SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.11.1.11
Vol.11 (23), No.1, (2015), 131–137

SOMEWHAT ω-CONTINUOUS FUNCTIONS

C. CARPINTERO, N. RAJESH, E. ROSAS AND S. SARANYASRI

Abstract. In this paper we introduce and study a new classes of functions called somewhat ω-continuous functions as a generalization of the somewhat continuous functions.

1. Introduction and preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized closed sets. Recently, as generalization of closed sets, the notion of ω-closed sets were introduced and studied by Hdeib [3]. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A subset A is said to be ω-closed [3] if it contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset W of a space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U \backslash W$ is countable. The family of all ω-open subsets of a topological space (X, τ) forms a topology on X which is finer than τ. The set of all ω-open sets of (X, τ) is denoted by $\omega O(X)$. The set of all ω-open sets of (X, τ) containing a point $x \in X$ is denoted by $\omega O(X, x)$. The intersection of all ω-closed sets containing S is called the ω-closure of S and is denoted by $\omega \operatorname{Cl}(S)$. The ω-interior of S is defined by the union of all ω-open sets contained in S and is denoted by $\omega \operatorname{Int}(S)$. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be ω-continuous if for each $U \in \sigma$, $f^{-1}(U)$ is an ω-open set in X. clearly every continuous function is an ω-continuous function.

2010 *Mathematics Subject Classification.* 54B05,54C08.
Key words and phrases. ω-open sets, somewhat ω-continuous functions, somewhat continuous functions.

2. Somewhat ω-continuous functions

Definition 2.1. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat ω-continuous if for $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$, there exists an ω-open set V in X such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.

Observe that every ω-continuous function is somewhat ω-continuous, but the converse is not true in general.

Example 2.2. Let $X = \mathbb{R}$ with topology the standard topology τ and $Y = \{a, b\}$ with topology $\sigma = \{\emptyset, Y, \{a\}, \{b\}\}$. Define $F : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows:

$$F(x) = \begin{cases} a & \text{if } x \geq 0 \\ b & \text{if } x < 0. \end{cases}$$

It is easy to see that F is somewhat ω-continuous but is not ω-continuous.

Definition 2.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat continuous [2] if for $U \in \sigma$ and $f^{-1}(U) \neq \emptyset$, there exists an open set V in X such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.

It is clear that every somewhat continuous function is somewhat ω-continuous. But the converse is not true in general.

Example 2.4. Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $Y = \mathbb{R}$ with topology $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}\}$. Define $F : (\mathbb{R}, \tau) \to (\mathbb{R}, \sigma)$ as the identity function. It is easy to see that F is somewhat ω-continuous but is not somewhat continuous.

The following example shows that the composition of two somewhat ω-continuous need not be somewhat ω-continuous, even if one of then is ω-continuous and not continuous, the composition need not be somewhat ω-continuous.

Example 2.5. Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with topologies $\beta = \{\emptyset, Y, \{b\}, \{a, c\}\}$ and $\sigma = \{\emptyset, Y, \{a, b\}\}$. Define $f : (X, \tau) \to (Y, \beta)$ by

$$f(x) = \begin{cases} a & \text{if } x \in \mathbb{Q}, \\ c & \text{if } x \in (\mathbb{R} - \mathbb{Q}) - \{e\}, \\ a & \text{if } x = e, \end{cases}$$

where e is an irrational number. Let $g : (Y, \beta) \to (Y, \sigma)$ the identity map. Observe that f is ω-continuous and not continuous, g is somewhat ω-continuous but the composition gf is not somewhat ω-continuous, because $(gf)^{-1}(\{a, b\}) = \mathbb{Q} \cup \{e\}$ and there are not exists ω-open set V in X such that $V \subseteq (gf)^{-1}(\{a, b\})$.

The next proposition states that if the map g is continuous and f is somewhat ω-continuous, then the composition is somewhat ω-continuous.

Proposition 2.6. *If $f : (X, \tau) \to (Y, \sigma)$ is somewhat ω-continuous and $g : (X, \sigma) \to (Z, \eta)$ is continuous, then $g \circ f : (X, \tau) \to (Z, \eta)$ is somewhat ω-continuous.*

Definition 2.7. A subset M of a topological space (X, τ) is said to be ω-dense in X if there is no proper ω-closed set C in X such that $M \subset C \subset X$.

Example 2.8. Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$. It is easy to see that $\mathbb{R} - \mathbb{Q}$ is an ω-dense set in X, but $\mathbb{Q}$ is not an ω-dense set in X.

