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PENCILS OF EULER TRIPLES, I

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Abstract. In this paper we will study the families of triples (u, v, w) of elements in a commutative ring R with the property that $v\,w + n = \widetilde{u}^2$, $w\,u + n = \widetilde{v}^2$ and $u\,v + n = \widetilde{w}^2$ for some n, $\widetilde{u}$, $\widetilde{v}$, $\widetilde{w} \in R$. These families of triples are built from the famous Euler triples $(a, b, a + b \pm 2\,t)$, where $n = t^2 - a\,b$. The idea is to use an extra element $m \in R$ and take either

$$A = (a,\, a\,m^2 + 2\,t\,m + b,\, a\,m^2 + 2\,(t + a)\,m + a + b + 2\,t)$$

or

$$B = (a,\, a\,m^2 + 2\,t\,m + b,\, a\,m^2 + 2\,(t - a)\,m + a + b - 2\,t).$$

In this way we get two parametrized families of Euler triples called pencils that are here explored in great detail. Many properties and identities for their components and various sums and products are established. Besides the above blowing up method that gives the infinite family of Euler triples from a single Euler triple, we also show the importance of the associated pencils $\widetilde{A}$ and $\widetilde{B}$ built from triples $(\widetilde{u}, \widetilde{v}, \widetilde{w})$. Many properties explain various methods how one can get squares and cubes from the four pencils, describe values of determinants of matrices with rows from the pencils and give factorizations of some expressions from members of the pencils A, B, $\widetilde{A}$ and $\widetilde{B}$. The role of the parameter m is examined extensively.

1. Introduction

Let R be a ring and let S denote its squares, i. e., $S = \{x^2 : x \in R\}$.

For an $n \in R$ and a natural number m, we say that $a \in R^m$ is an $S(n)$-*m-tuple* provided $a_i\, a_j + n$ is a square for each pair of different indices i and j.

In the special case when R is the ring of integers $\mathbb{Z}$ and $n \in \mathbb{Z}$, then the $S(n)$-m-tuples are called $D(n)$-m-tuples and Diophantine m-tuples when $n = 1$ because the Greek mathematician Diophantus of Alexandria first found that $(\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16})$ is the Diophantine quadruple of rationals ([3],

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[1]). The famous open question in this part of number theory is the existence of Diophantine quintuples of natural numbers.

In this paper we shall explore some general properties of families of $S(n)$-triples (called pencils) which are built from the well-known Euler $S(n)$-triples X and Y whose description follows (see [10]). There is some evidence that Diophantus has known them too.

Let $a, b, t \in R$. Put $n = t^2 - a\,b$. Let X and Y denote the triples $(a, b, a + b \pm 2\,t)$. Then X and Y are $S(n)$-triples. Indeed, if Z is either X or Y, then its components satisfy: $Z_2\,Z_3 + n = \widetilde{Z}_1^2$, $Z_3\,Z_1 + n = \widetilde{Z}_2^2$ and $Z_1\,Z_2 + n = \widetilde{Z}_3^2$, where $\widetilde{X} = (t + b, t + a, t)$ and $\widetilde{Y} = (t - b, t - a, t)$.

In order to get families of triples from the triples X and Y we use an additional parameter $m \in R$. In other words, we consider functions $\alpha, \beta \colon R^4 \to R^3$ (called *pencils of Euler $S(n)$-triples*) defined as follows. For $a, b, t, m \in R$, let $\alpha(a, b, t, m)$ and $\beta(a, b, t, m)$ be the triples A and B whose components are $A_1 = B_1 = a$, $A_2 = B_2 = a\,m^2 + 2\,t\,m + b$ and

$$A_3 = a\,m^2 + 2\,(t + a)\,m + a + b + 2\,t,$$

$$B_3 = a\,m^2 + 2\,(t - a)\,m + a + b - 2\,t.$$

Notice that $\alpha(a, b, t, 0) = X$ and $\beta(a, b, t, 0) = Y$. Therefore, the pencils α and β provide the parameterized versions of the Euler's $S(n)$-triples X and Y. Also, $A_2 = X_1\,m^2 + 2\,\widetilde{X}_3\,m + X_2$, $B_2 = Y_1\,m^2 + 2\,\widetilde{Y}_3\,m + Y_2$, $A_3 = X_1\,m^2 + 2\,\widetilde{X}_2\,m + X_3$ and $B_3 = Y_1\,m^2 + 2\,\widetilde{Y}_2\,m + Y_3$.

The following example shows how extensive the pencils α and β of Euler $S(n)$-triples can be. Recall that for every natural number n, the triple $(F_{2n}, F_{2n+2}, F_{2n+4})$ of three consecutive Fibonacci numbers with even indices is the Diophantine triple. When $n = 1$, we have the triple $\mathfrak{r} = (1, 3, 8)$ (that can be enlarged with the number 120 to the famous Fermat Diophantine quadruple $\mathfrak{F} = (1, 3, 8, 120)$ [2]). Using the above blowing up method, from the triple $\mathfrak{r}$, we get the pencils

$$\mathfrak{a} = \mathfrak{a}(m) = (1, m^2 + 4\,m + 3, m^2 + 6\,m + 8)$$

and

$$\mathfrak{b} = \mathfrak{b}(m) = (1, m^2 + 4\,m + 3, m^2 + 2\,m)$$

of infinitely many Diophantine triples like $\mathfrak{a}(2) = (1, 15, 24)$, $\mathfrak{a}(\frac{1}{2}) = (1, \frac{21}{4}, \frac{45}{4})$ and $\mathfrak{b}(-15) = (1, 168, 195)$. Also, $\mathfrak{b}(3 + 2\,i) = (1, 20 + 20\,i, 11 + 16\,i)$ (since the parameter m can even be a complex number; $i = \sqrt{-1}$).

The Euler $S(n)$-triples are quite common among the $S(n)$-triples. They are usually the first three terms of an $S(n)$-quadruple whose fourth component can be figured out from the initial Euler triple. For example, the following could be located in the reference [9] (when n is the square k^2

of an integer): [9, Theorems 3 – 7] from the generalized Fibonacci numbers for $k = p(p^2+1)v_{n+3}$, $p(p^2-1)h_{n+3}$, $(p+1)g_{n+2}$, $(p-1)g_{n+2}$, 1 and p; $(F_{2n}, F_{2n+8}, 5\,F_{2n+4})$ and $(F_{2n}, F_{2n+8}, 9\,F_{2n+4})$ for $k = 3$ [9, Example 1]; $(P_{2n}, P_{2n+2}, 2\,P_{2n})$ and $(P_{2n}, P_{2n+2}, 2\,P_{2n+2})$ from Pell numbers for $k = 2$ [9, Corollary 2]; $(P_{2n}, P_{2n+4}, 4\,P_{2n+2})$ and $(P_{2n}, P_{2n+4}, 8\,P_{2n+2})$ for $k = 4$ [9, Corollary 2]; $(P_{2n}, P_{2n+8}, 32\,P_{2n+4})$ and $(P_{2n}, P_{2n+8}, 36\,P_{2n+4})$ for $k = 24$ [9, Corollary 2]; $(n, n+2, 4(n+1))$ the Jones polynomial Euler $S(1)$-triple that includes for $n = \frac{1}{16}$ the first three terms of the original Diophantine quadruple (see [11] and [12]); $\left(\binom{n-1}{3}, \binom{n+3}{3}, 6\,n\right)$ and $\left(\binom{n-1}{3}, \binom{n+3}{3}, \frac{2\,n(n^2+2)}{3}\right)$ from the binomial coefficients for $k = 1$ [9]; and $((n-1)(n-2), (n+1)(n+2), 4\,n^2)$ and $(1, n^4 - 3\,n^2 + 1, n^2(n^2-1))$ from the quadratic polynomials for $k = n$ [9, Corollary 3].

