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GÖDEL FORM OF FUZZY TRANSITIVE RELATIONS

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Abstract. The concepts of fuzzy transitivity of a fuzzy relation on a given universe and the measure of fuzzy transitivity are studied with the use of Gödel fuzzy implicator.

1. Introduction and preliminaries

The assumption of transitivity is an essential component of definitions of equivalence as well as orderings. But as a matter of fact, when equivalence relations are used for modelling indistinguishability or approximate equality, it is the assumption of transitivity that causes paradoxical situations to come forth. In history, this fact was recorded first of all by Poincaré in 1902 [20]. Later on, Menger pointed out that the failure of crisp relations in modelling approximate equality is not only due to assumption of transitivity but the crisp nature of definition of relations also contributes towards the emergence of paradoxical situations. This is why the development of the notion fuzzy set theory and later of fuzzy relations brought hope for the formulation of better ways of defining transitivity.

Several fuzzy versions of transitivity were developed and extensively used in the past four decades (see [26], [9] and [22]). Unfortunately, every new form of fuzzy transitivity was still accompanied by paradoxical situations (see [10], [17] and [18]). Studying transitivity pointwisely on the domain $X \times X \times X$, in terms of fuzzy logical operators has also been done such as [13], [14] and [15]. Beg and Ashraf [3],[5],[6],[7] and [8] reformulated the definition of fuzzy transitivity in the similar settings but they advanced in a different dimension. The major difference in their approach is that they have defined the fuzzy set of transitivity of a fuzzy relation R on a universe X as a fuzzy relation $tr(R)$ on the same universe. In this way, several new results were proved by comparing the two relations R and $tr(R)$. A fuzzy

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measure [12] is then applied to this fuzzy set of transitivity in order to obtain the degree or measure of transitivity of a given fuzzy relation R.

A pair of mappings consisting of a fuzzy implicator and fuzzy conjunction are building blocks for the concept introduced by Beg and Ashraf [3]. Consequently its properties show high dependence on the properties of the fuzzy logical operators used in place of fuzzy conjunction and a fuzzy implicator. This paper is written specifically for the study of properties of the fuzzy set of transitivity and the measure of fuzzy transitivity under the use of the Gödel fuzzy implicator. The reason behind this particular selection has been the way Gödel fuzzy implicator takes a zero value, which is different from other fuzzy implicators. It consequently opens new venues for nontransitive fuzzy relations.

A *fuzzy set* A in a universe X is a mapping from X to $[0,1]$. For any $x \in X$, the value $A(x)$ is called the *degree of membership* of x in A. Moreover, $F(X)$ will stand for *the set of all fuzzy subsets of* X. Given a crisp universe X, a *fuzzy binary relation* is a fuzzy subset of $X \times X$. Fuzzy binary relations will be called fuzzy relations throughout this paper.

Given a crisp universe X, and $A, B \in F(X)$, A is said to be a *subset* of B (in Zadeh's sense [25]) denoted by $A \subseteq B$, if and only if $A(x) \leq B(x)$ for all $x \in X$.

Definition 1.1. [19] *The triangular norm (t-norm) T and triangular conorm (t-conorm) S are increasing, associative, commutative and $[0,1]^2 \to [0,1]$ mappings satisfying $T(1,x) = x$ and $S(x,0) = x$, for all $x \in [0,1]$.*

A popular choice for the t-norm is: The minimum operator $M : M(x,y) = \min(x,y)$.

The corresponding dual t-conorm is: The maximum operator $M^* : M^*(x,y) = \max(x,y)$.

Definition 1.2. [11] *A negator N is an order-reversing $[0,1] \to [0,1]$ mapping such that $N(0) = 1$ and $N(1) = 0$. A strictly decreasing negator satisfying $n(n(x)) = x$, for all $x \in [0,1]$ is called a strong negator.*

Definition 1.3. [23] *Given a t-norm T, a T-equivalence relation on a set X is a fuzzy relation E on X that satisfies:*

(i) $E(x,x) = 1$, *for all* $x \in X$; *(Reflexivity),*
(ii) $E(x,y) = E(y,x)$, *for all* $x, y \in X$; *(Symmetry),*
(iii) $T(E(x,y), E(y,z)) \leq E(x,z)$ *for all* $x, y, z \in X$. *(T-transitivity).*