Proposition 2.9. *A subset M of a topological space (X, τ) is ω-dense in X if for any nonempty ω-open set U in X, $U \cap M \neq \emptyset$*

Theorem 2.10. *For a surjective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *f is somewhat ω-continuous.*
(2) *If C is a closed subset of Y such that $f^{-1}(C) \neq X$, then there is a proper ω-closed subset D of X such that $D \supset f^{-1}(C)$.*
(3) *If M is an ω-dense subset of X, then $f(M)$ is a dense subset of Y.*

Proof. (1)⇒(2): Let C be a closed subset of Y such that $f^{-1}(C) \neq X$. Then $Y \backslash C$ is an open set in Y such that $f^{-1}(Y \backslash C) = X \backslash f^{-1}(C) \neq \emptyset$. By (1), there exists an ω-open set V in X such that $V \neq \emptyset$ and $V \subset f^{-1}(Y \backslash C)$ $= X \backslash f^{-1}(C)$. This means that $X \backslash V \supset f^{-1}(C)$ and $X \backslash V = D$ is a proper ω-closed set in X.
(2)⇒(3): Let M be an ω-dense set in X. Suppose that $f(M)$ is not dense in Y. Then there exists a proper closed set C in Y such that $f(M) \subset C \subset Y$. Clearly $f^{-1}(C) \neq X$. By (2), there exists a proper ω-closed set D such that $M \subset f^{-1}(C) \subset D \subset X$. This is a contradiction of the fact that M is ω-dense in X.
(3)⇒(1): Suppose that f is not a somewhat ω-continuous function, then there exists a nonempty open set U in Y such that ω-interior of $f^{-1}(U)$ is empty, that is $X \backslash f^{-1}(U)$ is ω-dense in X, while $f(X \backslash f^{-1}(U)) = Y \backslash U$ is not dense in Y. This is a contradiction. □

Lemma 2.11. [1] *Let A and B be subsets of a space (X, τ).*

(1) *If $A \in \omega O(X, \tau)$ and $B \in \tau$, then $A \cap B \in \omega O(B, \tau_B)$;*
(2) *If $A \in \omega O(B, \tau_B)$ and $B \in \omega O(X, \tau)$, then $A \in \omega O(X, \tau)$.*

Proposition 2.12. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $X = A \cup B$, where $A, B \in \tau$. If the restriction functions $f_{|A} : (A, \tau_{|A}) \to (Y, \sigma)$ and $f_{|B} : (B, \tau_{|B}) \to (Y, \sigma)$ are somewhat ω-continuous, then f is somewhat ω-continuous.*

Proof. Let U be any open subset of Y such that $f^{-1}(U) \neq \emptyset$. Then $(f_{|A})^{-1}(U) \neq \emptyset$ or $(f_{|B})^{-1}(U) \neq \emptyset$ or both $(f_{|A})^{-1}(U) \neq \emptyset$ and $(f_{|B})^{-1}(U) \neq \emptyset$. Suppose $(f_{|A})^{-1}(U) \neq \emptyset$. Since $f_{|A}$ is somewhat ω-continuous, there exists an ω-open set V in A such that $V \neq \varnothing$ and $V \subset (f_{|A})^{-1}(U) \subseteq f^{-1}(U)$. Since V is ω-open in A and A is open in X, V is ω-open in X. Thus we find that f is somewhat ω-continuous. The proof of other cases are similar. □

Definition 2.13. If X is a set and τ and σ are topologies on X, then τ is said to be ω-equivalent (equivalent [2]) to σ provided if $U \in \tau$ and $U \neq \emptyset$, then there is an ω-open (open [2]) set V in (X, σ) such that $V \neq \emptyset$ and $V \subset U$ and if $U \in \sigma$ and $U \neq \emptyset$, then there is an ω-open (open) set V in (X, τ) such that $V \neq \emptyset$ and $U \supset V$.

Example 2.14. Let $X = \mathbb{R}$ with topologies $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $\sigma = \{\emptyset, \mathbb{R}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}\}$. It is easy to see that τ and σ are ω-equivalent but τ and σ are not equivalent.

Now, consider the identity function $f : (X, \tau) \to (Y, \sigma)$ and assume that τ and σ are ω-equivalent. Then $f : (X, \tau) \to (Y, \sigma)$ and $f^{-1} : (Y, \sigma) \to (X, \tau)$ are somewhat ω-continuous. Conversely, if the identity function $f : (X, \tau) \to (X, \sigma)$ is somewhat ω-continuous in both directions, then τ and σ are ω-equivalent.