Some of our results have been discovered earlier for the Euler $S(-4)$-triples $(P_{2n+1}, 2\,P_{2n+1}, P_{2n+3})$ and $(P_{2n+1}, 2\,P_{2n+3}, 2\,P_{2n+3})$ and the $S(8)$-triples $(Q_{2n+1}, 2\,Q_{2n+1}, Q_{2n+3})$ and $(Q_{2n+1}, 2\,Q_{2n+3}, 2\,Q_{2n+3})$ from the Pell-Lucas numbers in [4] and for the pencils on these triples in [5] and for the pencils built on the $S(-1)$-triples $(F_{2n+1}, F_{2n+3}, F_{2n+5})$ and $(F_{2n+1}, 5\,F_{2n+3}, F_{2n+5})$ and the $S(5)$-triples $(L_{2n+1}, L_{2n+3}, L_{2n+5})$ and $(L_{2n+1}, 5L_{2n+3}, L_{2n+5})$ from the Lucas numbers in [6].

The present paper investigates the common properties of all pencils of Euler $S(n)$-triples (like $\mathfrak{a}$ and $\mathfrak{b}$ and all others listed above). In another paper [7], the author will consider a similar blowing up procedure for the Euler $S(n)$-quadruples. However, the case of the triples is in many respects much richer in properties and possibilities even though the triples are not as valuable in the subject of $S(n)$-m-tuples as are the quadruples and the hypothetical quintuples.

The $D(n)$-m-tuples are mostly studied in number theory. There are over 200 papers that consider various aspects of this notion (see the complete list on the Web page [8]). The main questions and open problems deal with the existence and the estimates on the number of these m-tuples. The present paper is different because it searches for additional (intrinsic) properties that a given pencil of Euler $S(n)$-triples might have besides its existence and the defining property. Our goal is to explore the properties of the pencils α and β and to understand their structure better.

In order to help the reader follow our notation and counter the dry abstract ring setting, we give many concrete examples for the favorite triple $\mathfrak{r} = (1, 3, 8)$ and the associated pencils $\mathfrak{a}$ and $\mathfrak{b}$ of Diophantine triples.

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2. The functions $\widetilde{\alpha}$ and $\widetilde{\beta}$

The first property of the triples A and B is that they are $S(n)$-triples. In other words, the triples A and B have the form (u, v, w) with $v\,w+n$, $w\,u+n$ and $u\,v+n$ being perfect squares. In particular, $\mathfrak{a}_2\mathfrak{a}_3+1=(m^2+4m+3)(m^2+6m+8)+1=(m^2+5m+5)^2$.

Property 1. *For $a, b, t, m \in R$, the following relations are true:*

$$A_2\,A_3+n=[a\,m^2+(a+2\,t)\,m+b+t]^2,$$
$$B_2\,B_3+n=[a\,m^2-(a-2\,t)\,m+b-t]^2,$$
$$A_3A_1+n=(a\,m+a+t)^2,\ B_3\,B_1+n=(a\,m-a+t)^2,$$
$$A_1A_2+n=B_1\,B_2+n=(a\,m+t)^2.$$

Proof. In order to prove the first equality, note that the difference

$$[a\,m^2+(a+2\,t)\,m+b+t]^2-(A_2\,A_3+n)$$

is $t^2-a\,b-n=0$. The other equalities are proved similarly. □

The functions α and β determine the functions $\widetilde{\alpha}, \widetilde{\beta} : R^4 \to R^3$, where $\widetilde{A}=\widetilde{\alpha}(a,b,t,m)$ and $\widetilde{B}=\widetilde{\beta}(a,b,t,m)$ have the components

$$\widetilde{A}_1=a\,m^2+(a+2\,t)\,m+b+t,$$
$$\widetilde{A}_2=a\,m+a+t, \qquad \widetilde{A}_3=\widetilde{B}_3=a\,m+t,$$
$$\widetilde{B}_1=a\,m^2-(a-2\,t)\,m+b-t, \quad \widetilde{B}_2=a\,m-a+t,$$

given in the first property. Let E denote either A or B. With this notation, the coordinates of $\widetilde{E}$ satisfy the relations: $\widetilde{E}_1^2=E_2\,E_3+n$, $\widetilde{E}_2^2=E_3\,E_1+n$ and $\widetilde{E}_3^2=E_1\,E_2+n$. Notice that our selection of the components of $\widetilde{E}$ is suggested by the triangle geometry (where, for example, the join QR of Q and R is opposite the first vertex P).

Similarly, let ε denote either α or β. The function $\widetilde{\varepsilon}$ is just one of the eight possible choices because $k^2=(-k)^2$ so that we can also take the opposite values of the above components.

Notice that $\widetilde{\mathfrak{r}}=(5,3,2)$, $\widetilde{\mathfrak{a}} = (m^2+5m+5, m+3, m+2)$ and $\widetilde{\mathfrak{b}} = (m^2+3m+1, m+1, m+2)$.

Now that we know that A and B are $S(n)$-triples and we have the triples $\widetilde{A}$ and $\widetilde{B}$ it is possible to apply the above blowing up method from X and Y to A and B. The next property shows that such an extension of A and B does not give anything new. For example,

$$p^2+2(m+2)p+m^2+4m+3=(p+m)^2+4(p+m)+3$$

and

$$p^2+2(m+3)p+m^2+6m+8=(p+m)^2+6(p+m)+8$$

mean that blowing up $\mathfrak{a}(m)$ at the value p gives $\mathfrak{a}(p+m)$.

Property 2. *For $a, b, t, m, p \in R$,*

$$a\,p^2 + 2\,\widetilde{E}_3\,p + E_2 = a\,(p+m)^2 + 2\,\widetilde{Z}_3\,(p+m) + Z_2,$$
$$a\,p^2 + 2\,\widetilde{E}_2\,p + E_3 = a\,(p+m)^2 + 2\,\widetilde{Z}_2\,(p+m) + Z_3.$$

Proof. Let $E = A$. We have $a\,p^2 + 2\,\widetilde{A}_3\,p + A_2 = a\,p^2 + 2\,(a\,m + t)\,p + a\,m^2 + 2\,t\,m + b = a\,(m+p)^2 + 2\,t\,(m+p) + b$. This trinomial is equal $a\,(p+m)^2 + 2\,\widetilde{X}_3\,(p+m) + X_2$. The other identities are established similarly. □

For $a, b, t, m \in R$, let $m^A = \alpha(a, b, t, m)$, $m^{\widetilde{A}} = \widetilde{\alpha}(a, b, t, m)$, etc. This alternative notation for various triples is used in situations when only the fourth variable m takes other values while the other variables a, b and t have their usual values.

The next property shows that the functions α and $\widetilde{\alpha}$ can be figured out from the functions β and $\widetilde{\beta}$ and vice versa provided the ring R is commutative and has the multiplicative identity e. The interpretation of the first identity is the following: The first components of $\widetilde{\beta}(a, b, t, m+e)$ and $\widetilde{\alpha}(a, b, t, m)$ are equal. For example,

$$(m+1)_1^{\widetilde{\mathfrak{b}}} = (m+1)^2 + 3(m+1) + 1 = m^2 + 5m + 5 = \widetilde{\mathfrak{a}}_1.$$

Property 3. *Let the commutative ring R have the multiplicative identity e. For $a, b, t, m \in R$, we have:*

$$\widetilde{A}_1 = (m+e)_1^{\widetilde{B}}, \qquad \widetilde{A}_2 = (m+2e)_2^{\widetilde{B}}, \qquad A_3 = (m+2e)_3^{B} = (m+e)_2^{B}.$$

Also, if there is a $u \in R$ such that $2\,u = e$, then $\widetilde{A}_1 = (m+u)_2^{B} - a\,u^2$.