If $T = \min$, then E is called a similarity relation. A min transitive relation satisfies

$$\max y \in X(\min(E(x,y), E(y,z))) \leq E(x,z) \text{ for all } x, z \in X.$$

Commonly it is called a $\max - \min$ *transitive relation.*

Definition 1.4. [21] *A fuzzy implicator I is a binary operation on* $[0,1]$ *with order reversing first partial mappings and order preserving second partial mappings satisfying the boundary conditions;*

$$I(0,1) = I(0,0) = I(1,1) = 1 \text{ and } I(1,0) = 0.$$

The implicator to be used in this paper is the Gödel's implicator which is defined as:

$$I_g(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

Definition 1.5. [1] *Given two fuzzy relations R and S on X, the direct product or* $\sup -T$ *product of R and S is defined as:*

$$R \circ^T S(x,z) = \sup_{y \in X} T(R(x,y), S(y,z)).$$

Definition 1.6. [12, p.183] *Let (X, ρ) be a measurable space. A function* $m : \rho \to [0, \infty[$ *is a fuzzy measure if it satisfies the following properties:*

$m1$: $m(\varnothing) = 0$, and $m(X) = 1$;
$m2$: $A \subseteq B$ implies that $m(A) \leq m(B)$.

The concept of measure considers that $\rho \subseteq \{0,1\}^X$, but this consideration can be extended to a set $\Im$ of fuzzy subsets of X, i.e., $\Im \subseteq F(X)$, satisfying the properties of the measurable space $(F(X), \Im)$.

For any $A \in F(X)$, the following two measures will be used for the construction of results on measure of fuzzy transitivity:

1. $m_1(A) = plinth(A) = \inf_{x \in X} A(x)$,
2. $m_2(A) = \frac{|A|}{|X|} = \frac{\sum_{i=1}^{n} A(x_i)}{n}$ (in case of finite universes with n elements).

Definition 1.7. [4] *The fuzzy inclusion* Incl is a mapping $\text{Incl} : F(X) \times F(X) \to F(X)$ which assigns to every $A, B \in F(X)$, a fuzzy set $\text{Incl}(A,B) \in F(X)$, defined as:

$$\text{Incl}(A,B)(x) = I_g(A(x), B(x)), \text{ for all } x \in X.$$

The composition of fuzzy measure to this fuzzy set gives the measure of inclusion i.e.,

$$m\,\text{Incl}(A,B) = m(\text{Incl}(A,B)).$$

2. Fuzzy set of transitivity and measure of fuzzy transitivity

Definition 2.1. [3] *Let R be a fuzzy relation on X. The fuzzy set of transitivity $tr^{I,T}(R)$ is a fuzzy relation on X defined as:*

$$tr^{I,T}(R)(x,z) = \inf_{y \in X} I(T(R(x,y), R(y,z)), R(x,z)). \quad (1)$$

The transitivity function so defined assigns a degree of transitivity to the relation at each point of $X \times X$. Therefore, the given relation may have different degrees of transitivity at different points of $X \times X$. If $tr^{I,T}(R) \neq \emptyset$, then the relation R is called a *fuzzy transitive relation*, otherwise it is called a *nontransitive fuzzy relation.* If $tr^{I,T}(R)(x,z) \geq 0.5$, for all $x, z \in X$, then the relation R is called a *strong fuzzy transitive relation,* otherwise it is called a *weak fuzzy transitive relation.* A reflexive, symmetric and (strong or weak) fuzzy transitive relation is called a (*strong or weak* fuzzy equivalence relation). The superscripts I and T highlight the dependence of pointwise transitivity upon the implicator I and the t-norm T, whereas, R denotes the fuzzy relation for which the fuzzy relation of transitivity is being constructed.

Remark 2.2. As it can be observed from the definition of the fuzzy set of transitivity of a given relation that the given relation shows a zero transitivity only at the points where the implicator in use has a zero value. As can be observed from Definition 1.4, most of the fuzzy implicators assign a zero value only to the point $(1, 0)$. In the case when such implicators are used in the definition of fuzzy transitivity, this property can be interpreted as that a fuzzy relation would be nontransitive only when $R(x, y) = R(y, z) = 1$ and $R(x, z) = 0$. Intuitively speaking at any place where the link between the first and the third point diminishes while the link between first and second and between the second and third points exists to a nonzero degree, the transitivity should be considered zero. To obtain this consequence, this paper is dedicated to the study of fuzzy transitivity specifically with the use of the Gödel's implicator which allocates a value zero to all those points where the first variable is nonzero and the second variable is zero. The *min* and max t-norms will be used for conjunction and disjunction purposes. In this particularized atmosphere, we shall not use the superscripts, hence the fuzzy set of transitivity $tr(R)$ of a fuzzy relation R is defined as:

$$
\begin{aligned}
tr(R)(x,z) &= \inf_{y \in X} I_g(M(R(x,y), R(y,z)), R(x,z)) \\
&= \inf_{y \in X} \begin{cases} 1 & \text{if } \min(R(x,y), R(y,z)) \leq R(x,z), \\ R(x,z) & \text{if } \min(R(x,y), R(y,z)) > R(x,z). \end{cases} \qquad (2)
\end{aligned}
$$

Remark 2.3. It can be easily observed from (2), that for a $\max - \min$ transitive fuzzy relation R, $tr(R)(x, z) = 1$ for all $x, z \in X$. Hence the class of similarity relations is a subclass of the class of strong fuzzy equivalence relations.

Theorem 2.4. *Let R be a fuzzy relation on a universe X. The fuzzy set of inclusion (defined in terms of I_g) of $R \circ R$ into R is equal to the fuzzy set* *of transitivity of R i.e., for all $x, z \in X$,*

$$\text{Incl}(R \circ R, R)(x, z) = tr(R)(x, z).$$

Proof. Let $x, z \in X$, then using Definition 1.4 and 1.7, we get

$$\begin{aligned}\text{Incl}\,(R \circ R, R)(x, z) &= I_g(R \circ R(x, z), R(x, z))\\ &= I_g(\sup_{y \in X} \min(R(x, y), R(y, z)), R(x, z))\\ &= \inf_{y \in X} I_g(\min(R(x, y), R(y, z)), R(x, z))\\ &= tr(R)(x, z).\end{aligned}$$

□

Remark 2.5. Theorem 2.4 can be restated alternatively as: Calculating the transitivity at any point $(x, z) \in X^2$ is equivalent to calculating the degree of inclusion of $R \circ R$ into R at that point. This remark opens ways for the use of t-norms other than *min* along with the Gödel fuzzy implicator. Once we calculate the composition $R \circ R$ with the help of any t-norm, it stands as a fuzzy relation on X and the next task is to calculate the pointwise inclusion of $R \circ R$ into R, for which any implicator may be used.

Remark 2.6. It can be easily observed from (2), that for any $c \in [0, 1]$, $R(x, y) \geq c$ implies that $tr(R)(x, y) \geq c$ for all $x, y \in X$.

The converse of Remark 2.6 may not hold in general. The fuzzy relations with all the values less than some certain threshold value show large transitivity degrees at each point as can be observed by the following example.

Example 2.7. Consider two relations R and S defined on $X = \{1, 2, 3\}$ by:

$$R = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.2 \\ 0.1 & 0.2 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0.15 & 0.1 \\ 0.15 & 1 & 0.12 \\ 0.1 & 0.12 & 1 \end{bmatrix}.$$

Their fuzzy sets of transitivity are:

$$tr(R) = \begin{bmatrix} 1 & 1 & 0.1 \\ 1 & 1 & 1 \\ 0.1 & 1 & 1 \end{bmatrix}, \; tr(S) = \begin{bmatrix} 1 & 1 & 0.1 \\ 1 & 1 & 1 \\ 0.1 & 1 & 1 \end{bmatrix}.$$

This example also confirms the inclusions stated in Remark 2.6. We also observe that the only deciding factor for higher values of transitivity are the smaller variations within the data values.

Proposition 2.8. *Let R be a fuzzy relation, then the following inclusion holds:*

$$R \subseteq tr(R),$$

i.e., for all $x, z \in X$,

$$R(x,z) \leq tr(R)(x,z). \tag{3}$$

Proof. The proof follows from the Definition of the Gödel's implicator where $I_g(x,y) \geq y$, for all $x, y \in [0,1]$. □

Corollary 2.9. *The fuzzy transitivity relation* $tr(R)$ *of a reflexive, symmetric and fuzzy transitive relation* R *on a universe* X *is itself a reflexive, symmetric and fuzzy transitive relation on the same universe.*

Proof. Reflexivity is due to the fact that $R(x,x) = 1$ and $tr(R)(x,x) \geq R(x,x) = 1$.

