(\mathbf{x}) \to E$ as $n \to \infty$ for $\mathbf{x} \in \mathcal{U}$; the *basin of attraction* is the set of all $\mathbf{x} \in \mathcal{R}$ such that $T^n(\mathbf{x}) \to E$ as $n \to \infty$. A fixed point E is a global attractor on a set $\mathcal{K}$ if E is an attractor and $\mathcal{K}$ is a subset of the basin of attraction of E. If T is differentiable at a fixed point E, and if the Jacobian $J_T(E)$ has one eigenvalue with modulus less than one and a second eigenvalue with modulus greater than one, E is said to be a *saddle*. See [20] for additional definitions.

Next, we give some basic facts about the monotone maps in the plane, see [12, 14, 15, 21]. Now, we write System (1) in the form:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = T \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

where the map T is given as

$$T(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1 y}{A_1 + x} \\ \frac{\beta_2 x + \gamma_2 y}{y} \end{pmatrix}. \tag{3}$$

Now, we define a partial order $\preceq$ on $\mathbb{R}^2$ so that the positive cone in this partial order is the fourth quadrant. Specifically, for $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ we say that $\mathbf{u} \preceq \mathbf{v}$ if $u_1 \le v_1$ and $v_2 \le u_2$. Two points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_+$ are said to be *related* if $\mathbf{u} \preceq \mathbf{v}$ or $\mathbf{v} \preceq \mathbf{u}$.

Also, a strict inequality between points may be defined as $\mathbf{u} \prec \mathbf{v}$ if $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$. A stronger inequality may be defined as $\mathbf{u} \prec\prec \mathbf{v}$ if $u_1 < v_1$ and $v_2 < u_2$. A map $f : \operatorname{Int}\mathbb{R}^2_+ \to \operatorname{Int}\mathbb{R}^2_+$ is *strongly monotone* if $\mathbf{u} \prec \mathbf{v}$ implies that $f(\mathbf{u}) \prec\prec f(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \operatorname{Int}\mathbb{R}^2_+$. Clearly, being related is an invariant under iteration of a strongly monotone map. Differentiable strongly monotone maps have Jacobian with constant sign configuration

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}.$$

The mean value theorem and the convexity of $\mathbb{R}^2_+$ may be used to show that T is monotone, as in [4].

For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, define $Q_i(\mathbf{x})$ for $i = 1, \ldots, 4$ to be the usual four quadrants based at $\mathbf{x}$ and numbered in a counterclockwise direction, for example, $Q_1(\mathbf{x}) = \left\{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : x_1 \le y_1, x_2 \le y_2\right\}$. We now state three results for competitive maps in the plane.

The following definition is from [21].

Definition 2.1. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition $(O+)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition $(O-)$ if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by de Mottoni-Schiaffino for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [21].

Theorem 2.1. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which $(O+)$ holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead $(O-)$ holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [21], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 2.2. *Let $\mathcal{R} \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$, then T satisfies $(O+)$. If T is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$, then T satisfies $(O-)$.*

The next two results are from [14, 15].

Theorem 2.3. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \mathrm{int}\,(\mathcal{Q}_1(\overline{\mathrm{x}}) \cup \mathcal{Q}_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true:*

a. *The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.*
b. *The Jacobian matrix of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^{λ} associated with λ is not a coordinate axis.*

Then, there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^{λ} at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Theorem 2.4. *Let $\mathcal{I}_1, \mathcal{I}_2$ be intervals in $\mathbb{R}$ with endpoints a_1, a_2 and b_1, b_2 with endpoints respectively, with $a_1 < a_2$ and $b_1 < b_2$, where $-\infty \leq a_1 < a_2 \leq \infty$ and $-\infty \leq b_1 < b_2 \leq \infty$. Let T be a competitive map on an rectangle $\mathcal{R} = \mathcal{I}_1 \times \mathcal{I}_2$ and $\overline{\mathbf{x}} \in int\,(\mathcal{R})$. Suppose that the following hypotheses are satisfied:*