Proof. The first identity holds because the difference of $(m+e)_1^{\widetilde{B}}$ and $\widetilde{A}_1$ is $ae^2 + (2t - a + 2am)e - 2am - 2t = 0$.

For the proof of the second identity, note that the difference of $(m+2e)_2^{\widetilde{B}}$ and $\widetilde{A}_2$ is $2ae - 2a = 0$.

Since $(m+2e)_3^{B} = a\,(m+2e)^2 - 2\,(a-t)(m+2e) + a + b - 2\,t = a\,m^2 + 2\,(a+t)\,m + a + b + 2\,t$, it follows that $(m+2e)_3^{B} = A_3$.

Similarly, since $(m+u)_2^{B} = a\,m^2 + 2\,(t + a\,u)\,m + a\,u^2 + b + 2\,t\,u$, $\widetilde{A}_1 = a\,m^2 + (a + 2\,t)\,m + b + t$ and $2\,u = e$, we get $\widetilde{A}_1 = (m+u)_2^{B} - a\,u^2$. □

3. Identities for components

In the next property of the functions ε and $\widetilde{\varepsilon}$ we collect the obvious basic relationships among their components that are the consequence of the Euler observation that the equalities $b\,c + n = p^2$, $c\,a + n = q^2$ and $a\,b + n = r^2$

hold if and only if either $c = c_+ = a + b + 2\,r$, $p = b + r$ and $q = a + r$ or $c = c_- = a + b - 2\,r$, $p = b - r$ and $q = a - r$. In other words, each triple in the pencils α and β is an Euler triple.

Property 4. *For $a, b, t, m \in R$,*

$$A_3 = a + A_2 + 2\,\widetilde{A}_3, \quad \widetilde{A}_1 = A_2 + \widetilde{A}_3, \quad \widetilde{A}_2 = \widetilde{A}_3 + a,$$

$$B_3 = a + B_2 - 2\,\widetilde{B}_3, \quad \widetilde{B}_1 = B_2 - \widetilde{B}_3, \quad \widetilde{B}_2 = \widetilde{B}_3 - a.$$

The next set of identities are easy consequences of Property 4. In particular, the components of A are linear functions of the components of $\widetilde{A}$ and when for every $x \in R$ there is a $y \in R$ such that $x = 2\,y$ (i. e., when all elements of R are divisible by 2), then the components of $\widetilde{A}$ are linear functions of the halves of the components of A.

Property 5. *For $a, b, t, m \in R$,*

$$A_3 = \widetilde{A}_1 + \widetilde{A}_2, \quad B_3 = \widetilde{B}_1 - \widetilde{B}_2,$$

$$E_2 + E_3 = a + 2\,\widetilde{E}_1, \quad \widetilde{A}_2 + \widetilde{A}_3 = A_3 - A_2, \quad \widetilde{B}_2 + \widetilde{B}_3 = B_2 - B_3.$$

Proof. For the first relation, we add the second and the third formulae in the Property 4 and apply the first.

For the second relation, we subtract the last from the fifth formulae in the Property 4 and apply the fourth.

For the third relation in the case $E = A$, the sum of $A_2 = \widetilde{A}_1 - \widetilde{A}_3$ and $A_3 = \widetilde{A}_1 + \widetilde{A}_2$ gives $A_2 + A_3 = 2\,\widetilde{A}_1 + \widetilde{A}_2 - \widetilde{A}_3 = a + 2\,\widetilde{A}_1$.

Finally, $\widetilde{A}_2 + \widetilde{A}_3 = a + 2\,\widetilde{A}_3 = A_3 - A_2$. □

The most interesting cases of equalities among components of E and $\widetilde{E}$ are considered in the next property.

Property 6. *Let the commutative ring R have the identity e and no divisors of zero and let $a \in R$ have the inverse c. For $b, t, m, u \in R$,*

$$\alpha(a, b, t, u)_2 = A_3 \Leftrightarrow u = m + e \text{ or } u = -m - e - 2\,c\,t,$$

$$\widetilde{\alpha}(a, b, t, u)_2 = \widetilde{A}_3 \Leftrightarrow u = m - e,$$

$$\beta(a, b, t, u)_2 = B_3 \Leftrightarrow u = m - e \text{ or } u = -m + e - 2\,c\,t,$$

$$\widetilde{\beta}(a, b, t, u)_2 = \widetilde{B}_3 \Leftrightarrow u = m + e.$$

Proof. Since $\alpha(a, b, t, u)_2 - A_3 = (u - m - e)(2\,t + a(u + m + e))$, the first equivalence follows easily. The second is the consequence of the equality $\widetilde{\alpha}(a, b, t, u)_2 - \widetilde{A}_3 = a(u - m + e)$. □

4. Squares from products of components

The next two items are related to the first and the last identity of Property 1 and show that from the second and third components of u^A and A and of u^B and B it is possible to get complete squares in many ways. For example, $(u+m+1)_2^{\mathfrak{a}}\mathfrak{a}_3 + u^2 = [\mathfrak{a}_3 + u\widetilde{\mathfrak{a}}_2]^2$, i. e.,

$$[(u+m+1)^2 + 4(u+m+1) + 3](m^2+6m+8) + u^2 = \\ [(m^2+6m+8) + u(m+3)]^2.$$

Property 7. *If the commutative ring R has the multiplicative identity e, then for $a, b, t, m, u \in R$,*

$$\alpha(a,b,t,u+m+e)_2\, A_3 + n\, u^2 = (A_3 + u\, \widetilde{A}_2)^2,$$
$$A_2\, \alpha(a,b,t,u+m-e)_3 + n\, u^2 = (A_2 + u\, \widetilde{A}_3)^2,$$
$$\beta(a,b,t,u+m-e)_2\, B_3 + n\, u^2 = (B_3 + u\, \widetilde{B}_2)^2,$$
$$B_2\, \beta(a,b,t,u+m+e)_3 + n\, u^2 = (B_2 + u\, \widetilde{B}_3)^2.$$

Proof. The difference $[\alpha(a,b,t,u+m+e)_2\, A_3 + n\, u^2] - (A_3 + u\, \widetilde{A}_2)^2$ is zero because it reduces to the product $u^2\,(a\,b + n - t^2)$. The remaining three formulae have similar proofs. □

Notice that for $u = -e$ in the first formula of Property 7, in view of the first identity in the first group of Property 5, we get the first identity in Property 1. Similarly, for $u = e$ in the third formula of Property 7, in view of the last identity in the first group of Property 5, we get the last identity in Property 1.