For *symmetry* we observe that for all $x, z \in X$:

$$\begin{aligned} tr(R)(z,x) &= \inf_{y \in X} I_g(\min(R(z,y), R(y,x)), R(z,x)) \\ &= \inf_{y \in X} I_g(\min(R(y,z), R(x,y)), R(x,z)) \text{ (by symmetry of } R) \\ &= \inf_{y \in X} I_g(\min(R(x,y), R(y,z)), R(x,z)) \text{ (by commutative nature of } \min) \\ &= tr(R)(x,z). \end{aligned}$$

It can further be observed that the fuzzy set of transitivity remains non zero throughout its domain. Due to Proposition 2.8, if $tr(R)(x,z) > 0$ for all $x, z \in X$, then $tr(tr(R)(x,z)) > 0$ for all $x, z \in X$. □

Corollary 2.10. *If* $tr(R)(x,z) = 0$ *for some* $x, z \in X$, *then* $tr(tr(R)(x,z)) = 0$ *and* $tr(tr(tr(R)(x,z))) = 0$.

Proof. $tr(R)(x,z) = 0$ implies that there exists a $y \in X$ such that $R(x,y) \neq 0$ and $R(y,z) \neq 0$ and $R(x,z) = 0$. It further implies that $tr(R)(x,y) \neq 0$, $tr(R)(y,z) \neq 0$ and $tr(R)(x,z) = 0$. □

Definition 2.11. [3] *The measure of fuzzy transitivity is a mapping* $Tr : F(X \times X) \to [0,1]$ *defined as:*

$$Tr(R) = m(tr(R)),$$

where $F(X \times X)$ *denotes the set of all fuzzy relations on* X *and* m *is a fuzzy measure.*

A fuzzy relation R is called ϵ-fuzzy transitive if $Tr(R) = \epsilon$. A reflexive, symmetric and ϵ- transitive relation is called an ϵ-equivalence relation.

It can be easily observed that for a fuzzy transitive relation $\epsilon > 0$, if R is a strong fuzzy equivalence relation, then $\epsilon \geq 0.5$. We will use only two measures m_1 and m_2 (mentioned after Definition 1.6.) We shall call $Tr_1(R)$ and $Tr_2(R)$ according to as measure m_1 or m_2 is being used respectively.

Example 2.12. If we apply the measures m_1 and m_2 from Example 2.7, then we get

$$Tr_1(R) = m_1(tr(R)) = 0.1 \text{ and } Tr_2(R) = m_2(tr(R)) = 0.8$$

and

$$Tr_1(S) = m_1(tr(S)) = 0.1 \text{ and } Tr_2(S) = m_2(tr(S)) = 0.8.$$

Theorem 2.13. *Let R be an ϵ-equivalence relation on X. Then $tr(R)$ is a fuzzy equivalence relation with measure of transitivity greater than ϵ.*

Proof. Reflexivity and symmetry have already been proved in Corollary 2.9.

Next let $Tr(R) = m(tr(R)) = \epsilon$. Now from Proposition 2.8 it follows that

$$R \subseteq tr(R) \subseteq tr(tr(R)).$$

Applying the Sugeno's fuzzy measure and using its monotonic nature, we get

$$\epsilon \leq m(R) \leq m(tr(R)) \leq m(tr(tr(R))).$$

Hence the fuzzy set of transitivity of an ϵ-equivalence relation R is a fuzzy relation with measure of transitivity greater than ϵ. □

We further observe that the measure of transitivity is always greater than the measure of a given fuzzy relation.

Example 2.14. Let $X = \{1, 2, 3, 4\}$. If a relation R is defined on X as:

$$R = \begin{bmatrix} 1 & 0.9 & 1 & 0.6 \\ 0.9 & 1 & 0.6 & 1 \\ 1 & 0.6 & 1 & 0.8 \\ 0.6 & 1 & 0.8 & 1 \end{bmatrix}, \text{ then } tr(R) = \begin{bmatrix} 1 & 1 & 1 & 0.6 \\ 1 & 1 & 0.6 & 1 \\ 1 & 0.6 & 1 & 1 \\ 0.6 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{and } tr(tr(R)) = \begin{bmatrix} 1 & 1 & 1 & 0.6 \\ 1 & 1 & 0.6 & 1 \\ 1 & 0.6 & 1 & 1 \\ 0.6 & 1 & 1 & 1 \end{bmatrix} = tr(tr(tr(R)))\ldots$$

We observe that none of the $tr(R)$, $tr(tr(R))\ldots$ is a min transitive fuzzy relation on X. In fact the relation $tr(R)$ is a fixed point of tr, tr^2, tr^3, and so on.

