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ON THE GLOBAL BEHAVIOR OF THE RATIONAL SYSTEM $x_{n+1} = \frac{\alpha_1}{x_n + y_n}$ AND $y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + y_n}{y_n}$

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Dedicated to Mustafa Kulenović on the occasion of his 60^{th} birthday and for all his support and guidance

Abstract. We investigate the system of rational difference equations in the title, where the parameters and the initial conditions are positive real numbers. We show that the system is permanent and has a unique positive equilibrium which is locally asymptotically stable. We also find sufficient conditions to insure that the unique positive equilibrium is globally asymptotically stable.

1. Introduction

We show that the system of rational difference equations

$$\begin{cases} x_{n+1} = \dfrac{\alpha_1}{x_n + y_n} \\ \\ y_{n+1} = \dfrac{\alpha_2 + \beta_2 x_n + y_n}{y_n} \end{cases}, \quad n = 0, 1, \ldots \tag{1}$$

is permanent, where the parameters $\alpha_1, \alpha_2, \beta_2$ and the initial conditions x_0, y_0 of the system are positive real numbers. We actually show that there exist positive real numbers l_1, l_2, L_1, L_2 such that for every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1), we have

$$l_1 < x_n < L_1 \quad \text{and} \quad l_2 < y_n < L_2 \qquad \text{for n} \geq 3.$$

We show that the system has a unique positive equilibrium which is locally asymptotically stable. We also find sufficient conditions to insure that the unique positive equilibrium is globally asymptotically stable.

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For the last four years we have been interested in the boundedness character and the global behavior of systems of rational difference equations. This paper is part of a general project which involves the system of rational difference equations

$$\begin{cases} x_{n+1} &= \dfrac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\ \\ y_{n+1} &= \dfrac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{cases}, \quad n = 0, 1, \ldots$$

which includes 2401 special cases. In the numbering system which was introduced by Camouzis, Kulenović, Ladas, and Merino in ([6]), System (1) is #(12, 41). Related work has recently been given in ([1]-[11]) and ([14]-[19]).

The following well-known result is needed for the local asymptotic stability analysis of the equilibrium of System (1).

Theorem 1.1. *Let $F = (f, g)$ be a continuously differentiable function defined on an open set W in R^2, and let $(\bar{x}, \bar{y})$ in W be a fixed point of F.*

(1) *If all the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ have modulus less than one, then the equilibrium point $(\bar{x}, \bar{y})$ is locally asymptotically stable.*

(2) *If at least one of the eigenvalues of the Jacobian matrix $J_F(\bar{x}, \bar{y})$ has modulus greater than one, then the equilibrium point $(\bar{x}, \bar{y})$ is unstable.*

The following theorem gives necessary and sufficient conditions for the two roots of a quadratic equation to have modulus less than one.

Theorem 1.2. ([13]) *Assume p and q are real numbers. Then a necessary and sufficient condition for both roots of the equation*

$$\lambda^2 + p\lambda + q = 0$$

to have modulus less than 1 is that

$$|p| < 1 + q < 2.$$

The next theorem gives a sufficient condition to insure that there exists a unique positive equilibrium, and it is a global attractor. Let k be a positive integer. For $i \in \{1, \ldots, k\}$, assume $[a_i, b_i]$ is a closed and bounded interval, and let $F^i : [a_1, b_1] \times \ldots \times [a_k, b_k] \to [a_i, b_i]$ be a continuous function. For each $i, j \in \{1, \ldots, k\}$, let $M_{i,j} : [a_i, b_i] \to [a_i, b_i]$ and $m_{i,j} : [a_i, b_i] \to [a_i, b_i]$ be defined as follows: given $m_i, M_i \in [a_i, b_i]$
set

$$M_{i,j}(m_i, M_i) = \begin{cases} M_i, & \text{if } \mathrm{F^j} \text{ is increasing in } z_i \\ m_i, & \text{if } \mathrm{F^j} \text{ is non-increasing in } z_i \end{cases}$$

and

$$m_{i,j}(m_i, M_i) = M_{i,j}(M_i, m_i).$$

Theorem 1.3. ([12]) *Assume that each $i \in \{1, \ldots, k\}$, $[a_i, b_i]$ is a closed and bounded interval of real numbers, and the function*

$$F^i : C([a_1, b_1] \times \ldots \times [a_k, b_k], [a_i, b_i]),$$

satisfies the following conditions:

(1) *$F^i(z_1, \ldots, z_k)$ is weakly monotonic in each of its arguments.*
(2) *If $M_1, \ldots, M_k, m_1, \ldots, m_k$, where $m_i \le M_i$ for each $i \in \{1, \ldots, k\}$, is a solution of the system of 2k equations:*

$$\begin{cases} M_i = F^i(M_{1,i}(m_1, M_1), \ldots, M_{k,i}(m_k, M_k)) \\ m_i = F^i(m_{1,i}(m_1, M_1), \ldots, m_{k,i}(m_k, M_k)) \end{cases}, \quad i \in \{1, \ldots, k\}$$

then

$$M_i = m_i, \text{for all } i \in \{1, \ldots, k\}.$$

Then the system of k difference equations:

$$\begin{cases} x^1_{n+1} = F^1(x^1_n, \ldots, x^k_n) \\ x^2_{n+1} = F^2(x^1_n, \ldots, x^k_n) \\ \vdots \\ x^k_{n+1} = F^k(x^1_n, \ldots, x^k_n) \end{cases}, \quad n = 0, 1, \ldots$$

with initial condition $(x^1_0, \ldots, x^k_0) \in [a_1, b_1] \times \ldots \times [a_k, b_k]$, has exactly one equilibrium point $(\bar{x}^1, \ldots, \bar{x}^k)$, and it is a global attractor.

2. Local stability of system (1)

Lemma 2.1. *System* (1) *has a unique equilibrium $(\bar{x}, \bar{y})$. Moreover, $(\bar{x}, \bar{y})$ is locally asymptotically stable.*

Proof. Suppose $(\bar{x}, \bar{y})$ is a feasible equilibrium of System (1). That is

$$\bar{x} = \frac{\alpha_1}{\bar{x} + \bar{y}} \quad \text{and} \quad \bar{y} = \frac{\alpha_2 + \beta_2 \bar{x} + \bar{y}}{\bar{y}}.$$

Note that $\bar{x} < \sqrt{\alpha_1}$ and $\bar{y} = \dfrac{\alpha_1 - \bar{x}^2}{\bar{x}}$

and so

$$\frac{\alpha_1 - \bar{x}^2}{\bar{x}} = \bar{y} = \frac{\alpha_2 + \beta_2 \bar{x} + \bar{y}}{\bar{y}} = \frac{\alpha_2 + \beta_2 \bar{x} + \frac{\alpha_1 - \bar{x}^2}{\bar{x}}}{\frac{\alpha_1 - \bar{x}^2}{\bar{x}}}.$$

After simplifying we have

$$\alpha_2 \bar{x}^2 + \beta_2 \bar{x}^3 + \alpha_1 \bar{x} - \bar{x}^3 - (\alpha_1 - \bar{x}^2)^2 = 0.$$

Set

$$f(x) = x^4 + (1-\beta_2)x^3 - (2\alpha_1 + \alpha_2)x^2 - \alpha_1 x + \alpha_1^2. \tag{2}$$

Thus in order to show that there exists a unique equilibrium $(\bar{x}, \bar{y})$, it suffices to show $f(x) = 0$ has a unique positive solution less than $\sqrt{\alpha_1}$. By Descartes' rule of signs we know (2) has at most two positive roots. We also see that $f(0) = \alpha_1^2 > 0$ and $f(\sqrt{\alpha_1}) = -\alpha_1(\sqrt{\alpha_1}\beta_2 + \alpha_2) < 0$. Since $f(x)$ is a fourth degree polynomial with a positive leading coefficient we know that it has a minimum of two positive roots. Therefore there are exactly two positive roots; one root is less than $\sqrt{\alpha_1}$, and the other is greater than $\sqrt{\alpha_1}$. Thus the proof is complete. □

We shall now investigate the linearized stability of the equilibrium $(\bar{x}, \bar{y})$ of System (1).