1. *$T\,(int\,(\mathcal{R})) \subset int\,(\mathcal{R})$ and T is strongly competitive on $int\,(\mathcal{R})$.*
2. *The point $\overline{\mathbf{x}}$ is the only fixed point of T in $(Q_1\,(\overline{\mathbf{x}}) \cup Q_3\,(\overline{\mathbf{x}})) \cap int\,(\mathcal{R})$.*
3. *The map T is continuously differentiable in a neighborhood of $\overline{\mathbf{x}}$.*
4. *At least one of the following statements is true.*

   a. *T has no minimal period two orbits in $(Q_1\,(\overline{\mathbf{x}}) \cup Q_3\,(\overline{\mathbf{x}})) \cap int\,(\mathcal{R})$.*
   b. *$\det J_T\,(\overline{\mathbf{x}}) > 0$ and $T\,(\mathbf{x}) = \overline{\mathbf{x}}$ only for $\mathbf{x} = \overline{\mathbf{x}}$.*

5. *$\overline{\mathbf{x}}$ is a saddle point.*

Then, the following statements are true:

(i.) *The stable set $\mathcal{W}^s(\overline{\mathbf{x}})$ is connected and it is the graph of a continuous increasing curve with endpoints in $\partial\mathcal{R}$. $int\,(\mathcal{R})$ is divided by the closure of $\mathcal{W}^s(\overline{\mathbf{x}})$ into two invariant connected regions $\mathcal{W}_+$ ("below the stable set"), and $\mathcal{W}_-$ ("above the stable set"), where*

$$\mathcal{W}_- := \left\{\mathbf{x} \in \mathcal{R} \backslash \mathcal{W}^s(\overline{\mathbf{x}}) : \exists \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ with } \mathbf{x} \preceq_{se} \mathbf{x}'\right\},$$
$$\mathcal{W}_+ := \left\{\mathbf{x} \in \mathcal{R} \backslash \mathcal{W}^s(\overline{\mathbf{x}}) : \exists \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ with } \mathbf{x}' \preceq_{se} \mathbf{x}\right\}.$$

(ii.) *The unstable set $\mathcal{W}^u(\overline{\mathbf{x}})$ is connected and it is the graph of a continuous decreasing curve.*

(iii.) *For every $\mathbf{x} \in \mathcal{W}_+$, $T^n\,(\mathbf{x})$ eventually enters the interior of the invariant set $Q_4\,(\overline{\mathbf{x}}) \cap \mathcal{R}$, and for every $\mathbf{x} \in \mathcal{W}_-$, $T^n\,(\mathbf{x})$ eventually enters the interior of the invariant set $Q_2\,(\overline{\mathbf{x}}) \cap \mathcal{R}$.*

(iv.) *Let* $\mathbf{m} \in Q_2(\overline{\mathbf{x}})$ *and* $\mathbf{M} \in Q_4(\overline{\mathbf{x}})$ *be the endpoints of* $\mathcal{W}^u(\overline{\mathbf{x}})$*, where* $\mathbf{m} \preceq_{se} \overline{\mathbf{x}} \preceq_{se} \mathbf{M}$*. For every* $\mathbf{x} \in \mathcal{W}_-$ *and every* $\mathbf{z} \in \mathcal{R}$ *such that* $\mathbf{m} \preceq_{se} z$*, there exists* $m \in \mathbb{N}$ *such that* $T^m(\mathbf{x}) \preceq_{se} z$*, and for every* $\mathbf{x} \in \mathcal{W}_+$ *and every* $\mathbf{z} \in \mathcal{R}$ *such that* $\mathbf{z} \preceq_{se} \mathbf{M}$*, there exists* $m \in \mathbb{N}$ *such that* $\mathbf{M} \preceq_{se} T^m(\mathbf{x})$*.*

The following result gives a convergence result for a system in $\mathbb{R}^2$ when there exists an invariant rectangle, and the map of the system is an anti-competitive and satisfies certain conditions. See [10].

Theorem 2.5. *Let* T *be an anti-competitive map on a closed and bounded rectangular region* $\mathcal{R} \subset \mathbb{R}^2$*. Suppose that* T *has a unique fixed point* $\bar{\mathbf{e}}$ *in* $\mathcal{R}$ *and that* T *has no minimal period-two solutions. Then,* $\bar{\mathbf{e}}$ *is globally asymptotically stable on* $\mathcal{R}$*.*

3. Linearized stability analysis

Theorem 3.1. *System* (1) *has the unique positive equilibrium point* E *for all values of the parameters.*

i) *If* $A_1\gamma_2 - \beta_2\gamma_1 > 0$*, then* E *is locally asymptotically stable.*
ii) *If* $A_1\gamma_2 - \beta_2\gamma_1 < 0$*, then* E *is a saddle point.*
iii) *If* $A_1\gamma_2 - \beta_2\gamma_1 = 0$*, then* E *is non-hyperbolic.*