From Example 2.14, we observe that firstly $tr(R)$ is the fixed point of tr, tr^2, tr^3, and so on and secondly that although the transitive operator increases the transitivity at each point of $X \times X$, does not necessarily reach the transitive closure of R (i.e., the smallest min-transitive superset of R).

Beg and Ashraf [3] tried to obtain a min-transitive fuzzy relation by repeated application of tr on R. From Example 2.14, it is clear that the choice of the Gödel fuzzy implicator does not favor this hypothesis.

Theorem 2.15. *Let R be a fuzzy relation on X and $\circ$ stand for* $\sup - min$ *product. Then*

$$m\,\mathrm{Incl}(R \circ R, R) = Tr(R).$$

Proof. The proof follows from the equality of fuzzy sets established in Theorem 2.4 i.e.,

$$\mathrm{Incl}(R \circ R, R) = tr(R).$$

Applying measure on both sides, we get

$$m(\mathrm{Incl}(R \circ R, R)) = m(tr(R)) = Tr(R).$$

In case we use m_1 we obtain $\mathrm{Inc}(R \circ R, R) = Tr(R)$, where Inc is defined by $\mathrm{Incl}(R \circ R, R) = \inf_{x \in X} I(R \circ R(x), R(x))$ as given by Bandler and Kohout [2]. □

Theorem 2.16. *If $\epsilon \neq 0$ and R is an ϵ-equivalence relation with measure m_1 used in the calculation of measure of transitivity, then either R is* $\max - \min$ *transitive or $R(x, z) \geq \epsilon$, for all $(x, z) \in X^2$.*

Proof. Given R is an ϵ-equivalence relation,

$$\inf_{x,z \in X} [\inf_{y \in X} I_g(\min(R(x, y), R(y, z)), R(x, z))] \geq \epsilon.$$

It implies that for all $x, y, z \in X$,

$$I_g(\min(R(x, y), R(y, z)), R(x, z)) \geq \epsilon.$$

This implies either the relation is $\max - \min$ transitive or $R(x, z) \geq \epsilon$ for all $x, z \in X$. □

Remark 2.17. The class of strong fuzzy equivalence relations is much wider than the class of min-transitive equivalence relations. For example, if $R(x, y) = 0.9$, $R(y, z) = 0.8$ and $R(x, z) = 0.6$, then $tr^{I_g, M}(R)(x, z) = 0.6$ but the min-transitivity failed at this point.

Theorem 2.18. *The class of fuzzy transitive relations is closed under fuzzy intersection.*

Proof. Let R and S be the two fuzzy transitive relations. By hypothesis for all $(x, y, z) \in X^3$

$$I_g(\min(R(x, y), R(y, z)), R(x, z)) > 0$$
$$\text{and } I_g(\min(S(x, y), S(y, z)), S(x, z)) > 0.$$

On the contrary suppose that there exist $x, y, z \in X$ such that

$$\begin{aligned}
&I_g(M(M(R,S)(x,y), M(R,S)(y,z)), M(R,S)(x,z)) = 0,\\
&\Rightarrow \min(M(R,S)(x,y), M(R,S)(y,z)) \neq 0 \text{ and } M(R,S)(x,z) = 0,\\
&\Rightarrow M(R,S)(x,y) \neq 0 \text{ and } M(R,S)(y,z) \neq 0 \text{ and } M(R,S)(x,z) = 0,\\
&\Rightarrow R(x,y) \neq 0 \text{ and } S(x,y) \neq 0 \text{ and } R(y,z) \neq 0\\
&\qquad\qquad \text{and } S(y,z) \neq 0 \text{ and } R(x,z) = 0 \text{ or } S(x,z) = 0,\\
&\Rightarrow \text{ either } I_g(\min(R(x,y), R(y,z)), R(x,z)) = 0\\
&\qquad\qquad \text{or } I_g(\min(S(x,y), S(y,z)), S(x,z)) = 0,
\end{aligned}$$

a contradiction to the hypothesis. Thus $R \cap S$ is fuzzy transitive. □

Theorem 2.19. *Let (X, d) be an ultra-metric space. Define a fuzzy relation R on X as:*

$$R(x,y) = \frac{1}{1 + d(x,y)}.$$

Then R is reflexive, symmetric and $tr(R)(x,z) > 0.5$, for all $(x,z) \in X$.