Let

$$f(x,y) = \frac{\alpha_1}{x+y} \qquad \text{and} \qquad g(x,y) = \frac{\alpha_2 + \beta_2 x + y}{y}.$$

Then

$$\mathcal{J}_{(\bar{x},\bar{y})} = \begin{pmatrix} \dfrac{\partial f}{\partial \bar{x}} & \dfrac{\partial f}{\partial \bar{y}} \\ \\ \dfrac{\partial g}{\partial \bar{x}} & \dfrac{\partial g}{\partial \bar{y}} \end{pmatrix} = \begin{pmatrix} \dfrac{-\alpha_1}{(\bar{x}+\bar{y})^2} & \dfrac{-\alpha_1}{(\bar{x}+\bar{y})^2} \\ \\ \dfrac{\beta_2}{\bar{y}} & \dfrac{-(\alpha_2+\beta_2\bar{x})}{\bar{y}^2} \end{pmatrix} = \begin{pmatrix} \dfrac{-\bar{x}^2}{\alpha_1} & \dfrac{-\bar{x}^2}{\alpha_1} \\ \\ \dfrac{\beta_2}{\bar{y}} & \dfrac{1-\bar{y}}{\bar{y}} \end{pmatrix}.$$

The characteristic equation of the linearized equation of System (1) about the equilibrium $(\bar{x}, \bar{y})$ is

$$\lambda^2 + \frac{\bar{x}^2\bar{y} - \alpha_1(1-\bar{y})}{\alpha_1\bar{y}}\lambda + \frac{\bar{x}^2(\bar{y} - 1 + \beta_2)}{\alpha_1\bar{y}} = 0.$$

By Theorem 1.2 we see that both roots are real and lie within the unit disk. Therefore by Theorem 1.1, the unique positive equilibrium $(\bar{x}, \bar{y})$ is locally asymptotically stable.

3. Permanence

We say that System (1) is permanent if there exists real numbers l_1, L_1, l_2, and L_2 such that for every positive solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of System (1), there exists an integer $N \geq 0$, such that

$$l_1 < x_n < L_1 \quad \text{and} \quad l_2 < y_n < L_2 \quad \text{for every integer n} \geq N.$$

With this in mind, define l_1, L_1, l_2, and L_1 as follows:

(1) $l_1 = \dfrac{\alpha_1}{\alpha_1 + \alpha_2 + 1 + \beta_2\alpha_1}$

(2) $L_1 = \alpha_1$

(3) $l_2 = 1$

(4) $L_2 = \alpha_1 + 1 + \beta_2\alpha_1$.

Theorem 3.1. *System* (1) *is permanent. In particular, let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a positive solution of System* (1). *Then for every integer* $n \geq 4$, *we have*

$$l_1 < x_n < L_1 \quad \text{and} \quad l_2 < y_n < L_2.$$

Proof. Given a non-negative integer $n \geq 0$, note that

$$x_{n+1} = \frac{\alpha_1}{x_n + y_n} \in (0, \infty)$$

and

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + y_n}{y_n} = \frac{\alpha_2 + \beta_2 x_n}{y_n} + 1 \in (1, \infty).$$

Thus $y_n > 1 = l_2$ for $n \geq 1$.

Hence if $n \geq 1$, then

$$0 < x_{n+1} = \frac{\alpha_1}{x_n + y_n} < \frac{\alpha_1}{0 + 1} = \alpha_1$$

and so $x_n < L_1$ for $n \geq 2$.

Hence if $n \geq 2$, then

$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + y_n}{y_n} < \frac{\alpha_2 + \beta_2\alpha_1 + 1}{1} = L_2.$$

That is, for every integer $n \geq 3$ we have

$$l_2 < y_n < L_2.$$

If $n \geq 3$, then

$$x_{n+1} = \frac{\alpha_1}{x_n + y_n} > \frac{\alpha_1}{\alpha_1 + \alpha_2 + \beta_2\alpha_1 + 1} = l_1.$$

That is, for every integer $n \geq 4$ we have

$$l_1 < x_n < L_1$$

and the proof is complete. □

4. Global asymptotic stability of System (1)

The following theorem gives a sufficient condition for the unique equilibrium of System (1) to be globally asymptotically stable.

