OPEN PROBLEMS AND CONJECTURES ON RATIONAL SYSTEMS IN THREE DIMENSIONS

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Dedicated to Professor Mustafa Kulenović on the occasion of his 60th birthday

ABSTRACT. We present some open problems and conjectures on Rational Systems in three dimensions, or higher, with nonnegative parameters and with nonnegative initial conditions such that the denominators are always positive. We also employ the method of Full Limiting Sequences to confirm an outstanding conjecture on k^{th}-order rational difference equations.

1. INTRODUCTION

In this paper, we present some open problems and conjectures on rational systems in three dimensions of the form:

$$
\left.
\begin{aligned}
x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n + \delta_1 z_n}{A_1 + B_1 x_n + C_1 y_n + D_1 z_n} \\[2ex]
y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n + \delta_2 z_n}{A_2 + B_2 x_n + C_2 y_n + D_2 z_n} \\[2ex]
z_{n+1} &= \frac{\alpha_3 + \beta_3 x_n + \gamma_3 y_n + \delta_3 z_n}{A_3 + B_3 x_n + C_3 y_n + D_3 z_n}
\end{aligned}
\right\},
\qquad n = 0, 1, \ldots
\tag{1}
$$

with nonnegative parameters and with nonnegative initial conditions such that the denominators are always positive.

We also employ the method of Full Limiting sequences, see Theorem 1.8 in [27], to confirm Conjecture 1, in Section 2, about the global character of

solutions of the $(k+1)^{th}$-order rational difference equation :

$$x_{n+1} = \frac{\alpha}{1 + \prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \dots \tag{2}$$

which is derived from a rational system in higher dimensions.

A systematic work on the global character of rational systems in two dimensions of the form :

$$\left.\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n} \\[2ex] y_{n+1} &= \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n} \end{aligned}\right\}, \qquad n = 0, 1, \dots \tag{3}$$

was intitiated in [15]. Several special cases of system (3) have been investigated by Kulenović and Merino and their students, and also by Ladas and his collaborators and students. See [1]-[3], [5]-[7], [9]-[21], [25], [28]-[29], [31]-[40]. See also: [4], [8], [22]-[23], [26], [30], [41]-[42], and [45].

It is an amazing fact that system (1) contains,

$$(2^4 - 1) \times (2^4 - 1) \times (2^4 - 1) \times (2^4 - 1) \times (2^4 - 1) \times (2^4 - 1) = 11,390,625$$

special cases of rational systems in 3 dimensions and it is of paramount importance to understand the global character of solutions of each one of these special cases.

2. OPEN PROBLEMS AND CONJECTURES

Here we pose some open problems and conjectures on the global character of solutions of System (1).

We wish to determine the boundedness characterization of each special case of System (1). In the past work on systems of two rational difference equations, patterns emerged eventually after a large amount of work describing the boundedness characterizations on a case by case basis. We are especially interested in finding similar patterns of boundedness for System (1).

For each system with bounded solutions we wish to determine the global stability character of their equilibrium points and the periodic nature of their solutions.

For each system with unbounded solutions, we wish to determine the way that their solutions are unbounded, the stable and unstable manifolds of their equilibrium points, any invariants and whether there exists any periodic trichotomies.

We now pose an open problem for the simpler system :

$$x_{n+1} = \frac{\alpha_1}{y_n}$$

$$\left.y_{n+1} = \frac{\alpha_2}{z_n}\right\}, \qquad n = 0, 1, \dots \qquad (4)$$

$$z_{n+1} = \frac{\alpha_3 + \beta_3 x_n + \gamma_3 y_n + \delta_3 z_n}{A_3 + B_3 x_n + C_3 y_n + D_3 z_n}$$

System (4) contains 225 of the 11,390,625 rational systems included in System (1).

By eliminating the variables x_n and y_n from the third equation in (4), we see that $\{z_n\}$ satisfies a third order rational difference equation of the form:

$$z_{n+1} = \frac{\gamma + \alpha z_{n-1} + \delta z_n z_{n-1} + \alpha_1 \beta z_{n-1} z_{n-2}}{C + A z_{n-1} + D z_n z_{n-1} + \alpha_1 B z_{n-1} z_{n-2}}, \quad n = 0, 1, \dots \qquad (5)$$

with nonnegative parameters and nonnegative initial conditions such that the denominators are always positive.