Proof. The equilibrium point $E = (\overline{x}, \overline{y})$ of System (1) satisfies the following system of equations:

$$\overline{x} = \frac{\gamma_1 \overline{y}}{A_1 + \overline{x}}, \;\; \overline{y} = \frac{\beta_2 \overline{x} + \gamma_2 \overline{y}}{\overline{y}}, \tag{4}$$

i.e.

$$\overline{x}^2 = \gamma_1 \overline{y} - A_1 \overline{x}, \;\; \overline{y}^2 = \beta_2 \overline{x} + \gamma_2 \overline{y}.$$

This implies that System (1) has a unique positive equilibrium point $E = (\overline{x}, \overline{y})$, which is an intersection of the following two parabolas:

$$\overline{y} = \frac{1}{\gamma_1}\left(\overline{x}^2 + A_1\overline{x}\right), \;\; \overline{x} = \frac{1}{\beta_2}\left(\overline{y}^2 - \gamma_2\overline{y}\right). \tag{5}$$

The map T associated to System (1) is (3) and the Jacobian matrix of T at the equilibrium point $E = (\overline{x}, \overline{y})$ is

$$J_T(\overline{x}, \overline{y}) = \begin{pmatrix} -\frac{\gamma_1 \overline{y}}{(A_1+\overline{x})^2} & \frac{\gamma_1}{A_1+\overline{x}} \\ \frac{\beta_2}{\overline{y}} & -\frac{\beta_2 \overline{x}}{\overline{y}^2} \end{pmatrix} = \begin{pmatrix} -\frac{\overline{x}^2}{\gamma_1 \overline{y}} & \frac{\overline{x}}{\overline{y}} \\ \frac{\beta_2}{\overline{y}} & -\frac{\beta_2 \overline{x}}{\overline{y}^2} \end{pmatrix}. \tag{6}$$

The corresponding characteristic equation is of the form:

$$\lambda^2 - p\lambda + q = 0,$$

where

$$p = TrJ_T(E) = -\frac{\overline{x}^2}{\gamma_1\overline{y}} - \frac{\beta_2\overline{x}}{\overline{y}^2} < 0,$$
$$q = DetJ_T(E) = \frac{\beta_2\overline{x}^3}{\gamma_1\overline{y}^3} - \frac{\beta_2\overline{x}}{\overline{y}^2} = \frac{\beta_2\overline{x}}{\gamma_1\overline{y}^3}\left(\overline{x}^2 - \gamma_1\overline{y}\right) = -\frac{A_1\beta_2\overline{x}^2}{\gamma_1\overline{y}^3} < 0.$$

i) It is obvious that $1+q<2$. On other hand:
$|p| < 1+q \Leftrightarrow \frac{\overline{x}^2}{\gamma_1\overline{y}} + \frac{\beta_2\overline{x}}{\overline{y}^2} < 1 + \frac{\beta_2\overline{x}^3}{\gamma_1\overline{y}^3} - \frac{\beta_2\overline{x}}{\overline{y}^2}$
$\overset{(4)}{\Leftrightarrow} \frac{\gamma_1\overline{y}-A_1\overline{x}}{\gamma_1\overline{y}} + \frac{\beta_2\overline{x}}{\overline{y}^2} < 1 + \frac{\beta_2\overline{x}^3}{\gamma_1\overline{y}^3} - \frac{\beta_2\overline{x}}{\overline{y}^2} \Leftrightarrow -A_1\overline{y}^2 + 2\beta_2\gamma_1\overline{y} < \beta_2\overline{x}^2$
$\overset{(4)}{\Leftrightarrow} -A_1\left(\beta_2\overline{x}+\gamma_2\overline{y}\right) + 2\beta_2\gamma_1\overline{y} < \beta_2\left(\gamma_1\overline{y} - A_1\overline{x}\right) \Leftrightarrow A_1\gamma_2 > \beta_2\gamma_1,$
which implies that E is locally asymptotically stable if $A_1\gamma_2 > \beta_2\gamma_1$.

ii) Now, we check conditions for E to be a saddle point.

(1) $p^2 - 4q > 0$ because $q<0$.