When the commutative ring R with the multiplicative identity e have a $\frac{1}{2}$ (i.e., if there exists and element $h \in R$ with $2\,h = e$), from the second and the third components of u^E and the first component of $\widetilde{E}$ and also from the first component of $u^{\widetilde{E}}$ and the second and third components of E it is possible to get complete squares too. We state the identities for $E = A$ only. The proofs are similar to the proof of Property 7.

$$64\,\widetilde{\alpha}(a,b,t,m+h)_1\, A_3 + a^2 = (8\, A_3 - a)^2,$$
$$64\,\widetilde{A}_1\, \alpha(a,b,t,m+h)_2 + a^2 = 64\,\widetilde{A}_1\, \alpha(a,b,t,m-h)_3 + a^2 = (8\,\widetilde{A}_1 + a)^2,$$
$$64\,\widetilde{\alpha}(a,b,t,m-h)_1\, A_2 + a^2 = (8\, A_2 - a)^2.$$

For the pencil $\mathfrak{a}$, the first identity above gives the following:

$$64\,\widetilde{\mathfrak{a}}\left(m+\tfrac{1}{2}\right)_1 \mathfrak{a}_3 + 1 = 64\Big[\left(m+\tfrac{1}{2}\right)^2 + 5\,m + \tfrac{15}{2}\Big]\,(m^2+6\,m+8) + 1$$
$$= \left(8\,m^2 + 48\,m + 63\right)^2 = \left[8(m^2+6\,m+8) - 1\right]^2 = (8\,\mathfrak{a}_3 - 1)^2.$$

5. Squares from symmetric functions

Let σ_1, σ_2, $\sigma_3 : R^3 \to R$ be the basic symmetric functions defined for $x = (a, b, c)$ by $x_{\sigma_1} = a + b + c$, $x_{\sigma_2} = bc + ca + ab$, $x_{\sigma_3} = a\,b\,c$. Let $\sigma_2^*, \sigma_1^* : R^3 \to R$ be defined by $x_{\sigma_2^*} = b\,c - c\,a + a\,b$ and $x_{\sigma_1^*} = a - b + c$. Note that $x_{\sigma_1^*}$ is the determinant of the 1×3 matrix $[a, b, c]$ (see [13]).

The linear expressions involving these symmetric functions of the triples E and $\widetilde{E}$ are the source of many complete squares as the following properties show.

For the triple $\mathfrak{r}$, the first formula gives

$$1\cdot(2\cdot(1 + 3 + 8) - 3\cdot 1) + 4\cdot 1 = 25 = (3 + 2)^2$$

while the last becomes

$$(u+v)(v\mathfrak{r}_{\sigma_2}+u)-v(u\mathfrak{r}_{\sigma_2}-v) = 36\,v^2+12uv+u^2 = ((\widetilde{\mathfrak{r}}_1+1)v+u)^2.$$

Property 8. *For $a, b, t, m, u, v \in R$,*

$$a(2E_{\sigma_1} - 3a) + 4n = (\widetilde{E}_2 + \widetilde{E}_3)^2,$$

$$u\,(v\,E_{\sigma_1} + u) + v^2(E_{\sigma_2} + n) = [v\,(\widetilde{E}_1 + a) + u]^2,$$

$$a^2\,u[v(E_{\sigma_1} - 2a) + u] + a\,v^2(E_{\sigma_3} + a\,n) = a^2[v\,\widetilde{E}_1 + u]^2,$$

$$a^2(u + v)(v\,E_{\sigma_2} + a^2\,u) - a\,v(u\,E_{\sigma_3} - a\,n\,v) = a^2[v\,(\widetilde{E}_1 + a) + a\,u]^2.$$

Proof. Let $E = A$. We shall prove only the second formula. The sum

$$u\,(v\,A_{\sigma_1} + u) + v^2(A_{\sigma_2} + n) - [v\,(\widetilde{A}_1 + a) + u]^2$$

is zero because it simplifies into the product of v^2 and $a\,b + n - t^2$. □

In particular, when the ring R have the multiplicative identity e and $v = e$ and $u = 0$ and $v = e$ and u satisfies $u^2 = -n$, from the second and the third identity, we get the following statement.

Property 9. $E_{\sigma_2} + n = (\widetilde{E}_1 + a)^2, E_{\sigma_2} + u\,E_{\sigma_1} = (\widetilde{E}_1 + a + u)^2$ *and* $a(E_{\sigma_3} + a\,n) = a^2\,\widetilde{E}_1^2$.

For example, when applied to the triple $\mathfrak{r}$, we get

$$\mathfrak{r}_{\sigma_2} + 1 = (3\cdot 8 + 8\cdot 1 + 1\cdot 3) + 1 = 36 = (5 + 1)^2 = (\widetilde{\mathfrak{r}}_1 + 1)^2,$$

$$(3\cdot 8 + 8\cdot 1 + 1\cdot 3) \pm i(1 + 3 + 8) = 35 \pm 12i = (5 + 1 \pm i)^2,$$

$$\mathfrak{r}_{\sigma_3} + 1 = 1\cdot 3\cdot 8 + 1 = 25 = 5^2 = \widetilde{\mathfrak{r}}_1^2.$$

Another source of squares provides many products of components of the triples E and $\widetilde{E}$ or symmetric functions of them. Therefore, the following result is similar to those from the previous section. For the pencils $\mathfrak{a}$ and $\mathfrak{b}$, the first identity becomes

$$\begin{aligned}\mathfrak{a}_{\sigma_1}\,\mathfrak{b}_{\sigma_1} + 1 &= [(m+2)^2 + (m+3)^2 - 1][(m+2)^2 + (m+1)^2 - 1] + 1\\ &= (2\,m^2 + 10\,m + 12)(2\,m^2 + 6\,m + 4) + 1 = [2(m^2 + 4\,m + 3) + 1]^2.\end{aligned}$$

Property 10.

$$A_{\sigma_1}B_{\sigma_1} + 4\,n - 3\,a^2 = (2\,E_2 + a)^2,$$
$$4(\widetilde{A}_{\sigma_1}\widetilde{B}_{\sigma_1} + n) + 5\,a^2 = (2\,A_3 - 3\,a)^2,$$
$$A_{\sigma_2}B_{\sigma_2} + 4\,n\,\widetilde{E}_3^2 = (E_2^2 + \widetilde{E}_3^2 + a^2 - 3\,n)^2,$$
$$4\,a\,\widetilde{E}_{\sigma_3} + n^2 = (2\,a\,\widetilde{E}_1 + n)^2, \qquad a(E_{\sigma_3} + a\,n) = (a\,\widetilde{E}_1)^2,$$
$$\widetilde{A}_{\sigma_3}\,\widetilde{B}_{\sigma_3} + (n\,\widetilde{E}_3)^2 = [\widetilde{E}_3^2(E_2 - a)]^2.$$

Proof. We shall prove only the third identity. By Property 9, we know that $E_{\sigma_2} = (\widetilde{E}_1 + a)^2 - n$. Hence,

$$\begin{aligned}A_{\sigma_2}B_{\sigma_2} + 4\,n\,\widetilde{E}_3^2 = \big[&a^2\,m^4 + 4\,a\,t\,m^3 + \big(2\,a\,b + 4\,t^2 + a^2\big)\,m^2 +\\ &2\,t\,(2\,b + a)\,m + a^2 - 2\,t^2 + b^2 + 3\,ab\big]^2 = (E_2^2 + \widetilde{E}_3^2 + a^2 - 3\,n)^2.\end{aligned}$$

□

For the sums σ_1^* and σ_2^*, we have the following property. Of course, it is nice that the added part $n - a^2$ on the left hand side does not depend on the fourth variable m. For the triple $\mathfrak{r}$, the last identity is

$$1\cdot(1 - 3 + 8) + (3\cdot 8 - 8\cdot 1 + 1\cdot 3) + 1 - 1^2 = 5^2.$$

Property 11. *For $a, b, t, m \in R$, the following identities hold.*

$$4\,a^2 - A_{\sigma_1^*}B_{\sigma_1^*} = (2\,\widetilde{E}_3)^2,$$
$$4[\widetilde{A}_{\sigma_1^*}\widetilde{B}_{\sigma_1^*} + n - 2\,a\,\widetilde{E}_3] + 5\,a^2 = (2\,E_2 - a)^2,$$
$$A_{\sigma_2^*}B_{\sigma_2^*} + 4[a^4 + (a^2 - n)^2 - a\,n\,E_2] = (\widetilde{A}_1\,\widetilde{B}_1 + 3\,a^2 - n)^2,$$
$$\widetilde{A}_{\sigma_2^*}\widetilde{B}_{\sigma_2^*} - 2\,a\,n\,\widetilde{A}_1 = n^2,$$
$$a\,E_{\sigma_1^*} + E_{\sigma_2^*} - a^2 + n = \widetilde{E}_1^2.$$

Proof. We shall prove only the last two identities.