Open Problem 1. *Investigate the global character of solutions of the 225 special cases which are included in Eq. (5). For each special case, determine its boundedness character, its periodic behavior, and the global stability character of its equilibrium points.*

The following conjectures deal with some higher order analogues of System (4) and equation (5).

Assume $\alpha > 0$. Then we pose the following three conjectures :

Conjecture 1. *Every positive solution of the difference equation:*

$$z_{n+1} = \frac{\alpha}{1 + \prod_{i=0}^{k} z_{n-i}}, \quad n = 0, 1, \dots$$

has a finite limit.

This result was established, in [24], for the case in which $k = 1$.

Conjecture 2. *Every positive solution of the difference equation:*

$$z_{n+1} = \frac{\alpha}{1 + \prod_{i=1}^{k} z_{n-i}}, \quad n = 0, 1, \dots$$

converges to a (not necessarily prime) period $(k + 3)$ solution.

Conjecture 3. *Every positive solution of the difference equation :*

$$z_{n+1} = \frac{\alpha}{1 + z_{n-l} z_{n-k}}, \quad n = 0, 1, \dots$$

converges to a (not necessarily prime) period $(l + k + 2)$ *solution.*

We now present some other conjectures about special cases of System (1), which are included in the following simpler system :

$$
\left.
\begin{aligned}
x_{n+1} &= \gamma_1 y_n \\[2mm]
y_{n+1} &= \delta_2 z_n \\[2mm]
z_{n+1} &= \frac{\alpha_3 + \beta_3 x_n + \gamma_3 y_n + \delta_3 z_n}{A_3 + B_3 x_n + C_3 y_n + D_3 z_n}
\end{aligned}
\right\}, \qquad n = 0, 1, \dots \qquad (6)
$$

This system reduces to the following third order rational difference equation :

$$
z_{n+1} = \frac{\alpha + \beta z_n + \gamma z_{n-1} + \delta z_{n-2}}{A + B z_n + C z_{n-1} + D z_{n-2}}, \quad n = 0, 1, \dots, \qquad (7)
$$

This equation was investigated in [16]. In [16], the authors posed several interesting open problems and conjectures on the difference equation (7). There are several noteworthy conjectures, which have not yet been resolved.

Conjecture 4. *Assume $C > 0$ and that an equilibrium point $\bar{z}$ of the difference equation*

$$
z_{n+1} = \frac{\alpha + \beta z_n + \gamma z_{n-1}}{A + B z_n + C z_{n-1}}, \quad n = 0, 1, \dots, \qquad (8)
$$

is locally asymptotically stable. Show that $\bar{x}$ is a global attractor of all positive solutions of equation (8).

To prove Conjecture 4, it is necessary that following conjecture be confirmed.

Conjecture 5. *For the difference equation*

$$
z_{n+1} = \frac{\alpha + \beta z_n}{A + z_n + C z_{n-1}}, \quad n = 0, 1, \dots. \qquad (9)
$$

assume $A \geq 0$, all other parameters positive, and nonnegative initial conditions such that the denominators are never zero. Show that every solution of (9) converges.

For the most recent account of the progress made on Conjecture 4, see [5], [16], [41] and [46].

Another interesting conjecture is the following period-six trichotomy conjecture :

Conjecture 6. *Assume that $\alpha, C \in [0, \infty)$. Then the following period-six trichotomy result is true for the rational equation*

$$z_{n+1} = \frac{\alpha + z_n}{C z_{n-1} + z_{n-2}}, \quad n = 0, 1, \dots. \tag{10}$$

  (a) *Every positive solution of equation (10) converges to its positive equilibrium, if and only if, $\alpha C^2 > 1$.*
  (b) *Every positive solution of equation (10) converges to a not necessarily prime period-six solution of equation (10), if and only if, $\alpha C^2 = 1$.*
  (c) *Equation (10) has positive unbounded solutions, if and only if, $\alpha C^2 < 1$.*

The only part of this conjecture which has been resolved is the case for which $\alpha C^2 = 0$. This was resolved in [43].