(2) $|p| > |1+q| \Leftrightarrow p^2 > (1+q)^2 \Leftrightarrow (p-1-q)(p+1+q) > 0$.

1° $p-1-q = -\frac{\overline{x}^2}{\gamma_1\overline{y}} - \frac{\beta_2\overline{x}}{\overline{y}^2} - 1 - \frac{\beta_2\overline{x}^3}{\gamma_1\overline{y}^3} + \frac{\beta_2\overline{x}}{\overline{y}^2} = -\frac{\overline{x}^2}{\gamma_1\overline{y}} - 1 - \frac{\beta_2\overline{x}^3}{\gamma_1\overline{y}^3} < 0,$

2° We see that **i)** implies: $p+1+q>0 \Leftrightarrow A_1\gamma_2 - \beta_2\gamma_1 > 0$, from which

$$p+1+q<0 \Leftrightarrow A_1\gamma_2 < \beta_2\gamma_1.$$

iii) It is obvious that

$$|p| = |1+q| \Leftrightarrow A_1\gamma_2 = \beta_2\gamma_1,$$

i.e. now E is a non-hyperbolic equilibrium point. □

4. Period-two solutions

In this section, we present the results for the existence of period-two solutions of System (1).

Lemma 4.1. *If $A_1\gamma_2 \neq \beta_2\gamma_1$, then System* (1) *has no minimal period-two solutions.*

Proof. System (1) can be reduced to the following second-order difference equation:

$$x_{n+2} = \gamma_1\frac{\gamma_2A_1x_{n+1} + \gamma_2x_nx_{n+1} + \beta_2\gamma_1x_n}{\left(A_1+x_{n+1}\right)x_{n+1}\left(A_1+x_n\right)}, \tag{7}$$

or to the following second-order difference equation:

$$y_{n+2} = \frac{\gamma_2y_ny_{n+1}^2 - \gamma_2^2y_ny_{n+1} + \beta_2^2\gamma_1y_n + \beta_2\gamma_2A_1y_{n+1}}{\left(y_ny_{n+1} + \beta_2A_1 - \gamma_2y_n\right)y_{n+1}}. \tag{8}$$

Now, we prove that both of the difference equations (7) and (8) have no minimal period-two solutions. Assume that this is not true for equation (7), that is that

$$\varphi, \psi, \varphi, \psi, \ldots, (\varphi \neq \psi)$$

is a minimal period-two solution of (7). Then, we have:

$$\begin{aligned} \varphi &= \gamma_1 \frac{\gamma_2 A_1 \psi + \gamma_2 \varphi\psi + \beta_2\gamma_1\varphi}{(A_1+\psi)\,\psi\,(A_1+\varphi)}, \\ \psi &= \gamma_1 \frac{\gamma_2 A_1 \varphi + \gamma_2 \psi\varphi + \beta_2\gamma_1\psi}{(A_1+\varphi)\,\varphi\,(A_1+\psi)}, \end{aligned} \tag{9}$$

from which:

$$\begin{aligned} \varphi\psi\,(A_1+\psi)\,(A_1+\varphi) &= \gamma_1\left(\gamma_2 A_1\psi + \gamma_2\varphi\psi + \beta_2\gamma_1\varphi\right), \\ \varphi\psi\,(A_1+\psi)\,(A_1+\varphi) &= \gamma_1\left(\gamma_2 A_1\varphi + \gamma_2\psi\varphi + \beta_2\gamma_1\psi\right). \end{aligned}$$

By subtraction, we obtain

$$(\psi-\varphi)\,(A_1\gamma_2 - \beta_2\gamma_1) = 0,$$

and this implies that $\psi = \varphi$, which is a contradiction.

Now, assume that

$$\chi, \phi, \chi, \phi, \ldots, (\chi \neq \phi)$$

is a minimal period-two solution of equation (8). Then, we have:

$$\begin{aligned} \chi &= \frac{\gamma_2\chi\phi^2 - \gamma_2^2\chi\phi + \beta_2^2\gamma_1\chi + \beta_2\gamma_2 A_1\phi}{(\chi\phi + \beta_2 A_1 - \gamma_2\chi)\,\phi}, \\ \phi &= \frac{\gamma_2\phi\chi^2 - \gamma_2^2\phi\chi + \beta_2^2\gamma_1\phi + \beta_2\gamma_2 A_1\chi}{(\phi\chi + \beta_2 A_1 - \gamma_2\phi)\,\chi}. \end{aligned}$$