Proposition 2.15. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat ω-continuous surjection and τ^* be a topology for X, which is ω-equivalent to τ. Then $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat ω-continuous.*

Proof. Let V be an open subset of Y such that $f^{-1}(V) \neq \emptyset$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat ω-continuous, there exists a nonempty ω-open set U in (X, τ) such that $U \subset f^{-1}(V)$. But by hypothesis τ^* is ω-equivalent to τ. Therefore, there exists an ω-open set U^* in (X, τ^*) such that $U^* \subset U$. But $U \subset f^{-1}(V)$. Then $U^* \subset f^{-1}(V)$; hence $f : (X, \tau^*) \to (Y, \sigma)$ is somewhat ω-continuous. □

Proposition 2.16. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat ω-continuous surjection and σ^* be a topology for Y, which is equivalent to σ. Then $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat ω-continuous.*

Proof. Let V^* be an open set of (Y, σ^*) such that $f^{-1}(V^*) \neq \emptyset$. Since σ^* is equivalent to σ, there exists a nonempty open set V in (Y, σ) such that $V \subset V^*$. Now $\emptyset \neq f^{-1}(V) \subset f^{-1}(V^*)$. Since $f : (X, \tau) \to (Y, \sigma)$ is somewhat ω-continuous, there exists a nonempty ω-open set U in (X, τ) such that $U \subset f^{-1}(V)$. Then $U \subset f^{-1}(V^*)$; hence $f : (X, \tau) \to (Y, \sigma^*)$ is somewhat ω-continuous. □

Definition 2.17. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat ω-open provided that if $U \in \tau$ and $U \neq \emptyset$, then there exists an ω-open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.

Definition 2.18. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be somewhat open [2] provided that if $U \in \tau$ and $U \neq \emptyset$, then there exists an open set V in Y such that $V \neq \emptyset$ and $V \subset f(U)$.

It is clear that every somewhat open is somewhat ω-open. But the converse is not true in general.

Example 2.19. Let $X = \mathbb{R}$ with topology $\tau = \{\emptyset, \mathbb{R}, \mathbb{Q}\}$ and $Y = \{a, b, c\}$ with topology $\sigma = \{\emptyset, Y, \{a\}\}$. Define $F : (\mathbb{R}, \tau) \to (Y, \sigma)$ as follows:

$$F(x) = \begin{cases} a & \text{if } x \in \mathbb{R} - \mathbb{Q} \\ b & \text{if } x \in \mathbb{Q}. \end{cases}$$

It is easy to see that F is somewhat ω-open but is not somewhat open.

Proposition 2.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be an open function and $g : (Y, \sigma) \to (Z, \gamma)$ somewhat ω-open. Then $g \circ f : (X, \tau) \to (Z, \gamma)$ is somewhat ω-open.*

Proposition 2.21. *For a bijective function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *f is somewhat ω-open.*
(2) *If C is a closed subset of X, such that $f(C) \neq Y$, then there is an ω-closed subset D of Y such that $D \neq Y$ and $D \supset f(C)$.*

Proof. Since f is somewhat ω-open if and only if f^{-1} is somewhat ω-continuous, then (1)$\Leftrightarrow$(2) is obvious by Theorem 2.10. $\square$

Proposition 2.22. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(1) *f is somewhat ω-open.*
(2) *If A is an ω-dense subset of Y, then $f^{-1}(A)$ is a dense subset of X.*

Proof. (1)$\Rightarrow$(2): Suppose A is an ω-dense set in Y. We want to show that $f^{-1}(A)$ is a dense subset of X. Suppose not, then there exists a closed set B in X such that $f^{-1}(A) \subset B \subset X$. Since f is somewhat ω-open and $X \backslash B$ is open, there exists a nonempty ω-open set C in Y such that $C \subset f(X \backslash B)$. Therefore, $C \subset f(X \backslash B) \subset f(f^{-1}(Y \backslash A)) \subseteq Y \backslash A$. That is, $A \subset Y \backslash C \subset Y$. Now, $Y \backslash C$ is an ω-closed set and $A \subset Y \backslash C \subset Y$. This implies that A is not an ω-dense set in Y, which is a contradiction. Therefore, $f^{-1}(A)$ must be a dense set in X.
(2)$\Rightarrow$(1): Suppose A is a nonempty open subset of X. We want to show that $\omega\,\mathrm{Int}(f(A)) \neq \emptyset$. Suppose $\omega\,\mathrm{Int}(f(A)) = \emptyset$. Then, $\omega\,\mathrm{Cl}(Y \backslash f(A)) =$