Related to Conjecture 8 is the following conjecture :

Conjecture 7. *For the difference equation*

$$z_{n+1} = \frac{\alpha + z_n}{C z_{n-1} + z_{n-2}}, \quad n = 0, 1, \dots. \tag{11}$$

assume positive parameters and nonnegative initial conditions such that the denominators are never zero. The difference equation (11), has unbounded solutions in some range of the parameters.

This is the only special case of (7) whose boundedness character has not been established yet. We present another conjecture about the boundedness character of (7).

Conjecture 8. *Assume $\alpha, \beta, \gamma \in [0, \infty)$. Then every positive solution of the difference equation*

$$z_{n+1} = \frac{\alpha + \beta z_n + \gamma z_{n-1}}{z_{n-2}}, \quad n = 0, 1, \dots. \tag{12}$$

is bounded, if and only if, $\beta = \gamma$.

For the most recent work on the boundedness character of (7) see [16], [43] and [44].

3. Confirmation of Conjecture 1

In this section, we establish the following theorem, which confirms the above Conjecture 1.

Theorem 1. *Every solution of the $(k+1)^{th}$-order difference equation*

$$x_{n+1} = \frac{\alpha}{1 + \prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \dots, \tag{13}$$

has a finite limit.

For the proof of this theorem we need the following lemma, which is Theorem 1.8 of [27].

Lemma 1. *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \tag{14}$$

where $f \in C[J^{k+1}, J]$ for some interval of real numbers J and some non-negative integer k. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (14). Set $I = \lim_{n\to\infty} \inf x_n$ and $S = \lim_{n\to\infty} \sup x_n$, and suppose that $I, S \in J$. Let $\mathcal{L}_0$ be a limit point of the sequence $\{x_n\}_{n=-k}^{\infty}$. Then the following statements are true:

1. *There exists a solution $\{L_n\}_{n=-\infty}^{\infty}$ of (14), called a full limiting sequence of $\{x_n\}_{n=-k}^{\infty}$, such that $L_0 = \mathcal{L}_0$, and such that for every $N \in \{\ldots, -1, 0, 1, \ldots\}$, L_N is a limit point of $\{x_n\}_{n=-k}^{\infty}$. In particular,*

$$I \leq L_N \leq S \quad \text{for all} \quad N \in \{\ldots, -1, 0, 1, \ldots\}.$$

2. *For every $i_0 \in \{\ldots, -1, 0, 1, \ldots\}$, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-m}^{\infty}$ such that*

$$L_N = \lim_{i\to\infty} x_{r_i+N} \quad \text{for all} \quad N \geq i_0.$$

The result that every solution of the difference equation (13) has a finite limit will follow as a corollary of the following theorem.

Theorem 2. *Every solution of the $(k+2)$-order difference equation*

$$x_{n+1} = \frac{x_n\left(1 + \prod_{i=1}^{k+1} x_{n-i}\right)}{1 + \prod_{i=0}^{k} x_{n-i}}, \quad n = 0, 1, \ldots, \tag{15}$$

has a finite limit.

This is because every solution of the difference equation (13) converges if every solution of difference equation (15) converges. This is true because every solution of (13) is a solution of (15). This will be demonstrated below in the case for which $k = 2$.

For simplicity of the presentation, we will just establish Theorem 1 for the third order difference equation,

$$x_{n+1} = \frac{\alpha}{1 + x_n x_{n-1} x_{n-2}}, \quad n = 0, 1, \ldots. \tag{16}$$

The proof for the general case is similar but lengthier.

Notice that, for the difference equation (16), we have

$$x_n(1 + x_{n-1} x_{n-2} x_{n-3}) = \alpha, \quad n = 1, 2, \ldots. \tag{17}$$

Using (17), we can embed the difference equation (16) into the difference equation

$$x_{n+1} = \frac{x_n \left(1 + x_{n-1} x_{n-2} x_{n-3}\right)}{1 + x_n x_{n-1} x_{n-2}}, \quad n = 1, 2, \ldots \qquad (18)$$

which is a special case of the difference equation (15).

To establish Theorem 2, for the case where $k = 2$, we need the following lemma.