This implies:

$$\begin{aligned} (\chi\phi + \beta_2 A_1 - \gamma_2\chi)\,\phi\chi &= \gamma_2\chi\phi^2 - \gamma_2^2\chi\phi + \beta_2^2\gamma_1\chi + \beta_2\gamma_2 A_1\phi, \\ (\phi\chi + \beta_2 A_1 - \gamma_2\phi)\,\chi\phi &= \gamma_2\phi\chi^2 - \gamma_2^2\phi\chi + \beta_2^2\gamma_1\phi + \beta_2\gamma_2 A_1\chi. \end{aligned}$$

By subtracting, we obtain:

$$(A_1\gamma_2 - \beta_2\gamma_1)\,(\phi - \chi) = 0,$$

from which $\phi = \chi$. It is a contradiction. □

Lemma 4.2. *If $A_1\gamma_2 = \beta_2\gamma_1$, then System* (1) *has continuum of minimal period-two solutions A_x and B_x, for $x > 0$, of the form:*

$$\begin{aligned} A_x &= \left(x, \tfrac{-(A_1+x)(xA_1-\gamma_1\gamma_2)+\sqrt{(A_1+x)^2(xA_1-\gamma_1\gamma_2)^2+4x^2\gamma_1\gamma_2A_1(A_1+x)}}{2x\gamma_1}\right), \\ B_x &= (x_B, y_B), \end{aligned} \tag{10}$$

where

$$\begin{aligned} x_B &= \frac{-(A_1+x)(xA_1-\gamma_1\gamma_2)+\sqrt{(A_1+x)^2(xA_1-\gamma_1\gamma_2)^2+4x^2\gamma_1\gamma_2A_1(A_1+x)}}{2x(A_1+x)}, \\ y_B &= \frac{(x+A_1)(xA_1+\gamma_1\gamma_2)+\sqrt{(A_1+x)^2(xA_1-\gamma_1\gamma_2)^2+4x^2\gamma_1\gamma_2A_1(A_1+x)}}{2\gamma_1(x+A_1)}. \end{aligned} \tag{11}$$

Proof. Minimal period-two solution of System (1) is a fixed point of the map T^2, which is equivalent to the following system

$$\begin{cases} u(x,y) = x \\ v(x,y) = x \end{cases} \tag{12}$$

i.e.

$$\begin{cases} \frac{(x\beta_2+y\gamma_2)\gamma_1(x+A_1)}{y\left(A_1^2+xA_1+y\gamma_1\right)} = x \\ \frac{y^2\beta_2\gamma_1+(y\gamma_2+x\beta_2)\gamma_2(x+A_1)}{(x\beta_2+y\gamma_2)(x+A_1)} = y \end{cases}$$

$$\Leftrightarrow \begin{cases} (x\beta_2+y\gamma_2)\gamma_1(x+A_1) = xy\left(A_1^2+xA_1+y\gamma_1\right) \\ y^2\beta_2\gamma_1+(y\gamma_2+x\beta_2)\gamma_2(x+A_1) = y(x\beta_2+y\gamma_2)(x+A_1) \end{cases} \tag{13}$$

$$\Leftrightarrow \begin{cases} (x\beta_2+y\gamma_2)\gamma_1(x+A_1) = xy\left(A_1^2+xA_1+y\gamma_1\right) \\ y\left(-y\gamma_1+xA_1+x^2\right)(\beta_2\gamma_1-\gamma_2A_1) = 0. \end{cases} \tag{14}$$

It is easy to see that $x > 0$ and $y > 0$. Since $x \neq \overline{x}, y \neq \overline{y}$, from the second equation in (14) we obtain $A_1\gamma_2 = \beta_2\gamma_1$. Now, the first equation in (14) is of the form:

$$xy^2\gamma_1 + y(A_1+x)(xA_1-\gamma_1\gamma_2) - x\gamma_2A_1(A_1+x) = 0. \tag{15}$$

So, System (1) has infinitely many minimal period-two solutions which are located along the third order curve given by (15), i.e.

$$\mathcal{G} = \left\{(x,y) \in (0,\infty)^2 : xy^2\gamma_1 + y(A_1+x)(xA_1-\gamma_1\gamma_2) - x\gamma_2A_1(A_1+x) = 0\right\}. \tag{16}$$

It is easy to see that $A_x, B_x \in \mathcal{G}$ and that $T(A_x) = B_x, T(B_x) = A_x$ if $A_1\gamma_2 = \beta_2\gamma_1$. □

5. Global results

In this section, we present results about basins of attraction of System (1).