Since $\widetilde{A}_{\sigma_2^*} = 2\,a\,\widetilde{A}_1 + n$ and $\widetilde{B}_{\sigma_2^*} = n$ (see Property 36), we get

$$\widetilde{A}_{\sigma_2^*}\widetilde{B}_{\sigma_2^*} - 2\,a\,n\,\widetilde{A}_1 = n(2\,a\,\widetilde{A}_1 + n) - 2\,a\,n\,\widetilde{A}_1 = n^2.$$

Finally, let $E = A$. The sum $\widetilde{A}_1^2 - a\,A_{\sigma_1^*} - A_{\sigma_2^*} - n + a^2$ expands to $t^2 - a\,b - n = 0$. □

The following six cases are of the same type and use symmetric functions of both E and $\widetilde{E}$. Note that the versions for A and B can be different as in Property 7. For the triple $\mathfrak{r}$, the second formula gives

$$(3{\cdot}8+8{\cdot}1+1{\cdot}3)+2(3{\cdot}2+2{\cdot}5+5{\cdot}3)+3 = 100 = (8+2)^2.$$

Property 12. *For $a, b, t, m \in R$, we have*

$$4\,a\,n(\widetilde{E}_{\sigma_3} - E_{\sigma_3}) + n^2(4\,n - 3a^2) = n^2(\widetilde{E}_2 + \widetilde{E}_3)^2,$$

$$A_{\sigma_2} + 2\,\widetilde{A}_{\sigma_2} + 3\,n = (A_3 + \widetilde{A}_3)^2, \qquad B_{\sigma_2} + 2\,\widetilde{B}_{\sigma_2} + 3\,n = (B_2 + \widetilde{B}_2)^2,$$

$$4(A_{\sigma_2} - \widetilde{A}_{\sigma_2} + 3\,n) - 3\,a^2 = (2A_2 - a)^2,$$

$$4(B_{\sigma_2} - \widetilde{B}_{\sigma_2} + 3\,n) - 3\,a^2 = (2B_3 - a)^2.$$

Proof. In order to prove the second formula, note that $A_{\sigma_2} + 2\,\widetilde{A}_{\sigma_2} + 3\,n$ is the sum $\sum_{i=0}^{4} k_i\, m^i$, where $k_4 = a^2$, $k_3 = 2a(3a+2t)$, $k_2 = 11a^2 + 2a\,(9t+b) + 4t^2$, $k_1 = 2(3a+2t)(a+3t+b)$ and $k_0 = (a+3t+b)^2$. This sum we recognize as the square of $am^2 + (3a+2t)m + a + 3t + b$. In other words, the square of $A_3 + \widetilde{A}_3$. $\square$

6. Determinants of matrices from components

For triples $a, b, c \in R^3$, let $[a, b, c]$ and $|a, b, c|$ denote the 3×3 matrix whose rows are a, b and c and its determinant. The following property describes the determinants of the matrices whose rows are values of the functions α, β, $\widetilde{\alpha}$ and $\widetilde{\beta}$ and shows their invariance with respect to the translation of the fourth variable.

Let $\widetilde{\sigma}$ and $\widetilde{\pi}$ be the cyclic sum and the cyclic product. For example, $\widetilde{\sigma}(a_1) = a_1 + a_2 + a_3$ and $\widetilde{\pi}(a_1) = a_1\, a_2\, a_3$.

For $i = 1, 2, 3$, $x_i = (a_i, b_i, t_i, m_i) \in R^4$ and $w \in R$, let $x = (x_1, x_2, x_3)$ and let h_x^w, f_x^w and g_x^w denote $(m_2 - m_3)\, a_1\, w$,

$$(m_2 - m_3)[(2\,m_1 - m_2 - m_3)\, t_1\, w + b_1]\, a_2\, a_3\, w,$$

and $[2\,(m_2 - m_3)\, t_2\, t_3\, w + b_2\, t_3 - b_3\, t_2]\, a_1$. If the element w is absent from these expressions, we use h_x, f_x and g_x in this case.

Property 13. *For $i = 1, 2, 3$, an element w of the commutative ring R and $x_i = (a_i, b_i, t_i, m_i) \in R^4$, let $x = (x_1, x_2, x_3)$, $x_i^w = (a_i, b_i, t_i, m_i + w)$, $U_i^w = \alpha(x_i^w)$, $V_i^w = \beta(x_i^w)$, $\widetilde{U}_i^w = \widetilde{\alpha}(x_i^w)$ and $\widetilde{V}_i^w = \widetilde{\beta}(x_i^w)$. The values of $|U_1^w, U_2^w, U_3^w|$, $-|V_1^w, V_2^w, V_3^w|$, $-2\,|\widetilde{U}_1^w, \widetilde{U}_2^w, \widetilde{U}_3^w|$, and $2\,|\widetilde{V}_1^w, \widetilde{V}_2^w, \widetilde{V}_3^w|$ are all independent of w and are equal to $\Delta = 2\,[\widetilde{\sigma}(f_x + g_x) - \widetilde{\pi}(h_x)]$.*

Proof. Let $M_U = [U_1^w, U_2^w, U_3^w] = [u_{ij}]$. Then $|M_U| = f + g + h$, where $f = u_{11}(u_{22}\,u_{33} - u_{32}\,u_{23})$, $g = u_{21}(u_{32}\,u_{13} - u_{12}\,u_{33})$ and $h = u_{31}(u_{12}\,u_{23} - u_{22}\,u_{13})$. Since $f = a_1\,(2\,\lambda_1\,w^2 + 2\,\mu_1\,w + \nu_1)$, where $\lambda_1 = (m_2 - m_3)a_2\,a_3 - t_3\,a_2 + t_2\,a_3$, $\mu_1 = (m_2^2 + m_2 - m_3^2 - m_3)\,a_2\,a_3 - (2\,m_3\,t_3 + t_3 + b_3)a_2 + (2\,m_2\,t_2 + t_2 + b_2)a_3, \nu_1 = (m_2 - m_3)(m_2 + m_3 + 2\,m_2\,m_3)\,a_2\,a_3 - (2(2\,m_2\,m_3 + m_3 - m_2^2)\,t_3 + 2\,m_2\,b_3 + b_3)a_2 + (2(2\,m_2\,m_3 + m_2 - m_3^2)t_2 + 2\,m_3\,b_2 + b_2)\,a_3 + 2(2(m_2 - m_3)\,t_2\,t_3 - b_3\,t_2 + b_2\,t_3)$ and the parts g and h have analogous values (with indices cyclicly permuted), it follows that the sum $f + g + h$ has the above form Δ while the coefficients of w^2 and w are zero. The other determinants are computed similarly. □

The following special case of the previous property is perhaps easier to understand and appreciate.

Property 14. *For $a, b, t, u, v, w, z \in R$, the values of the determinants*

$$-|(u+z)^A, (v+z)^A, (w+z)^A|, \qquad |(u+z)^B, (v+z)^B, (w+z)^B|,$$

$$2\,|(u+z)^{\widetilde{A}}, (v+z)^{\widetilde{A}}, (w+z)^{\widetilde{A}}| \;\text{ and }\; -2\,|(u+z)^{\widetilde{B}}, (v+z)^{\widetilde{B}}, (w+z)^{\widetilde{B}}|$$

are all independent of b, t and z and are equal two times the value of the determinant of the Vandermonde matrix of the triple (au, av, aw).