Lemma 2. *Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of* (18). *Set*

$$m = \min \{x_{-2}, x_{-1}, x_0, x_1\}$$

and

$$M = \max \{x_{-2}, x_{-1}, x_0, x_1\} \ .$$

Then

$$m \le x_n \le M, \quad \text{for all} \quad n \ge -2.$$

Proof. Clearly

$$m \le x_n \le M, \quad \text{for all} \quad -2 \le n \le 1.$$

Notice that

$$x_2 = \frac{x_1 \left(1 + x_0 x_{-1} x_{-2}\right)}{1 + x_1 x_0 x_{-1}}.$$

and

$$\frac{\partial}{\partial w} \left(\frac{w \left(1 + xyz\right)}{1 + wxy} \right) = \frac{xyz + 1}{\left(wxy + 1\right)^2} > 0 \qquad (19)$$

and

$$\frac{\partial}{\partial z} \left(\frac{w \left(1 + xyz\right)}{1 + wxy} \right) = \frac{wxy}{wxy + 1} > 0. \qquad (20)$$

By (19) and (20) it follows that

$$m = \frac{m \left(1 + x_0 x_{-1} m\right)}{1 + m x_0 x_{-1}} \le \frac{x_1 \left(1 + x_0 x_{-1} x_{-2}\right)}{1 + x_1 x_0 x_{-1}} \le \frac{M \left(1 + x_0 x_{-1} M\right)}{1 + M x_0 x_{-1}} = M.$$

and by induction, that

$$m \le x_n \le M \quad \text{for all} \quad n \ge -2.$$

$\square$

To complete the proof, it suffices to show that

$$\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n.$$

For the sake of contradiction, assume that

$$\liminf_{n \to \infty} x_n < \limsup_{n \to \infty} x_n.$$

By part 1. of Lemma 1, there exists a full limiting sequence $\{L_n\}_{n=-\infty}^{\infty}$ of $\{x_n\}_{n=-2}^{\infty}$ with $L_0 = I = \lim_{n\to\infty} \inf x_n$.

We claim that $L_n = I$ for all $n \in \mathbb{Z}$.

We first show that $L_{-n} = L_0$ for all $n = 0, 1, \ldots$. We will show $L_{-1} = L_0$. For the sake of contradiction, assume that $L_{-1} > L_0$. Then, by (19) and (20), (18) is increasing with respect to the first and last argument. Hence

$$L_0 = \frac{L_{-1}\left(1 + L_{-2}L_{-3}L_{-4}\right)}{1 + L_{-1}L_{-2}L_{-3}} > \frac{L_0\left(1 + L_{-2}L_{-3}L_0\right)}{1 + L_0L_{-2}L_{-3}} = L_0.$$

This is a contradiction. It follows by induction that $L_{-n} = L_0$ for all $n = 0, 1, \ldots$.

It follows that $L_n = I$, for all $n \in \mathbb{N}$, since every point of (18) is an equilibrium point and since $L_{-n} = L_0$ for all $n = 0, 1, \ldots$.

By part 2 of Lemma 1, there exists a subsequence $\{x_{r_i}\}_{i=0}^{\infty}$ of $\{x_n\}_{n=-2}^{\infty}$ such that

$$\lim_{i\to\infty} x_{r_i-j} = L_{-j}$$

for every $-1 \leq j \leq 2$. It also follows that $L_{-j} = L_0$ for every $-1 \leq j \leq 2$ from what was just proved. So as $L_0 = \lim_{n\to\infty} \inf x_n$, there exists a positive integer s such that $r_s \geq 0$ and

$$\max\left\{x_{r_s-2}, x_{r_s-1}, x_{r_s}, x_{r_s+1}\right\} \leq \frac{1}{2}\left(\lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \inf x_n\right).$$

It follows by Lemma 2 that

$$x_n \leq \frac{1}{2}\left(\lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \inf x_n\right)$$

for all $n \geq r_{s+1}$. Thus, using the assumption $\lim_{n\to\infty} \inf x_n < \lim_{n\to\infty} \sup x_n$, it follows that

$$\lim_{n\to\infty} \sup x_n \leq \frac{1}{2}\left(\lim_{n\to\infty} \sup x_n + \lim_{n\to\infty} \inf x_n\right) < \lim_{n\to\infty} \sup x_n.$$

$\square$

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