Lemma 5.1. *The map* T^2 *is injective and* $\det J_{T^2(x,y)} > 0$ *for all* $x \geq 0$ *and* $y > 0$.

Proof. 1) The map T^2 is of the form:

$$\begin{aligned} T^2(x,y) &= (u(x,y), v(x,y)) \\ &= \left(\frac{(x\beta_2+y\gamma_2)\gamma_1(x+A_1)}{y\left(A_1^2+xA_1+y\gamma_1\right)}, \frac{y^2\beta_2\gamma_1+(y\gamma_2+x\beta_2)\gamma_2(x+A_1)}{(x\beta_2+y\gamma_2)(x+A_1)} \right). \end{aligned} \tag{17}$$

Now, we prove that the map T^2 is injective. Indeed, $T(x_1,y_1) = T(x_2,y_2)$ implies that

$$A_1(y_1 - y_2) = x_1 y_2 - x_2 y_1 = 0, \tag{18}$$

from which we obtain $y_1 = y_2$ and $x_1 = x_2$, i.e. $(x_1,y_1) = (x_2,y_2)$. This implies that T is injective and that T^2 is injective, too.

2) For the map $T^2(x,y) = (u(x,y), v(x,y))$ we have:

$$J_{T^2}(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

where

$$\begin{aligned} u_x &= \frac{\left(\beta_2 A_1^3 + y^2\gamma_1\gamma_2 + 2x\beta_2 A_1^2 + x^2\beta_2 A_1 + 2xy\beta_2\gamma_1 + y\beta_2\gamma_1 A_1\right)\gamma_1}{y\left(y\gamma_1 + xA_1 + A_1^2\right)^2}, \\ u_y &= -\frac{\gamma_1(x+A_1)\left(y^2\gamma_1\gamma_2 + x\beta_2 A_1^2 + x^2\beta_2 A_1 + 2xy\beta_2\gamma_1\right)}{y^2\left(y\gamma_1 + xA_1 + A_1^2\right)^2}, \\ v_x &= -y^2\beta_2\gamma_1 \frac{2x\beta_2 + y\gamma_2 + \beta_2 A_1}{(x\beta_2 + y\gamma_2)^2 (x+A_1)^2}, \quad v_y = \frac{y\beta_2\gamma_1(2x\beta_2 + y\gamma_2)}{(x+A_1)(x\beta_2 + y\gamma_2)^2}. \end{aligned}$$

After some simplifications, we obtain:

$$\det J_{T^2} = \frac{\beta_2^2\gamma_1^2 A_1^2}{\left(y\gamma_1 + xA_1 + A_1^2\right)^2 (x\beta_2 + y\gamma_2)} > 0$$

for all $x \geq 0$ and $y > 0$, and the Jacobian matrix of $T^2(x,y)$ is invertible. $\square$

Remark 5.1. By Lemma 5.1 and Theorems 2.1 and 2.2 we see that the map T^2 satisfies condition $(O+)$ and consequently, the sequences $\{x_{2n}\}$, $\{x_{2n+1}\}$, $\{y_{2n}\}$, $\{y_{2n+1}\}$ of every solution of System (1) are eventually monotone.

Lemma 5.2. *The map T^2 associated to System* (1) *satisfied the following:*

$$T^2(x,y) = (\overline{x}, \overline{y}) \quad \textit{only for} \ (x,y) = (\overline{x}, \overline{y}).$$

Proof. Since T^2 is injective, then

$$T^2(x,y) = (\overline{x}, \overline{y}) = T^2(\overline{x}, \overline{y}) \Rightarrow (x,y) = (\overline{x}, \overline{y}).$$

$\square$

Proof of Theorem 1.1.