Proof. For all $x, y, z \in X$ we have:

(i) *Reflexivity*: $R(x,x) = \frac{1}{1+d(x,x)} = \frac{1}{1+0} = 1$,
(ii) *Symmetry*: $R(x,y) = \frac{1}{1+d(x,y)} = \frac{1}{1+d(y,x)} = R(y,x)$,
(iii) *Strong Fuzzy transitivity*: Suppose on contrary that there exist $x, y, z \in X$ such that

$$tr(x,y,z) = I_g(M(R(x,y), R(y,z)), R(x,z)) < 0.5.$$

This implies that

$$\begin{aligned}
&I_g(M(\frac{1}{1+d(x,y)}, \frac{1}{1+d(y,z)}), \frac{1}{1+d(x,z)}) < 0.5,\\
&\Leftrightarrow M(\frac{1}{1+d(x,y)}, \frac{1}{1+d(y,z)}) \geq 0.5 \text{ and } \frac{1}{1+d(x,z)} < 0.5,\\
&\Leftrightarrow \frac{1}{1+d(x,y)} > 0.5 \text{ and } \frac{1}{1+d(y,z)} > 0.5 \text{ and } 1 + d(x,z) > 2,\\
&\qquad \text{i.e., } d(x,y) < 1 \text{ and } d(y,z) < 1 \text{ and } d(x,z) > 1. \qquad (4)
\end{aligned}$$

The inequality (4) implies that $\max(d(x,y), d(y,z)) < 1$. Since d is an ultra-metric it follows that, $d(x,z) \leq \max(d(x,y), d(y,z)) < 1$. But (4) also implies that $d(x,z) > 1$. Hence these inequalities can not hold simultaneously, a contradiction. □

3. Conclusion

In this paper, we studied the concepts of fuzzy set of transitivity and measure of fuzzy transitivity with the help of the Gödel fuzzy implicator. The two dimensional approach to the study of transitivity of a fuzzy relation initially with a point wise character and then globally, gives better insight into the transitive nature of a given fuzzy relation. The definition of the measure of fuzzy transitivity opens new ways to allocate a single degree of transitivity to a given fuzzy relation.

Now let us turn towards one of the paradoxical situations discussed by Klawonn in [18]. According to him: In a metric space (X, d) if we define $x \approx y$ (read as x is indistinguishable to y) $\Leftrightarrow d(x, y) \leq \delta$ for some $\delta > 0$, then for any $a, b, c \in X$,

$d(a, b) \leq \delta$ and $d(b, c) \leq \delta$, implies that $d(a, c) \in]0, 2\delta[$.

Hence $a \approx b$ and $b \approx c$, but a and c may or may not be indistinguishable by this criteria. Hence transitivity can not be obtained however small a δ is selected.

In order to analyze this situation in light of the Gödel form of fuzzy transitivity, let us assume that d is a $[0, 1]-$valued metric on X. Define a fuzzy relation on X as: $R(x, y) = N(d(x, y))$, where N is a strong negator. We observe that at any point $(a, c) \in X$:

$$tr(R)(a, c) = \inf_{b \in X} \begin{cases} 1 & \text{if } \min(R(a, b),\ R(b, c)) \leq R(a, c) \\ R(a, c) & \text{otherwise.} \end{cases}$$

It is obvious that, $tr(R)(a, c) = 1$ implies that $d(a, c) \leq \delta$, other wise,

$$tr(R)(a, c) = R(a, c) = N(d(a, c)). \tag{5}$$

So, in a nutshell, in terms of If $a \approx b$ and $b \approx c$, and $tr(R)(a, c) = 1$, $a \approx c$. So, the paradox appears because of the requirement of 1-degree transitivity which is quite unnatural. As can be observed from (5), the greater the distance between a and c, be smaller is the value of $tr(R)(a, c)$. The smallest value of $tr(R)(a, c)$ is obtained for the greatest value of $d(a, c) = 2\delta$, in which case $tr(R)(a, c) = N(2\delta)$, which obviously depends directly on the selection of value of δ. A direct observation is that to attain the phenomenon of high transitivity one should define approximate equality for very small radii.

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