In the same way we have the following result that explains what happens when the fourth variable is multiplied by the same element of the ring R.

Property 15. *For $i = 1, 2, 3$, an element w of the commutative ring R and $x_i = (a_i, b_i, t_i, m_i) \in R^4$, let $x = (x_1, x_2, x_3)$, $y_i^w = (a_i, b_i, t_i, w\,m_i)$, $P_i^w = \alpha(y_i^w)$, $Q_i^w = \beta(y_i^w)$, $\widetilde{P}_i^w = \widetilde{\alpha}(y_i^w)$ and $\widetilde{Q}_i^w = \widetilde{\beta}(y_i^w)$. The values of $|P_1^w, P_2^w, P_3^w|$, $-|Q_1^w, Q_2^w, Q_3^w|$, $-2\,|\widetilde{P}_1^w, \widetilde{P}_2^w, \widetilde{P}_3^w|$, and $2\,|\widetilde{Q}_1^w, \widetilde{Q}_2^w, \widetilde{Q}_3^w|$ are all equal to $2\,[\widetilde{\sigma}(f_x^w + g_x^w) - \widetilde{\pi}(h_x^w)]$.*

Again, perhaps it easier to comprehend the following special case of the previous property.

Property 16. *For $a, b, t, u, v, w, z \in R$, the values of the determinants $-|(u\,z)^A, (v\,z)^A, (w\,z)^A|$, $|(u\,z)^B, (v\,z)^B, (w\,z)^B|$, $2\,|(u\,z)^{\widetilde{A}}, (v\,z)^{\widetilde{A}}, (w\,z)^{\widetilde{A}}|$ and $-2\,|(u\,z)^{\widetilde{B}}, (v\,z)^{\widetilde{B}}, (w\,z)^{\widetilde{B}}|$ are all independent of b and t and are equal to the product of $2\,z^3$ and the value of the determinant of the Vandermonde matrix of the triple (au, av, aw).*

Of course, we could try to get similar results for other variables a, b and t. In all these cases the equality of the four determinants analogous to the ones from the previous four properties are retained. However, for the variables a and t the common values are rather complicated. Therefore, we describe only the simpler case of the variable b.

For $i = 1, 2, 3$ and $a_i, b_i, t_i, m_i \in R$, let $L = \widetilde{\sigma}((m_2 - m_3)\, a_2\, a_3 - (t_2 - t_3)\, a_1)$ and $L_* = \widetilde{\sigma}((m_2 - m_3)\, b_1\, a_2\, a_3 - (t_2\, b_3 - t_3\, b_2)\, a_1)$.

Property 17. *For $i = 1, 2, 3$, an element w of the commutative ring R and $a_i, b_i, t_i, m_i \in R$, let $u_i^w = (a_i, b_i + w, t_i, m_i), v_i^w = (a_i, b_i\, w, t_i, m_i), H_i^w = \alpha(u_i^w), K_i^w = \beta(u_i^w), M_i^w = \alpha(v_i^w), N_i^w = \beta(v_i^w)$, etc.*

The values of $|H_1^w, H_2^w, H_3^w|$, $-|K_1^w, K_2^w, K_3^w|$, $-2\,|\widetilde{H}_1^w, \widetilde{H}_2^w, \widetilde{H}_3^w|$, and $2\,|\widetilde{K}_1^w, \widetilde{K}_2^w, \widetilde{K}_3^w|$ are all equal to $\Delta + 2\,w\,L$.

The values of $|M_1^w, M_2^w, M_3^w|$, $-|N_1^w, N_2^w, N_3^w|$, $-2\,|\widetilde{M}_1^w, \widetilde{M}_2^w, \widetilde{M}_3^w|$ and $2\,|\widetilde{N}_1^w, \widetilde{N}_2^w, \widetilde{N}_3^w|$ are all equal to $\Delta + 2\,[w\,L_* - L_*]$.

Once more, the following special case of the previous property has a rather simple form.

Let Δ_1 and Δ_2 denote the determinants

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + w & b_2 + w & b_3 + w \\ t_1 & t_2 & t_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1\, w & b_2\, w & b_3\, w \\ t_1 & t_2 & t_3 \end{vmatrix}.$$

Property 18. *If $m_1 = m_2 = m_3 = m \in R$ in the Property 17, then the values of $|H_1^w, H_2^w, H_3^w|$,$-|K_1^w, K_2^w, K_3^w|$,$-2\,|\widetilde{H}_1^w, \widetilde{H}_2^w, \widetilde{H}_3^w|$, and $2\,|\widetilde{K}_1^w, \widetilde{K}_2^w, \widetilde{K}_3^w|$ are all equal to $2\,\Delta_1$.*

The values of $|M_1^w, M_2^w, M_3^w|$, $-|N_1^w, N_2^w, N_3^w|$, $-2\,|\widetilde{M}_1^w, \widetilde{M}_2^w, \widetilde{M}_3^w|$ and $2\,|\widetilde{N}_1^w, \widetilde{N}_2^w, \widetilde{N}_3^w|$ are all equal to $2\,\Delta_2$.

There are analogous versions of the previous property that describe the same special cases of Property 17 for the first and third variables a and t. The determinants similar to Δ_1 and Δ_2 have the first and the third row longer, respectively.

7. Cubes and squares from determinants

The following property shows that the functions α and β and the determinants can easily give us lots of cubes. For $a, b, t, m, u \in R$, let $*_A = \alpha(a, b, t, -m), u_A = \alpha(a, b, t, u\, m)$, etc. This notation is used here only in order to get nice one line relationships. For the pencils $\mathfrak{a}$ and $\mathfrak{b}$ and $i = \sqrt{-1}$, the Property 19 gives the identities

$$2\,|\widetilde{\mathfrak{b}}(\pm i\, m), \widetilde{\mathfrak{b}}(-m), \widetilde{\mathfrak{b}}| = -2\,|\widetilde{\mathfrak{a}}(\pm i\, m), \widetilde{\mathfrak{a}}(-m), \widetilde{\mathfrak{a}}| =$$
$$|\mathfrak{a}(\pm i\, m), \mathfrak{a}(-m), \mathfrak{a}| = -|\mathfrak{b}(\pm i\, m), \mathfrak{b}(-m), \mathfrak{b}| = (2\, m)^3.$$

Property 19. *Let $a, b, t, m, u \in R$, where R is a commutative ring with the multiplicative identity e. If $u^2 = -e$ then*

$$|u_A, *_A, A| = -2\,|u_{\widetilde{A}}, *_{\widetilde{A}}, \widetilde{A}| = -|u_B, *_B, B| = 2\,|u_{\widetilde{B}}, *_{\widetilde{B}}, \widetilde{B}| = (2\, a\, m)^3,$$

and if $u^2 = e$ then

$$|u_A, *_A, A| = -2\,|u_{\widetilde{A}}, *_{\widetilde{A}}, \widetilde{A}| = -|u_B, *_B, B| = 2\,|u_{\widetilde{B}}, *_{\widetilde{B}}, \widetilde{B}| = 0.$$

Proof. Let $U = a\,m^2$, $V = 2\,m\,t$ and $W = 2\,m(a+t)$. Then $[u_A, *_A, A]$ is

$$\begin{bmatrix} a & u^2\,U + u\,V + b & u^2\,U + u\,W + X_3 \\ a & U - V + b & U - W + X_3 \\ a & U + V + b & U + W + X_3 \end{bmatrix},$$

where $X_3 = a + b + 2\,t$. Its determinant is

$$2\,a\,U(e - u^2)(W - V) = 4\,(a\,m)^3(e - u^2).$$

Hence, for $u^2 = -e$ it is a cube and for $u^2 = e$ it is zero. □

In order to get squares from the above property, it suffices to consider the values $m = 2\,a$ or $m = a$. More precisely, if $u^2 = -e$, then

$$|\alpha(a, b, t, 2\,u\,a),\ \alpha(a, b, t, -2\,a),\ \alpha(a, b, t, 2\,a)| = (2\,a)^6,$$

$$-|\widetilde{\alpha}(a, b, t,\ u\,a),\ \widetilde{\alpha}(a, b, t,\ -a),\ \widetilde{\alpha}(a, b, t,\ a)| = (2\,a^3)^2,$$

and similarly for the function β.