Case 1: $A_1\gamma_2 > \beta_2\gamma_1$

Since $f(x,y)$ is decreasing in x and increasing in y, while $g(x,y)$ is increasing in x and decreasing in y, we have:

$$\begin{aligned} 0 \le x_{n+1} &= \frac{\gamma_1 y_n}{A_1 + x_n} \le \frac{\gamma_1 U_2}{A_1} \le U_1, \\ \gamma_2 \le y_{n+1} &= \frac{\beta_2 x_n + \gamma_2 y_n}{y_n} = \frac{\beta_2 x_n}{y_n} + \gamma_2 \le \frac{\beta_2 U_1}{\gamma_2} + \gamma_2 \le U_2, \end{aligned}$$

which implies:

$$U_1 \ge \frac{\gamma_1 U_2}{A_1} \ge \frac{\gamma_1}{A_1}\left(\frac{\beta_2 U_1}{\gamma_2} + \gamma_2\right) \Leftrightarrow U_1 \ge \frac{\gamma_1\gamma_2^2}{A_1\gamma_2 - \beta_2\gamma_1} > 0,$$

and

$$U_2 \ge \frac{\beta_2 U_1}{\gamma_2} + \gamma_2 \ge \frac{\beta_2\gamma_1\gamma_2}{A_1\gamma_2 - \beta_2\gamma_1} + \gamma_2 = \frac{A_1\gamma_2^2}{A_1\gamma_2 - \beta_2\gamma_1} > 0,$$

i.e.

$$U_1 \ge \frac{\gamma_1\gamma_2^2}{A_1\gamma_2 - \beta_2\gamma_1}, U_2 \ge \frac{A_1\gamma_2^2}{A_1\gamma_2 - \beta_2\gamma_1}. \tag{19}$$

This shows that

$$[0, U_1] \times [\gamma_2, U_2]$$

is an attractive box.

Now, we show that $\mathcal{R} = [0, U_1^*] \times [\gamma_2, U_2^*]$, where $U_1^* = \frac{\gamma_1\gamma_2^2}{A_1\gamma_2 - \beta_2\gamma_1}$ and $U_2^* = \frac{A_1\gamma_2^2}{A_1\gamma_2 - \beta_2\gamma_1}$, is an invariant box, i.e. $T(\mathcal{R}) \subset \mathcal{R}$. Indeed, suppose that $(x,y) \in \mathcal{R}$. Then, we have:

$$\begin{aligned} 0 \le f(x,y) &= \frac{\gamma_1 y}{A_1 + x} \le \frac{\gamma_1 y}{A_1} \le \frac{\gamma_1 U_2^*}{A_1} = U_1^*, \\ \gamma_2 \le g(x,y) &= \frac{\beta_2 x}{y} + \gamma_2 \le \frac{\beta_2 U_1^*}{y} + \gamma_2 = U_2^*. \end{aligned}$$

By Lemma 4.1 and Theorem 2.5 equilibrium $E = (\overline{x}, \overline{y})$ is globally asymptotically stable.

Case 2: $A_1\gamma_2 < \beta_2\gamma_1$

It is easy to check that $E = (\overline{x}, \overline{y})$ is a saddle point for T^2 as well. System (1) can be decomposed into the system of the even-indexed and odd-indexed terms as follows:

$$\begin{cases} x_{2n+1} = \frac{\gamma_1 y_{2n}}{A_1+x_{2n}}, \\ x_{2n} = \frac{\gamma_1 y_{2n-1}}{A_1+x_{2n-1}}, \\ y_{2n+1} = \frac{\beta_2 x_{2n}}{y_{2n}} + \gamma_2, \\ y_{2n} = \frac{\beta_2 x_{2n-1}}{y_{2n-1}} + \gamma_2. \end{cases}$$

The existence of the set $\mathcal{C}$ with stated properties follows from Lemmas 5.1, 4.1, 4.2 and 5.2 and Theorems 2.3 and 2.4.

Case 3: $A_1\gamma_2 = \beta_2\gamma_1$

Lemma 5.3. *Assume that* $\beta_2\gamma_1 = A_1\gamma_2$. *Then, the following statements are true.*

i) *All periodic points* $A_x, B_x \in \mathcal{G}$, *given by* (10) *and* (11), *are non-hyperbolic fixed points for the map* T^2, *and in both of them the corresponding Jacobian matrix of the the map* T^2 *has eigenvalues* $\lambda_1 = 1$ *and* $\lambda_2 \in (0,1)$.

ii) *Eigenvectors corresponding to the eigenvalues* λ_1 *and* λ_2 *are not parallel to coordinate axes.*