Theorem 4.1. *Suppose that either*

$$0 < \alpha_2 \leq \frac{\alpha_1 \beta_2^2}{1+\beta_2} - 2\sqrt{\frac{\alpha_1 \beta_2^2}{1+\beta_2}}$$

or

$$\frac{\alpha_1 \beta_2^2}{1+\beta_2} \leq \alpha_2.$$

Then the unique equilibrium point $(\bar{x}, \bar{y})$ *is globally asymptotically stable.*

Proof. The proof will be by Theorem 1.3. For $(x, y) \in [0, \infty) \times (0, \infty)$, set

$$f(x, y) = \frac{\alpha_1}{x+y} \qquad \text{and} \qquad g(x, y) = \frac{\alpha_2 + \beta_2 x + y}{y}$$

and let $\mathcal{R} = [a, b] \times [c, d] = [0, \alpha_1] \times [1, \alpha_2 + \beta_2 \alpha_1 + 1]$. Let $T : [0, \infty) \times (0, \infty) \rightarrow (0, \infty) \times (0, \infty)$ be given by $T(x, y) : (f(x, y), g(x, y))$.

We shall first show that $T[\mathcal{R}] \subset \mathcal{R}$. Suppose $(x, y) \in \mathcal{R}$. It suffices to show that

$$f(x, y) \in [a, b] \quad \text{and} \quad g(x, y) \in [c, d].$$

(1) We shall first show that $a < f(x, y)$.
Note that

$$a = 0 < \frac{\alpha_1}{x+y} = f(x, y).$$

(2) We shall next show that $f(x, y) \leq b$.
We have

$$f(x, y) = \frac{\alpha_1}{x+y} \leq \frac{\alpha_1}{a+c} = \frac{\alpha_1}{0+1} = \alpha_1 = b.$$

(3) We shall next show that $c < g(x, y)$.

$$c = 1 < \frac{\alpha_2 + \beta_2 x}{y} + 1 = \frac{\alpha_2 + \beta_2 x + y}{y} = g(x, y).$$

(4) Finally, we shall show that $g(x, y) \leq d$.
Now

$$g(x, y) = \frac{\alpha_2 + \beta_2 x + y}{y} \leq \frac{\alpha_2 + \beta_2 b + 1}{1} = \alpha_2 + \beta_2 \alpha_1 + 1 = d.$$

Thus $T[\mathcal{R}] \subset \mathcal{R}$.

Clearly f is strictly decreasing in x and strictly decreasing in y, and g is strictly increasing in x and strictly decreasing in y. So to apply Theorem

1.3, suppose $(m_1, M_1, m_2, M_2) \in [0, \alpha_1]^2 \times [1, \alpha_2 + 1 + \beta_2\alpha_1]^2$ is a solution of the system of equations

$$\begin{cases} m_1 = \dfrac{\alpha_1}{M_1 + M_2} & , \quad M_1 = \dfrac{\alpha_1}{m_1 + m_2} \\ \\ m_2 = \dfrac{\alpha_2 + \beta_2 m_1 + M_2}{M_2} & , \quad M_2 = \dfrac{\alpha_2 + \beta_2 M_1 + m_2}{m_2} \end{cases}$$

with

$$0 \leq m_1 \leq M_1 \leq \alpha_1 \qquad \text{and} \qquad 1 \leq m_2 \leq M_2 \leq \alpha_2 + 1 + \beta_2\alpha_1.$$

It suffices to show that

$$m_1 = M_1 \qquad \text{and} \qquad m_2 = M_2.$$

For the sake of contradiction, suppose that this is not the case.

Now

$$m_1 M_1 + m_1 M_2 = \alpha_1 = M_1 m_1 + M_1 m_2$$

and so $m_1 M_2 = M_1 m_2$. Since $m_1 = \dfrac{\alpha_1}{M_1 + M_2}$, we see m_1 is positive, and so as $m_1 M_2 = M_1 m_2$, we have

$$0 < m_1 < M_1 \qquad \text{and} \qquad 1 < m_2 < M_2.$$

Hence

$$M_2 = \frac{m_2}{m_1} M_1.$$

We also have

$$\alpha_2 + \beta_2 m_1 + M_2 = m_2 M_2 = \alpha_2 + \beta_2 M_1 + m_2.$$

Therefore $\beta_2 m_1 + M_2 = \beta_2 M_1 + m_2$, and hence

$$M_2 - m_2 = \beta_2 M_1 - \beta_2 m_1.$$

Thus

$$\beta_2(M_1 - m_1) = M_2 - m_2 = \frac{m_2}{m_1} M_1 - m_2 = \frac{m_2}{m_1}(M_1 - m_1).$$