Another method to get cubes using the functions α and β and the determinants is described in the following property.

Property 20. *For $a, b, t, m \in R$ and $u \in \{a, b, t\}$,*

$$|2(m+u)^A, -(m-u)^A, A| = |2(m+u)^B, (m-u)^B, B| =$$
$$|2(m+u)^{\widetilde{A}}, 2(m-u)^{\widetilde{A}}, \widetilde{A}| = |2(m+u)^{\widetilde{B}}, -2(m-u)^{\widetilde{B}}, \widetilde{B}| = (2\,a\,u)^3,$$

$$|2(3\,m)^A, -(5\,m)^A, A| = |2(3\,m)^B, (5\,m)^B, B| =$$
$$|2(3\,m)^{\widetilde{A}}, 2(5\,m)^{\widetilde{A}}, \widetilde{A}| = |2(3\,m)^{\widetilde{B}}, -2(5\,m)^{\widetilde{B}}, \widetilde{B}| = (4\,a\,m)^3.$$

Proof. Let M denote the matrix with rows $2(m+u)^A$, $-(m-u)^A$, A. In order to compute the determinant $|M|$, we use elementary row operations to get zeros in the first and the second row of the first column. Hence, $|M|$ is

$$a \begin{vmatrix} 8\,u\,\widetilde{A}_3 & 8\,u\,\widetilde{A}_2 \\ 2\,u\,\widetilde{A}_3 - a\,u^2 & 2\,u\,\widetilde{A}_2 - a\,u^2 \end{vmatrix} = (2\,a\,u)^3.$$

The other determinants are computed similarly. □

An interesting source of squares from the functions α, β, $\widetilde{\alpha}$ and $\widetilde{\beta}$ and the determinants is described in the following property.

For $a, b, t, u, v, w \in R$, let $M(u, v, w)$ denote the 6×6 matrix

$$\begin{pmatrix} (u^{\widetilde{A}})_1 & (u^{\widetilde{A}})_2 & (u^{\widetilde{A}})_3 & (u^{\widetilde{B}})_1 & (u^{\widetilde{B}})_2 & (u^{\widetilde{B}})_3 \\ (v^{\widetilde{A}})_1 & (v^{\widetilde{A}})_2 & (v^{\widetilde{A}})_3 & (v^{\widetilde{B}})_1 & (v^{\widetilde{B}})_2 & (v^{\widetilde{B}})_3 \\ (w^{\widetilde{A}})_1 & (w^{\widetilde{A}})_2 & (w^{\widetilde{A}})_3 & (w^{\widetilde{B}})_1 & (w^{\widetilde{B}})_2 & (w^{\widetilde{B}})_3 \\ (u^{A})_1 & (u^{A})_2 & (u^{A})_3 & (u^{B})_1 & (u^{B})_2 & (u^{B})_3 \\ (v^{A})_1 & (v^{A})_2 & (v^{A})_3 & (v^{B})_1 & (v^{B})_2 & (v^{B})_3 \\ (w^{A})_1 & (w^{A})_2 & (w^{A})_3 & (w^{B})_1 & (w^{B})_2 & (w^{B})_3 \end{pmatrix}.$$

Property 21. *For $a, b, t, u, v, w \in R$,*

$$|M(u, v, w)| = \left[4\, a^3 (v-w)(w-u)(u-v)\right]^2.$$

Proof. Let us replace the fifth column of the matrix $M(u, v, w)$ with the difference of the fifth and the second column. The fifth column of the new matrix $M^*(u, v, w)$ has the $-2\,a$ in the first three rows and the 0 in the last three rows. Since the minors the first three elements in this fifth column of $M^*(u, v, w)$ have the determinants

$$\Gamma(v-w)(v_3^{\widetilde{A}}\, w_3^{\widetilde{A}} + m), \;\; \Gamma(u-w)(w_3^{\widetilde{A}}\, u_3^{\widetilde{A}} + m), \;\; \Gamma(u-v)(u_3^{\widetilde{A}}\, v_3^{\widetilde{A}} + m),$$

where $\Gamma = 8\, a^3\, (v-w)(w-u)(u-v)$ and $m = n - a^2$. Hence, we see that the determinant $|M(u, v, w)|$ is $\left[4\, a^3 (v-w)(w-u)(u-v)\right]^2$. $\square$

The following special case of the previous property gives a method to get the sixth powers (hence, both the squares and the cubes).

Property 22. *For $a, b, t, m \in R$,*

$$|M(m, 3\,m, 5\,m)| = (4\, a\, m)^6.$$

Notice that the right hand sides of the identities in the previous three properties do not depend on the variables b and t.

8. Replacement of m with $-m-u$ in sums

The following results show that it is possible to replace the fourth variable m with $-m-u$ for a suitable value $u \neq -2m$ without changing the basic symmetric sums of components. The first and the last identities for the triple $\mathfrak{b}$ are

$$(-m-3)^{\mathfrak{b}}_{\sigma_3} = [(-m-3)^2 + 4(-m-3) + 3] \\ [(-m-3)^2 + 2(-m-3)] = m(m+1)(m+2)(m+3) = \widetilde{\mathfrak{b}}_{\sigma_3},$$

$$\widetilde{\mathfrak{b}}_{\sigma_2^*} = (m+1)(m+2) - (m+2)(m^2+3m+1) + (m^2+3m+1)(m+1) = 1.$$

Property 23. *Let $a, b, t, m \in R$, where R is a commutative ring with the multiplicative identity e. If a has the inverse c and $u_\pm = 2ct \pm e$ and $v_\pm = -m - u_\pm$, then*

$$E_{\sigma_3} = (v_\pm)^E_{\sigma_3}, \qquad \widetilde{E}_{\sigma_3} = (v_\pm)^{\widetilde{E}}_{\sigma_3}, \qquad E_{\sigma_2} = (v_\pm)^E_{\sigma_2}, \qquad E_{\sigma_1} = (v_\pm)^E_{\sigma_1},$$
$$\widetilde{E}_{\sigma_1} = (v_\pm \mp 2e)^{\widetilde{E}}_{\sigma_1}, \qquad \widetilde{E}_{\sigma_1^*} = (v_\pm)^{\widetilde{E}}_{\sigma_1^*}, \qquad \widetilde{A}_{\sigma_2^*} = (v_+)^{\widetilde{A}}_{\sigma_2^*}, \qquad \widetilde{B}_{\sigma_2^*} = n.$$

Proof. Let $E = A$. In the first identity, for a $u \in R$, the difference of $(-m-u)^A_{\sigma_3}$ and A_{σ_3} factors as

$$a(u + 2m)(au - a - 2t)[au^2 + (2am - a - 2t)u + 2(am^2 + b + t)].$$

We conclude that for $au - a - 2t = 0$, i. e., for $u = u_+$, this difference is zero. The proofs of the other identities are similar. □

9. Identities for the operations $\odot$, $\triangleright$ and $\triangleleft$

Let us introduce three binary operations $\odot$, $\triangleright$ and $\triangleleft$ on the set R^3 of triples of elements of the ring R by the rules

$$(a,\, b,\, c) \odot (u,\, v,\, w) = (a\, u,\, b\, v,\, c\, w),$$
$$(a,\, b,\, c) \triangleright (u,\, v,\, w) = (a\, v,\, b\, w,\, c\, u),$$
$$(a,\, b,\, c) \triangleleft (u,\, v,\, w) = (a\, w,\, b\, u,\, c\, v).$$

For the functions $U = A \triangleright B$ and $V = A \triangleleft B$ the following analogues of Property 8 are true.