Proof. i) From (16), we obtain

$$y_{\mathcal{G}-}(x) = \frac{-(A_1+x)(xA_1-\gamma_1\gamma_2)-\sqrt{(A_1+x)^2(xA_1-\gamma_1\gamma_2)^2+4x^2\gamma_1\gamma_2A_1(A_1+x)}}{2x\gamma_1}, x \neq 0,$$

$$y_{\mathcal{G}+}(x) = \frac{-(A_1+x)(xA_1-\gamma_1\gamma_2)+\sqrt{(A_1+x)^2(xA_1-\gamma_1\gamma_2)^2+4x^2\gamma_1\gamma_2A_1(A_1+x)}}{2x\gamma_1}, x \neq 0,$$

and $y_{\mathcal{G}-}(x) < 0$, $y_{\mathcal{G}+}(x) > 0$ for $x > 0$.

The curve $y_{\mathcal{G}+}(x)$ is decreasing in x, that is $y'_{\mathcal{G}+}(x) < 0$. Indeed, from (12) we have:

$$u_x + u_y y' = 1,$$
$$v_x + v_y y' = y'.$$

If $(x,y) \in \mathcal{G}$, then

$$y'_{\mathcal{G}+}(x) = \frac{1-\Gamma}{\Lambda} = \frac{\Theta}{1-\Omega}, \tag{20}$$

where

$$\Gamma := u_x(x,y), \Lambda := u_y(x,y), \Theta := v_x(x,y), \Omega := v_y(x,y).$$

Since $\Theta < 0$ for $x > 0$, it is sufficient to prove that $\Omega < 1$. Namely, if $\beta_2\gamma_1 = A_1\gamma_2$, then we have:

$$\Omega = \frac{y\beta_2\gamma_1(2x\beta_2+y\gamma_2)}{(x+A_1)(x\beta_2+y\gamma_2)^2} = \frac{yA_1\gamma_2(2x\beta_2+y\gamma_2)}{(x+A_1)(x\beta_2+y\gamma_2)^2} < 1$$
$$\Leftrightarrow x\left(x^2\beta_2^2 + y^2\gamma_2^2 + x\beta_2^2A_1 + 2xy\beta_2\gamma_2\right) > 0,$$

which is satisfied for all $x > 0$ and $y > 0$.

Since $\Gamma > 0, \Lambda < 0$ and $y'_{\mathcal{G}+}(x) = \dfrac{1-\Gamma}{\Lambda} = \dfrac{\Theta}{1-\Omega} < 0$, we have that $\Gamma \in (0,1)$. The characteristic polynomial of the matrix

$$J_{T^2}^{\mathcal{G}}(x,y) = \begin{pmatrix} \Gamma & \Lambda \\ \Theta & \Omega \end{pmatrix}$$

is of the form:

$$P(\lambda) = \lambda^2 - (\Gamma + \Omega)\lambda + (\Gamma\Omega - \Lambda\Theta) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

On other hand, (20) implies that $(1-\Gamma)(1-\Omega) = \Lambda\Theta$, i.e.

$$P(1) = 1 - (\Gamma + \Omega) + (\Gamma\Omega - \Lambda\Theta) = 0.$$

This means that $\lambda_1 = 1$. So, $\lambda_1 + \lambda_2 = 1 + \lambda_2 = \Gamma + \Omega < 2$, from which follows that $\lambda_2 < 1$. Since $\lambda_1\lambda_2 = \Gamma\Omega - \Lambda\Theta > 0$ and $\lambda_1 = 1$, we have that $\lambda_2 > 0$.

ii) The eigenvectors $\mathbf{v}_1 = (v_1^1, v_2^1)$ and $\mathbf{v}_2 = (v_1^2, v_2^2)$, corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \Gamma\Omega - \Lambda\Theta$, are of the following form:

$$\mathbf{v}_1 = \begin{pmatrix} 1-\Omega \\ \Theta \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} \Gamma - 1 \\ \Theta \end{pmatrix}.$$

Since $\Gamma \in (0,1)$, $\Lambda < 0$, $\Theta < 0$ and $\Omega \in (0,1)$, we see that the eigenvectors are not parallel to coordinate axes. □

Therefore all conditions of Theorem 2.3 are satisfied for the map T^2 with $\mathcal{R} = [0,\infty) \times (0,\infty)$.

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Department of Mathematics
University of Tuzla
Univerzitetska 4, 75000 Tuzla
Bosnia and Herzegovina
E–mail: mehmed.nurkanovic@untz.ba
zehra.nurkanovic@untz.ba