So as $M_1 \neq m_1$, we have

$$\beta_2 = \frac{m_2}{m_1} \neq 0.$$

That is,

$$m_2 = \beta_2 m_1 \quad \text{and} \quad M_2 = \beta_2 M_1.$$

Recall that

$$m_1 = \frac{\alpha_1}{M_1 + M_2} = \frac{\alpha_1}{M_1 + \beta_2 M_1} = \frac{\alpha_1}{(1 + \beta_2) M_1}$$

and so

$$m_1 M_1 = \frac{\alpha_1}{1+\beta_2}.$$

Thus

(1) $M_1 = \dfrac{\alpha_1}{1+\beta_2} \cdot \dfrac{1}{m_1}$.
(2) $m_2 = \beta_2 m_1$.
(3) $M_2 = \beta_2 M_1 = \dfrac{\alpha_1 \beta_2}{1+\beta_2} \cdot \dfrac{1}{m_1}$.

In particular, since $m_2 = \beta_2 m_1$, we see that

$$\frac{1}{\beta_2} m_2 M_2 = m_1 M_2 = \frac{\alpha_1 \beta_2}{1+\beta_2}$$

and so

$$m_2 M_2 = \frac{\alpha_1 \beta_2^2}{1+\beta_2}.$$

Thus

$$\begin{aligned}\frac{\alpha_1 \beta_2^2}{1+\beta_2} &= m_2 M_2 = \alpha_2 + \beta_2 m_1 + M_2 \\ &= \alpha_2 + \beta_2 m_1 + \beta_2 M_1 \\ &= \alpha_2 + \beta_2 m_1 + \frac{\alpha_1 \beta_2}{1+\beta_2} \cdot \frac{1}{m_1}\end{aligned}$$

and so

$$0 = \beta_2 m_1^2 + \left(\alpha_2 - \frac{\alpha_1 \beta_2^2}{1+\beta_2}\right) m_1 + \frac{\alpha_1 \beta_2}{1+\beta_2}.$$

We also have

$$\begin{aligned}\frac{\alpha_1 \beta_2^2}{1+\beta_2} = m_2 M_2 &= \alpha_2 + \beta_2 M_1 + m_2 = \alpha_2 + \beta_2 M_1 + \beta_2 m_1 \\ &= \alpha_2 + \beta_2 M_1 + \frac{\alpha_1 \beta_2}{1+\beta_2} \cdot \frac{1}{M_1}\end{aligned}$$

and thus

$$0 = \beta_2 M_1^2 + \left(\alpha_2 - \frac{\alpha_1 \beta_2^2}{1+\beta_2}\right) M_1 + \frac{\alpha_1 \beta_2}{1+\beta_2}.$$

That is, m_1 and M_1 are the two distinct roots of the quadratic equation

$$\beta_2 z^2 + \left(\alpha_2 - \frac{\alpha_1 \beta_2^2}{1+\beta_2}\right) z + \frac{\alpha_1 \beta_2}{1+\beta_2} = 0.$$

Hence

$$0 < m_1 = \frac{\left(\dfrac{\alpha_1\beta_2^2}{1+\beta_2} - \alpha_2\right) - \sqrt{\left(\alpha_2 - \dfrac{\alpha_1\beta_2^2}{1+\beta_2}\right)^2 - \dfrac{4\alpha_1\beta_2^2}{1+\beta_2}}}{2\beta_2}$$

and

$$m_1 < M_1 = \frac{\left(\dfrac{\alpha_1\beta_2^2}{1+\beta_2} - \alpha_2\right) + \sqrt{\left(\alpha_2 - \dfrac{\alpha_1\beta_2^2}{1+\beta_2}\right)^2 - \dfrac{4\alpha_1\beta_2^2}{1+\beta_2}}}{2\beta_2}.$$

So by our hypothesis this is a contradiction, and the proof of the theorem is complete.

□

Extensive computer simulations lead us to the following conjecture:

Conjecture 4.1. *The unique positive equilibrium of System* (1) *is globally asymptotically stable for the entire range of the parameters.*

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