Property 24. *For $a, b, t, m \in R$, the following identities hold:*

$$n^3 + U_{\sigma_1} n^2 + U_{\sigma_2} n + U_{\sigma_3} = \widetilde{B}_1^2\, \widetilde{A}_2^2\, \widetilde{E}_3^2,$$
$$n^3 + V_{\sigma_1} n^2 + V_{\sigma_2} n + V_{\sigma_3} = \widetilde{A}_1^2 \widetilde{B}_2^2\, \widetilde{E}_3^2.$$

Proof. For $a, b, c, d \in R$, let $X = (a, b, c)$, $Y = (a, b, d)$, $u = X \triangleright Y$ and $v = X \triangleleft Y$. The above formulae for U and V follow from the following more general identities for u and v

$$n^3 + u_{\sigma_1} n^2 + u_{\sigma_2} n + u_{\sigma_3} = (b\, d + n)(a\, b + n)(a\, c + n),$$
$$n^3 + v_{\sigma_1} n^2 + v_{\sigma_2} n + v_{\sigma_3} = (a\, d + n)(a\, b + n)(b\, c + n).$$

□

Let $\diamond$ be either $\triangleright$ or $\triangleleft$. Let $\Phi = E \diamond E$. The following property gives another method of getting the complete squares using the triples Φ. For the pencil $\mathfrak{a}$ this formula in the case when the operation $\diamond$ is $\triangleright$ and $\mathfrak{c} = \mathfrak{a} \triangleright \mathfrak{a} = (f, f\, g, g)$, with $f = \widetilde{\mathfrak{a}}_2\, \widetilde{\mathfrak{b}}_2$ and $g = \widetilde{\mathfrak{a}}_3(\widetilde{\mathfrak{a}}_2 + 1)$, is as follows:

$$u^2(\mathfrak{c}_{\sigma_1} + 1) + v(u\, \mathfrak{c}_{\sigma_2} + v\, \mathfrak{c}_{\sigma_3}) = (\widetilde{\mathfrak{a}}_2\, \widetilde{\mathfrak{a}}_3[u + v(\widetilde{\mathfrak{a}}_1 - 1)])^2.$$

Property 25. *For $a, b, t, m, u, v \in R$,*

$$u^2(\Phi_{\sigma_1} + n) + v(u\,\Phi_{\sigma_2} + v\,\Phi_{\sigma_3}) = (u(\widetilde{E}_1 + a) + v\,E_{\sigma_3})^2.$$

Proof. For a triple $X = (u, v, w) \in R^3$, since $X \triangleright X = (u\,v, v\,w, w\,u)$ and $X \triangleleft X = (w\,u, u\,v, v\,w)$, the (basic) symmetric functions on these triples are the same. Hence, both products $\triangleright$ and $\triangleleft$ give the same left hand side in the above relation.

Let $E = A$ and $\diamond = \triangleright$. Since $\Phi_{\sigma_1} + n = (\widetilde{A}_1 + a)^2$, $\Phi_{\sigma_2} = 2\,A_{\sigma_3}\,(\widetilde{A}_1 + a)$ and $\Phi_{\sigma_3} = A^2_{\sigma_3}$, the above identity follows. □

Note that the left hand side of the identity in Property 25 is equal to $u^2(E_{\sigma_2} + n) + v\,E_{\sigma_3}(u\,E_{\sigma_1} + v\,E_{\sigma_3})$, its non-linear form.

Let $\underline{E} = E \odot E = (E_1^2, E_2^2, E_3^2)$ and $\underline{\widetilde{E}} = \widetilde{E} \odot \widetilde{E} = (\widetilde{E}_1^2, \widetilde{E}_2^2, \widetilde{E}_3^2)$. For the triple $\mathfrak{r}$, the next two identities in Property 26 are

$$(1+9+64)-2 = 72 = 2\cdot(5+1)^2, \qquad (25+9+4)-2 = 36 = (5+1)^2.$$

Property 26. *For $a, b, t, m \in R$,*

$$\underline{E}_{\sigma_1} - 2\,n = 2\,(\widetilde{E}_1 + a)^2, \qquad \underline{\widetilde{E}}_{\sigma_1} - 2\,n = (\widetilde{E}_1 + a)^2.$$

Proof. In order to prove the equality $\underline{\widetilde{A}}_{\sigma_1} - 2\,n = (\widetilde{A}_1 + a)^2$, note that the difference of its left and right sides is the product $2\,(t^2 - a\,b - n)$. The claim follows because $t^2 = a\,b + n$. □

All three symmetric functions of the triples $\underline{\widetilde{E}}$ appear in the following property. Of course, it is another method to get complete squares. For the triple $\mathfrak{r}$, this identity is $36\,u^2 + 360\,u\,v + 900\,v^2 = (6\,u + 5\cdot 6\,v)^2$.

Property 27. *For $a, b, t, m, u, v \in R$,*

$$u^2(\underline{\widetilde{E}}_{\sigma_1} - 2n) + u\,v(\underline{\widetilde{E}}_{\sigma_2} - n^2) + v^2\underline{\widetilde{E}}_{\sigma_3} = [u(\widetilde{E}_1 + a) + v\widetilde{E}_1(a\widetilde{E}_1 + n)]^2.$$

Proof. The difference of $u^2(\underline{\widetilde{A}}_{\sigma_1} - 2\,n) + u\,v\,(\underline{\widetilde{A}}_{\sigma_2} - n^2) + v^2\underline{\widetilde{A}}_{\sigma_3}$ and the square $[u(\widetilde{A}_1 + a) + v\,\widetilde{A}_1(a\,\widetilde{A}_1 + n)]^2$ has as a factor the expression $z = t^2 - a\,b - n$ from the previous proof. Since $z = 0$, we conclude that the formula is true. □

For the sums σ_1^* and σ_2^* the analogues of the previous property are the following relations.

Property 28. *For $a, b, t, m \in R$, $u = 4\,n + a^2$ and $v = n + a^2$,*

$$(u \pm 4\,a^2)\widetilde{E}_{\sigma_2^*} \mp a(u \pm a^2)\widetilde{E}_{\sigma_1^*} = (v \pm a^2)[2\,\widetilde{E}_3 \pm a]^2.$$

Proof. Let $E = B$. Since $B_{\sigma_2^*} = n$ and $B_{\sigma_1^*} = \widetilde{B}_1 + a$, we get

$$(u - 4\,a^2)\widetilde{B}_{\sigma_2^*} + a(u - a^2)\widetilde{B}_{\sigma_1^*} = (4\,n - 3\,a^2)n + 4\,a\,n(\widetilde{B}_1 + a)$$
$$= n[4\,n - 3\,a^2 + 4\,a\,\widetilde{B}_1 + 4\,a^2] = (v - a^2)(2\,\widetilde{E}_3 - a)^2.$$

The proof of the case $E = A$ is similar. □

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