Y. Therefore, by (2), $f^{-1}(Y\backslash f(A))$ is dense in X. But $f^{-1}(Y\backslash f(A)) \subseteq X\backslash A$. Now, $X\backslash A$ is closed. Therefore, $f^{-1}(Y\backslash f(A)) \subseteq X\backslash A$ gives $X = \mathrm{Cl}(f^{-1}(Y\backslash f(A))) \subseteq X\backslash A$. This implies that $A = \emptyset$, which is contrary to $A \neq \emptyset$. Therefore, $\omega\,\mathrm{Int}(f(A)) \neq \emptyset$. This proves that f is somewhat ω-open. □

Proposition 2.23. *Let $f : (X, \tau) \to (Y, \sigma)$ be somewhat ω-open and A be any open subset of X. Then $f_{|A} : (A, \tau_{|A}) \to (Y, \sigma)$ is somewhat ω-open.*

Proof. Let $U \in \tau_{|A}$ such that $U \neq \emptyset$. Since U is open in A and A is open in X, U is open in X and since by hypothesis $f : (X, \tau) \to (Y, \sigma)$ is somewhat ω-open function, there exists an ω-open set V in Y, such that $V \subset f(U)$. Thus, for any open set U of A with $U \neq \emptyset$, there exists an ω-open set V in Y such that $V \subset f(U)$ which implies $f_{|A}$ is a somewhat ω-open function. □

Proposition 2.24. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $X = A \cup B$, where $A, B \in \tau$. If the restriction functions $f_{|A}$ and $f_{|B}$ are somewhat ω-open, then f is somewhat ω-open.*

Proof. Let U be any open subset of X such that $U \neq \emptyset$. Since $X = A \cup B$, either $A \cap U \neq \emptyset$ or $B \cap U \neq \emptyset$ or both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. Since U is open in X, U is open in both A and B.

Case (i): Suppose that $A \cap U \neq \emptyset$, where $U \cap A$ is open in A. Since $f_{|A}$ is somewhat ω-open function, there exists an ω-open set V of Y such that $V \subset f(U \cap A) \subset f(U)$, which implies that f is a somewhat ω-open function.

Case (ii): Suppose that $B \cap U \neq \emptyset$, where $U \cap B$ is open in B. Since $f_{|B}$ is somewhat ω-open function, there exists an ω-open set V in Y such that $V \subset f(U \cap B) \subset f(U)$, which implies that f is also a somewhat ω-open function.

Case (iii): Suppose that both $A \cap U \neq \emptyset$ and $B \cap U \neq \emptyset$. Then by case (i) and (ii) f is a somewhat ω-open function. □

Remark 2.25. Two topologies τ and σ for X are said to be ω-equivalent if and only if the identity function $f : (X, \tau) \to (X, \sigma)$ is somewhat ω-open in both directions.

Proposition 2.26. *Let $f : (X, \tau) \to (Y, \sigma)$ be a somewhat open function. Let τ^* and σ^* be topologies for X and Y, respectively such that τ^* is equivalent to τ and σ^* is ω-equivalent to σ. Then $f : (X, \tau^*) \to (Y, \sigma^*)$ is somewhat ω-open.*

References

[1] K. Al-Zoubi and B. Al-Nashef, *The topology of ω-open subsets*, Al-Manarah, IX (2)(2003), 169–179.

[2] K. R. Gentry and H. B. Hoyle, *Somewhat continuous functions*, Czechslovak Math. J., 21 (96) (1971), 5–12. doi:Zbl 0222.54010.
[3] H. Z. Hdeib, *ω-closed mappings*, Rev. Colomb. Mat., 16 (1-2) (1982), 65–78.

(Received: June 5, 2014)

C. Carpintero and E. Rosas
Department of Mathematics
Universidad De Oriente
Nucleo De Sucre Cumana
Venezuela
carpintero.carlos@gmail.com
ennisrafael@gmail.com

N. Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India
nrajesh_topology@yahoo.co.in

S. Saranyasri
Department of Mathematics
M. R. K. Institute of Technology
Kattumannarkoil, Cuddalore–608 301
Tamilnadu, India
srisaranya_2010@yahoo.com